

# Complex dynamics effect on distributions

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**Abstract.** In this study, Lagrangian and Hamiltonian systems, which are mathematical models of mechanical systems, were introduced on the horizontal and the vertical distributions of tangent and cotangent bundles. Finally, some geometrical and physical results related to Lagrangian and Hamiltonian dynamical systems were deduced.

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**Key words:** tangent-cotangent bundles, Lagrangian-Hamiltonian systems.

## 1 Introduction

In modern differential geometry, the tangent-cotangent bundles of any differentiable manifold  $M$  are regarded as phase spaces of velocity-momentum of a given configuration space. Therefore Lagrangian and Hamiltonian systems in classical mechanics are explained by means of basic structures on the bundles so that this structures can be given by Liouville vector field  $C$ , Liouville form  $\lambda$ , tangent structure  $J$ , complex structure  $F$ , vertical distribution  $V$ , horizontal distribution  $H$  and semispray  $X$ . The following symbolical equation expresses the dynamical equations for both Lagrangian and Hamiltonian systems:

$$(1.1) \quad i_X \Phi = F$$

If one studies the Lagrangian systems, then equation (1.1) is the intrinsical form of Euler-Lagrange equations, where  $\Phi = \Phi_L = -dd_F L$  and  $F = dE_L$  such that  $L$  defined by  $L : TM \rightarrow \mathbf{R}$  is Lagrangian function,  $E_L$  given by  $E_L = C(L) - L$  is the energy function associated to  $L$ .

If one studies the Hamiltonian theory then equation (1.1) is the intrinsical form of Hamiltonian equations, where  $\Phi = \phi_H = -d\lambda$ ,  $F = dH$  and  $H$  is a Hamiltonian function such that  $H : T^*M \rightarrow \mathbf{R}$ . Mathematical expressions of mechanical systems are generally given by the Hamiltonian and Lagrangian systems. These expressions, in particular geometric expressions in mechanics and dynamics, are given in some studies

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[1, 2, 4, 6, 7]. It is well-known that the Lagrangian distribution on symplectic manifolds is used in geometric quantization, and a connection on any symplectic manifold is an important structure leading to a deformation quantization [3]. Para-complex analogues of the Lagrangians and Hamiltonians were obtained in the framework of para-Kählerian manifold and the geometric conclusions on a paracomplex dynamical systems were obtained [8].

As determined references and there in, real-complex-paracomplex geometry and mechanical-dynamical systems were analyzed successfully, but they have not dealt with complex dynamical systems on horizontal distribution and vertical distribution of tangent-cotangent bundles of any manifold  $M$ .

Therefore, here, Euler-Lagrange and Hamiltonian equations related to complex dynamical systems on the distributions used in obtaining geometric quantization have been obtained.

## 2 Preliminaries

In this study all the mappings are considered of the class  $C^\infty$ , expressed by the words "differentiable" or "smooth". The indices  $i, j, \dots$  run over set  $\{1, \dots, n\}$  and the Einstein convention of summarizing is adopted over all this paper.  $\mathbf{R}$ ,  $\mathcal{F}(TM)$ ,  $\chi(TM)$  and  $\chi(T^*M)$  denote the set of real numbers, the set of real functions on  $TM$ , the set of vector fields on  $TM$  and the set of 1-forms on  $T^*M$ , respectively.

### 2.1 Basic Structures

In this subsection, some definitions were derived taking into consideration the definitions given in [5]. Let  $TM$  be tangent bundle of a real manifold  $M$  of dimension  $n$ . Then we denote by  $x$  a point of  $M$  and by  $(U, \varphi)$  its local coordinate system such that  $\varphi(x) = (x^i)$ . Such that the projection  $\pi : TM \rightarrow M$ ,  $\pi(u) = x$ , a point  $u \in TM$  will be denoted by  $(x, y)$ , its local coordinates being  $(x^i, y^i)$ . Then we can consider both the natural basis  $(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i})$  and the dual basis  $(dx^i, dy^i)$  of the tangent space  $T_u TM$  and the cotangent space  $T_u^*(TM)$  at the point  $u \in TM$ , respectively. Consider the  $\mathcal{F}(TM)$ - and  $\mathcal{F}(T^*M)$ - linear mappings  $J : \chi(TM) \rightarrow \chi(TM)$  and  $J^* : \chi(T^*M) \rightarrow \chi(T^*M)$  given by

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \quad J\left(\frac{\partial}{\partial y^i}\right) = 0,$$

and

$$J^*(dx^i) = dy^i, \quad J^*(dy^i) = 0.$$

Assume that the tangent space  $V_u$  to the fibre  $\pi^{-1}(x)$  at the point  $u \in TM$  is locally spanned by  $\{\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}\}$ . Therefore, the mapping  $V : u \in TM \rightarrow V_u \subset T_u TM$  provides a regular distribution generated by the adapted basis  $\{\frac{\partial}{\partial y^i}\}$ . Consequently,  $V$  is an integrable distribution on  $TM$ . Then we say that  $V$  is a vertical distribution on  $TM$ . Let  $N$  be a nonlinear connection on  $TM$ . Suppose that  $N$  is characterized by  $v$ ,  $h$  vertical and horizontal projectors. Also we consider the vertical projector  $v : \chi(TM) \rightarrow \chi(TM)$  defined by  $v(X) = X$ ,  $\forall X \in \chi(VTM)$ ;  $v(X) = 0$ ,  $\forall X \in \chi(HTM)$ . Similarly, the mapping  $H : u \in TM \rightarrow H_u \subset T_u TM$  provides a regular

distribution determined by the adapted basis  $\{\frac{\delta}{\delta x^i}\}$ . Consequently,  $H$  is an integrable distribution on  $TM$ . Then we call that  $H$  is a horizontal distribution on  $TM$ . Consider that there is a  $\mathcal{F}(TM)$ -linear mapping  $h : \chi(TM) \rightarrow \chi(TM)$ , for which  $h^2 = h$ ,  $Ker h = \chi(VTM)$ . In the case, any vector field  $X \in \chi(TM)$  can be uniquely written as follows  $X = hX + vX = X^H + X^V$ . Therefore  $X^H$  and  $X^V$  are horizontal and vertical components of vector field  $X$ . And then, any vector field  $X$  can be uniquely written in the form

$$X = X^H + X^V$$

such that

$$X^H = X^i(\frac{\partial}{\partial x^i} - N_j^i(x, y)\frac{\partial}{\partial y^j}), \quad X^V = X^i N_j^i(x, y)\frac{\partial}{\partial y^j}$$

where  $N_j^i$  is the local coefficient of a nonlinear connection  $N$  on  $TM$ .

$(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$  is a local basis adapted to the horizontal distribution  $HTM$  and the vertical distribution  $VTM$ . Then  $(dx^i, \delta y^i)$  is the dual basis of  $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$  basis. So, we have

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_j^i(x, y)\frac{\partial}{\partial y^j},$$

and

$$\delta y^i = dy^i + N_j^i(x, y)dx^j.$$

Let  $F$  be an almost complex structure on  $TM$ . Then  $F^*$  is the dual structure of  $F$ . Hence regarding the operators  $h, v, F, F^*$  we find

$$\begin{aligned} h + v &= I; \quad F^2 = -I; \quad F^{*2} = -I; \\ h(\frac{\delta}{\delta x^i}) &= \frac{\delta}{\delta x^i}; \quad h(\frac{\partial}{\partial y^i}) = 0; \quad v(\frac{\delta}{\delta x^i}) = 0; \quad v(\frac{\partial}{\partial y^i}) = \frac{\partial}{\partial y^i}, \\ F(\frac{\delta}{\delta x^i}) &= -\frac{\partial}{\partial y^i}; \quad F(\frac{\partial}{\partial y^i}) = \frac{\delta}{\delta x^i}, \\ F^*(dx^i) &= -\delta y^i; \quad F^*(\delta y^i) = dx^i. \end{aligned}$$

### 3 Lagrangian Dynamical Systems

In this section, Euler-Lagrange equations for classical mechanics are derived by means of an almost complex structure  $F$  under the consideration of the basis  $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$  on distributions  $HTM$  and  $VTM$  of tangent bundle  $TM$  of manifold  $M$ . Let  $(x^i, y^i)$  be local coordinates in  $TM$ . Let a semispray be the vector field  $X$  given by

$$(3.1) \quad X = X^i \frac{\delta}{\delta x^i} + \dot{X}^i \frac{\partial}{\partial y^i}, \quad \dot{X}^i = X^j N_j^i$$

where the dot indicates the derivative with respect to time  $t$ . The vector field denoted by  $C = F(X)$  and expressed by

$$(3.2) \quad C = -X^i \frac{\partial}{\partial y^i} + \dot{X}^i \frac{\delta}{\delta x^i}$$

is called *Liouville vector field* on the bundle  $TM$ . The maps given by  $\mathbf{T}, \mathbf{P} : TM \rightarrow \mathbf{R}$  such that  $\mathbf{T} = \frac{1}{2}m_i(\dot{x}^i)^2$ ,  $\mathbf{P} = m_i g h$  are called *the kinetic energy* and *the potential energy of the mechanical system*, respectively. Here  $m_i, g$  and  $h$  stand for the mass of a mechanical system having  $m$  particles, the gravity acceleration and the distance

to the origin of a mechanical system on the tangent bundle  $TM$ , respectively. Then  $L : TM \rightarrow \mathbf{R}$  is a map that satisfies the conditions; i)  $L = \mathbf{T-P}$  is a *Lagrangian function*, ii) the function given by  $E_L = C(L) - L$  is a *Lagrangian energy*. The operator  $i_F$  induced by  $F$  and shown by

$$(3.3) \quad i_F \omega(X_1, X_2, \dots, X_r) = \sum_{i=1}^r \omega(X_1, \dots, F(X_i), \dots, X_r)$$

is said to be *vertical derivation*, where  $\omega \in \wedge^r TM$ ,  $X_i \in \chi(TM)$ . The *vertical differentiation*  $d_F$  is defined by

$$(3.4) \quad d_F = [i_F, d] = i_F d - d i_F$$

where  $d$  is the usual exterior derivation. For an almost complex structure  $F$ , the closed fundamental form is the closed 2-form given by  $\Phi_L = -d_F L$  such that

$$(3.5) \quad d_F : \mathcal{F}(TM) \rightarrow T^*M.$$

Then we have

$$(3.6) \quad \begin{aligned} \Phi_L &= -\left(\frac{\delta}{\delta x^j} dx^j + \frac{\partial}{\partial y^j} \delta y^j\right) \left(-\frac{\partial L}{\partial y^i} dx^i + \frac{\delta L}{\delta x^i} \delta y^i\right) \\ &= \frac{\delta(\partial L)}{\delta x^j \partial y^i} dx^j \wedge dx^i - \frac{\delta^2 L}{\delta x^j \delta x^i} dx^j \wedge \delta y^i + \frac{\partial^2 L}{\partial y^j \partial y^i} \delta y^j \wedge dx^i - \frac{\partial(\delta L)}{\partial y^j \delta x^i} \delta y^j \wedge \delta y^i. \end{aligned}$$

Let  $X$  be the second order differential equation (semispray) determined by **eq.** (1.1). Then we obtain

$$(3.7) \quad \begin{aligned} i_X \Phi_L &= \Phi_L(X) = X^i \frac{\delta(\partial L)}{\delta x^j \partial y^i} \delta_i^j dx^i - X^i \frac{\delta(\partial L)}{\delta x^j \partial y^i} dx^j - X^i \frac{\delta^2 L}{\delta x^j \delta x^i} \delta_i^j \delta y^i + \dot{X}^i \frac{\delta^2 L}{\delta x^j \delta x^i} dx^j \\ &+ \dot{X}^i \frac{\partial^2 L}{\partial y^j \partial y^i} \delta_i^j dx^i - X^i \frac{\partial^2 L}{\partial y^j \partial y^i} \delta y^j - \dot{X}^i \frac{\partial(\delta L)}{\partial y^j \delta x^i} \delta_i^j \delta y^i + \dot{X}^i \frac{\partial(\delta L)}{\partial y^j \delta x^i} \delta y^j. \end{aligned}$$

Since the closed 2-form  $\Phi_L$  on  $TM$  is the symplectic structure, we find

$$(3.8) \quad E_L = C(L) - L = -X^i \frac{\partial L}{\partial y^i} + \dot{X}^i \frac{\delta L}{\delta x^i} - L$$

and hence

$$(3.9) \quad dE_L = -X^i \frac{\delta(\partial L)}{\delta x^j \partial y^i} dx^j + \dot{X}^i \frac{\delta^2 L}{\delta x^j \delta x^i} dx^j - \frac{\delta L}{\delta x^j} dx^j - X^i \frac{\partial^2 L}{\partial y^j \partial y^i} \delta y^j + \dot{X}^i \frac{\partial(\delta L)}{\partial y^j \delta x^i} \delta y^j - \frac{\partial L}{\partial y^j} \delta y^j.$$

With the use of (1.1), considering (3.7) and (3.9), we obtain

$$(3.10) \quad \begin{aligned} &X^i \frac{\delta(\partial L)}{\delta x^j \partial y^i} \delta_i^j dx^i - X^i \frac{\delta(\partial L)}{\delta x^j \partial y^i} dx^j - X^i \frac{\delta^2 L}{\delta x^j \delta x^i} \delta_i^j \delta y^i + \dot{X}^i \frac{\delta^2 L}{\delta x^j \delta x^i} dx^j \\ &+ \dot{X}^i \frac{\partial^2 L}{\partial y^j \partial y^i} \delta_i^j dx^i - X^i \frac{\partial^2 L}{\partial y^j \partial y^i} \delta y^j - \dot{X}^i \frac{\partial(\delta L)}{\partial y^j \delta x^i} \delta_i^j \delta y^i + \dot{X}^i \frac{\partial(\delta L)}{\partial y^j \delta x^i} \delta y^j \\ &= -X^i \frac{\delta(\partial L)}{\delta x^j \partial y^i} dx^j + \dot{X}^i \frac{\delta^2 L}{\delta x^j \delta x^i} dx^j - \frac{\delta L}{\delta x^j} dx^j - X^i \frac{\partial^2 L}{\partial y^j \partial y^i} \delta y^j + \dot{X}^i \frac{\partial(\delta L)}{\partial y^j \delta x^i} \delta y^j - \frac{\partial L}{\partial y^j} \delta y^j \end{aligned}$$

or

$$(3.11) \quad X^i \frac{\delta(\partial L)}{\delta x^j \partial y^i} dx^j - X^i \frac{\delta^2 L}{\delta x^j \delta x^i} \delta y^j + \dot{X}^i \frac{\partial^2 L}{\partial y^j \partial y^i} dx^j - \dot{X}^i \frac{\partial(\delta L)}{\partial y^j \delta x^i} \delta y^j + \frac{\delta L}{\delta x^j} dx^j + \frac{\partial L}{\partial y^j} \delta y^j = 0.$$

If a curve denoted by  $\alpha : \mathbf{R} \rightarrow TM$  is considered to be an integral curve of  $X$ , i.e.  $X(\alpha(t)) = \frac{d\alpha(t)}{dt}$  then we have

$$(3.12) \quad \frac{d}{dt} \left( \frac{\partial L}{\partial y^i} \right) + \frac{\delta L}{\delta x^i} = 0, \quad \frac{d}{dt} \left( \frac{\delta L}{\delta x^i} \right) - \frac{\partial L}{\partial y^i} = 0,$$

which are named to be *Euler-Lagrange equations* deduced by means of an almost complex structure  $F$  on  $HTM$  horizontal and  $VTM$  vertical distributions.

Thus the triple  $(TM, \Phi_L, X)$  is a *mechanical system*, which is structured by means of an almost complex structure  $F$  and taking into account the basis  $\left\{ \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i} \right\}$  on the distributions  $HTM$  and  $VTM$ .

## 4 Hamiltonian Dynamical Systems

In this section, the Hamiltonian equations for classical mechanics are obtained on the distributions  $HT^*M$  and  $VT^*M$  of  $T^*M$ . Suppose that an almost complex structure, a Liouville form and a 1-form on  $T^*M$  are shown by  $P^*$ ,  $\lambda$  and  $\omega$ , respectively. Then we have

$$(4.1) \quad \omega = \frac{1}{2} (x^i dx^i + y^i \delta y^i)$$

and

$$(4.2) \quad \lambda = P^*(\omega) = \frac{1}{2} (-x^i \delta y^i + y^i dx^i).$$

As is well-known that if  $\phi$  is a closed 2-form on  $T^*M$ , then  $\phi_H$  is also a symplectic structure on  $T^*M$ . If the Hamiltonian vector field  $X_H$  associated with the Hamiltonian energy  $H$  is given by

$$(4.3) \quad X_H = X^i \frac{\delta}{\delta x^i} + Y^i \frac{\partial}{\partial y^i},$$

then we infer

$$(4.4) \quad \phi_H = -d\lambda = \delta y^i \wedge dx^i$$

and

$$(4.5) \quad i_{X_H} \phi = Y^i dx^i - X^i \delta y^i.$$

Moreover, the differential of Hamiltonian energy is written as follows:

$$(4.6) \quad dH = \frac{\delta H}{\delta x^i} dx^i + \frac{\partial H}{\partial y^i} \delta y^i.$$

By means of (1.1), using (4.5) and (4.6), the Hamiltonian vector field is calculated to be

$$(4.7) \quad X_H = -\frac{\partial H}{\partial y^i} \frac{\delta}{\delta x^i} + \frac{\delta H}{\delta x^i} \frac{\partial}{\partial y^i}.$$

Suppose that a curve

$$(4.8) \quad \alpha: I \subset \mathbf{R} \rightarrow T^*M$$

is an integral curve of the Hamiltonian vector field  $X_H$ , i.e.,

$$(4.9) \quad X_H(\alpha(t)) = \frac{d\alpha(t)}{dt}, \quad t \in I.$$

In the local coordinates, if one puts

$$(4.10) \quad \alpha(t) = (x^i(t), y^i(t)),$$

we obtain

$$(4.11) \quad \frac{d\alpha(t)}{dt} = \frac{dx^i}{dt} \frac{\delta}{\delta x^i} + \frac{dy^i}{dt} \frac{\partial}{\partial y^i}.$$

Taking into consideration (4.7), (4.9) and (4.11), we have the equations

$$(4.12) \quad \frac{dx^i}{dt} = -\frac{\partial H}{\partial y^i}, \quad \frac{dy^i}{dt} = \frac{\delta H}{\delta x^i},$$

which are named to be *Hamiltonian equations* deduced by means of an almost complex structure  $F^*$  on the horizontal distribution  $HT^*M$  and vertical distribution  $VT^*M$ .

Hence the triple  $(T^*M, \phi_H, X_H)$  is shown to be a *Hamiltonian mechanical system*, deduced by means of an almost complex structure  $F^*$  and using of basis  $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$  on the distributions  $HT^*M$  and  $VT^*M$ .

## 5 Conclusions

The physical meanings of the splitting  $TM = HTM \oplus VTM$  and of its dual, have been given in this paper. The Lagrangian and Hamiltonian dynamics have intrinsically been described by means of the almost complex structures  $F$  and  $F^*$ , as being functionals on distributions of tangent and cotangent bundles  $TM$  and  $T^*M$  of the manifold  $M$ , respectively. Also, we deduce that the Euler-Lagrange equations given by (3.12) turn into the Hamiltonian equations defined by (4.12), in view of the equalities  $x^i = \frac{\delta L}{\delta x^i}$ ,  $y^i = \frac{\partial L}{\partial y^i}$  and  $H = -L$ , and the corresponding converse.

## 6 Discussions

As is well-known, the geometry of Lagrangian and Hamiltonian formalisms gives a model for Relativity, Gauge Theory and electromagnetism in a very natural blending of the geometrical structures of the space with the characteristics properties of these physical fields. Therefore we consider that the equations (3.12) and (4.12) especially can be used in fields determined the above.

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