Ricci curvature and Weitzenböck formula on 3-dimensional Sasakian manifolds

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Abstract. In this paper we study a semiconformal map $\psi : M^3 \to N^2$, where M is endowed with a Sasakian structure (φ, ξ, η, g) and N^2 is an orientable surface endowed with a Hermitian structure (J, h). We will suppose that ψ is a $(\varphi, J)-$ holomorphic map.

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1 Introduction

The study of harmonic maps on contact metric manifolds was initiated by S. Ianus and A. M. Pastore ([14, 15]. This study was continued by C. Gherghe (see [10, 11]). Old and recent researches show the interest in the field of Ricci flow ([1]), harmonic morphisms ([4, 2, 6, 14, 15, 23]), f-structures, quaternionic and contact metric manifolds (see [1, 3, 5, 6, 8, 9, 14, 15, 16, 17, 18, 19, 20, 22]). The purpose of this paper is to study the semiconformal maps on some special almost contact metric manifolds of dimension three. In the next section we give some preliminaries on semiconformal maps, Kählerian manifolds and almost contact metric manifolds (see [1, 2, 5, 8]). In section 3 we obtain some formulas for the Ricci tensor of a semiconformal submersion on a 3-dimensional manifold and a Weitzenböck formula on 3-dimensional Sasakian manifold. Section 4 is devoted to some results on stability of harmonic morphisms in the geometry of Sasakian manifolds of dimension three.

2 Preliminaries

An almost complex structure on a manifold M is a tensor field J of type $(1, 1)$ such that $J^2 = -Id$. It is clear that such manifolds are of even real dimension 2m.

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Any 2-dimensional orientable surface has a native Kählerian structure ([7]). The analogue of Kählerian manifolds of odd dimensions are Sasakian manifolds. Let M to be a smooth manifold with odd dimension 2n+1. An *almost contact structure* on M is a triple (φ, ξ, η) where ξ is a vector field, η is a 1-form and φ is a (1,1) tensor field satisfying the following relations:

$$\varphi^2 = -Id + \xi \circ \eta$$

$$\eta(\xi) = 1$$

where Id is the identity endomorphism on TM. Then, we have $\varphi(\xi) = 0$ and $\eta \circ \varphi = 0$. Moreover, if g is a Riemannian metric associated on M, i.e. a metric which satisfying for any X and Y on $\Gamma(TM)$

$$g(\varphi(X), \varphi(Y)) = g(X, Y) - \eta(X)\eta(Y).$$

Then we say that (φ, ξ, η, g) is an almost contact metric structure. A manifold equipped with such structure is called *almost contact metric manifold* . The second fundamental form Φ on M is given by

$$\Phi(X, Y) = g(X, \varphi Y)$$

for any X and Y on $\Gamma(TM)$.

An *almost contact metric structure* (φ, ξ, η, g) is *normal* if the Nijenhuis tensor N^{φ} satisfies ([5]):

$$N^{\varphi} + 2d\eta \otimes \xi = 0.$$

A *contact manifold* is a smooth manifold M together with a 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. We say that $(M, \varphi, \xi, \eta, g)$ is a Sasakian manifold if it is a normal contact metric manifold such that $\Phi = d\eta$. An almost contact metric structure (φ, ξ, η, g) is Sasakian ([5]) if and only if

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X.$$

We further recall some definitions on harmonic morphisms (see [2]):

Let $\psi : M^m \to N^n$ be a smooth submersion between Riemannian manifolds. We recall that the tangent bundle of M splits as the Whitney sum of two distributions, the *vertical* one $\mathcal{V} = Ker(d\psi)$ and the orthogonal complementary distribution $\mathcal{H} = \mathcal{V}^{\perp}$ called *horizontal*: $TM = \mathcal{V} \oplus \mathcal{H}$ (see [8]). As usual, we denote by v and h the projections on the vertical and horizontal distributions. The sections of $\mathcal{V}$ (respectively $\mathcal{H}$) will be called *vertical (*respectively *horizontal) vector fields*. For any vector field E, vE and hE denote the vertical and the horizontal components of E, respectively.

We will use the following notations for the second fundamental forms of the horizontal and vertical distributions (see [2]): $A\,(\nabla_E F) = A^h\,(\nabla_E F) = \gtrapprox(\nabla_{hE} hF)$, $B\,(\nabla_E F) = B^v\,(\nabla_E F) = \approx(\nabla_{vE} vF)$ and $I(E, F) = I^h(E, F) = v[hE, hF]$ where $E, F \in \Gamma(TM)$.

We recall also that $\|I\|^2 = \sum_{a,b} |I(e_a, e_b)|^2 = \sum_{a,b} g(v[e_a, e_b], v[e_a, e_b])$ where $\{e_a\}$ is a local orthonormal frame for the horizontal distribution on M.

Definition 2.1 Let $\psi : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds and let $x \in M$. Then ψ is called horizontally weakly conformal or semiconformal at x if either

i)$d\psi_x = 0$, or

ii) $d\psi_x$ is surjective and there exists a number $\Lambda(x) > 0$ such that

$$h(d\psi_x(X), d\psi_x(Y)) = \Lambda(x)g(X, Y),$$

where $X, Y \in \mathcal{H}_x$. The function $\Lambda(x)$ is called the square dilation of ψ at x and $\lambda(x) = \sqrt{\Lambda(x)}$ is called the dilation of ψ at x. ψ is called horizontally weakly conformal on M if it is horizontally weakly conformal at every point of M. ψ is called horizontally conformal on M if is horizontally weakly conformal on M and $d\psi_x \neq 0, \forall x \in M$. If $d\psi_x \neq 0$ and $\lambda = 1$, then ψ is a Riemannian submersion (see [8]).

Let $\psi : (M, g) \to (N, h)$ be a smooth map between two Riemannian manifolds of dimension m and n, respectively. Its differential $d\psi$ can be viewed as a section of the bundle $T^*M \otimes \psi^{-1}(TN) \to M$ endowed with the Hilbert-Schmidt norm $\| \cdot \|$.

If $\{e_1, \dots e_m\}$ is an orthonormal local frame on M, the norm of $d\psi$ is given by

$$\|d\psi\|^2 := Tr_g(\psi^* h) = \sum_{i=1}^{m} h(d\psi(e_i), d\psi(e_i)).$$

The energy density of ψ is a smooth function $e(\psi) : M \to [0, \infty)$ defined by :

$$e(\psi)_x = \frac{1}{2}\|d\psi_x\|^2, \qquad x \in M.$$

For any compact domain $\Omega \subseteq M$, the energy of ψ over Ω is the integral of its energy density

$$E(\psi; \Omega) = \int_{\Omega} e(\psi)\vartheta_g$$

where ϑ_g is the volume measure associated to the Riemannian metric g. If M is compact and $\Omega = M$ we write just $E(\psi)$ for $E(\psi; \Omega)$. A smooth map $\psi : M \to N$ is said to be a harmonic map if

$$\frac{d}{dt}|_{t=0} E(\psi_t; \Omega) = 0$$

for all compact domains Ω and for all variations $\{\psi_t\}$ of ψ supported in Ω.

Definition 2.2 Let $\psi : M \to N$ be a smooth mapping between Riemannian manifolds. Then ψ is called a harmonic morphism if, for every harmonic function $h : V \to R$, defined on an open subset $V \subset N$ with $\psi^{-1}(V) \neq \varnothing$ the composition $ho\psi$ is harmonic on $\psi^{-1}(V)$.

We recall a result by Fuglede and Ishihara (see [2]) : A smooth map $\psi : M \to N$ between Riemannian manifolds is a harmonic morphism if and only if ψ is both harmonic and semiconformal.

Lemma 2.3 (see [2]) *The adjoints of A and B are given by*

$$A_E^* F = -h\left(\nabla_{hE} vF\right) \qquad \text{and} \qquad B_E^* F = -v\left(\nabla_{vE} hF\right)$$

We recall the results obtained in [2] for sectional curvatures $K^M(E,F)$ determined by a plane spanned by orthonormal vectors E, F at a point.

Proposition 2.4 ([2]) *Let $\psi : M^m \to N^n$ be a smooth horizontally conformal submersion with dilation λ. Let K^M, K^N and K^v the sectional curvatures for M, N and the fibres of ψ, respectively and let x be a point of M.*

1. *If U, V are orthonormal vertical vectors at x (so that m-n=2), then*

$$K^M(U,V) = K^v(U,V) + |B_U V|^2 - g(B_U U, B_V V)$$

2. *If X, U are unit horizontal and vertical vectors at x, respectively (so that m-n=1) then*

$$K^M(X,U) = \nabla d \ln\lambda(U,U) + d\ln\lambda(B_U U) - 2(U(\ln\lambda))^2 + |A^*_X U|^2 +$$
$$+g((\nabla_X B^*)_U X, U) - |B^*_U X|^2$$

3. *If X, Y are orthonormal horizontal vectors at x (so that n=2), then*

$$K^M(X,Y) = \lambda^2 K^N(\bar{X},\bar{Y}) + \nabla d\ln\lambda(X,X) + \nabla d\ln\lambda(Y,Y) -$$
$$-(X(\ln\lambda))^2 - (Y(\ln\lambda))^2 + |grad\ln\lambda|^2 - \frac{3}{4}|I(X,Y)|^2$$

where here $\bar{X} = d\psi(X)/(\lambda|X|)$.

3 Ricci curvature of a semiconformal map on Sasakian manifolds of dimension 3

Let ψ be a C^∞, semiconformal submersion $\psi : (M^3, g) \to (N^2, h)$ with dilation λ, where M and N are Riemannian manifolds of dimensions 3 and 2, respectively.

Let U be a unit vertical vector field, i.e. $d\psi(U) = 0$ and $g(U,U) = 1$, and $\mu = \nabla_U U$ the mean curvature of the fibres of ψ. We recall that $\|\mu\|^2 = g(\mu,\mu)$.
Let K^N be the Gauss curvature on manifold N and let θ be a 1-form such that $\theta = g(U,.) = U^b$, and let $\Omega = d\theta$ be the associated 2-form.

We have recalculated the Ricci tensor on vertical and horizontal vectors using the usual definition of the exterior derivative and we have obtained similar results as in [1]. Some terms differ just by constants.

The formula for the Ricci tensor evaluated on vertical vectors is as follows.

Lemma 3.1

$$Ric(U,U) = 2U(U(\ln\lambda)) - 2(U(\ln\lambda))^2 + 2d^*\Omega(U) + \frac{1}{2}(L_\mu g)(U,U) - \frac{1}{4}\|I\|^2$$

Proof. Using Proposition 2.4 (2) and the formula

$$Ric(U,U) = \sum_a g(R(U,e_a)e_a, U) = \sum_a K(e_a, U)$$

we have recalculate the components of this formula. From Proposition 2.4 (2) we obtain that

$$\sum_a \{K^M(e_a,U)\} = \sum_a \{\nabla d\ln\lambda(U,U) + d\ln\lambda(B_U U) - 2(U(\ln\lambda))^2 + \left|A^*_{e_a}U\right|^2 +$$

$$+g((\nabla_{e_a}B^*)_U e_a, U) - |B^*_U e_a|^2\}$$

We have that

$$\sum_a g((\nabla_{e_a}B^*)_U e_a, U) = \sum_a g(e_a, \nabla_{e_a}\mu) = -d^*\mu^b + \|\mu\|^2$$

$$d^*\Omega(U) = \frac{1}{2}(-d^*\mu^b + \|\mu\|^2 + \frac{1}{2}\|I\|^2)$$

and for the others components we have obtained similar results as in [1]. Combining all this, the formula follows. ◻

The formula for the Ricci tensor evaluated on horizontal vectors is given by:

Lemma 3.2

$$Ric(X,Y) = \{\lambda^2 K^N + \Delta\ln\lambda + \mu(\ln\lambda) - \frac{1}{4}\|I\|^2\}g(X,Y) + \frac{1}{2}(L_\mu g)(X,Y) -$$

$$-g(X,\mu)g(Y,\mu)$$

where X and Y are horizontal vectors.

Proof. Using Proposition 2.4 and calculating the components we obtain similar formulas as in [1]. ◻

Proposition 3.3([5]) *On a Sasakian manifold of dimension 3 we have:*

i)$\nabla_\xi \xi = 0$
ii)$\nabla_X \xi = -\varphi X$ *for any* $X \in \Gamma(TM)$
iii)$R(X,Y)\xi = \eta(Y)X - \eta(X)Y$ *for any* $X, Y \in \Gamma(TM)$.
iv)$Ric(\xi) = 2$ *if the dimension of the manifold is 3.*

Definition 3.4 A smooth map $\psi : (M^{2m+1}, \varphi, \xi, \eta, g)) \to (N^{2n}, h, J)$ from a Sasakian manifold to a Kählerian manifold is called $(\varphi, J)-$ holomorphic map if the following condition is satisfied

$$d\psi \circ \varphi = J \circ d\psi.$$

We consider a C^∞ semiconformal $(\varphi, J)-$ holomorphic submersion $\psi : (M^3, \varphi, \xi, \eta, g) \to (N^2, h, J)$ with dilation λ, where M is a Sasakian manifold of dimension three and N is a Kählerian manifold of dimension two.

Let the unit vertical vector field U be the characteristic vector field ξ and let $\mu = \nabla_U U = \nabla_\xi \xi = 0$ be the mean curvature of fibres of ψ.

Let K^N be the Gauss curvature of manifold N and let $\theta = \eta$ be a 1-form, $\eta = g(\xi, .)$. Let $\Omega = d\theta = d\eta = \Phi$ be the associated 2-form.

Let $\{e_1, e_2, e_3\} = \{e, \varphi e, \xi\}$ be an orthonormal local frame on the Sasakian manifold M, where $\{e, \varphi e\}$ is the othonormal frame for the horizontal space.

Lemma 3.5 *Let $\psi : (M^3, \varphi, \xi, \eta, g) \to (N^2, h, J)$ be a C^∞ $(\varphi, J)-$ holomorphic semiconformal submersion with dilation λ, where M is a Sasakian manifold of dimension* *three and N is a Kählerian manifold of dimension two. Let* $\{e_1, e_2, e_3\} = \{e, \varphi e, \xi\}$ *an orthonormal local frame for manifold M. Then*

i)$Ric(X, Y) = (K^h + 1)g(X, Y)$ *for any* X, Y *horizontal vector fields where* K^h *is the Gauss curvature on horizontal space.*

ii)$Ric(e, \xi) = Ric(\varphi e, \xi) = 0$.

iii)$Ric(\bar{Y}, \bar{Z}) = K^N h(\bar{Y}, \bar{Z}) = \lambda^2 K^N g(Y, Z)$ *for any vector fields Y and Z, where* $\bar{Y} = d\psi(Y)$ *.*

Proof. i)

$$\begin{aligned} Ric(e,e) &= \sum_i g(R(e,e_i)e_i, e) \\ &= g(R(e,e)e,e) + g(R(e,\varphi e)\varphi e, e) + g(R(e,\xi)\xi, e) \\ &= K(e,\varphi e) + g(\eta(\xi)e - \eta(e)\xi, e) \\ &= K^h + g(e,e) = K^h + 1. \end{aligned}$$

Analogously, $Ric(\varphi e, e) = 0$, $Ric(e, \varphi e) = 0$ and $Ric(\varphi e, \varphi e) = K^h + 1$. *ii)*

$$\begin{aligned} Ric(e,\xi) &= \sum_i g(R(e,e_i)e_i, e) \\ &= g(R(e,e)e,\xi) + g(R(e,\varphi e)\varphi e, \xi) + g(R(e,\xi)\xi, \xi) \\ &= -g(R(e,\varphi e)\xi, \varphi e) + g(\eta(\xi)e - \eta(e)\xi, \xi) \\ &= -g(\eta(\varphi e)e - \eta(e)\varphi e, \varphi e) + g(e,\xi) \\ &= 0 \end{aligned}$$

iii) Let $\{\bar{X}, J\bar{X}\}$ an orthonormal frame on the Kählerian manifold N. Then $Ric(\bar{X}, \bar{X}) = K^N$, $Ric(J\bar{X}, J\bar{X}) = K^N$, $Ric(\bar{X}, J\bar{X}) = 0$. □

We further obtain the formula for the Ricci tensor on 3-dimensional Sasakian manifolds:

Proposition 3.6 (The Ricci curvature on a 3-dimensional Sasakian manifold) *Let* $\psi : (M^3, \varphi, \xi, \eta, g) \to (N^2, h, J)$ *be a* C^∞, $(\varphi, J)-$ *holomorphic semiconformal submersion with dilation* λ *where M is a Sasakian manifold of dimension 3 and N is a Kählerian manifold of dimension 2. Then*

$$Ric(g) = \{\lambda^2 K^N + \Delta \ln \lambda - 2\}(g - \eta^2) + 2\eta^2.$$

Proof. From Lemma 3.1 and Lemma 3.5 we obtain $Ric(X, Y) = (K^h + 1)g(X, Y)$ and $Ric(X, Y) = \{\lambda^2 K^N + \Delta \ln \lambda - 2\}g(X, Y)$ for any X and Y horizontal vector fields. Then $Ric(\xi, \xi) = 2$ and $Ric(X, \xi) = 0$. Combining these relations we obtain the formula. □

Definition 3.7 (see [5]) A contact metric manifold M^{2n+1} is said to be η-Einstein if the Ricci operator Q is given by

$$Q = aId + b\eta \otimes \xi$$

for some functions a and b on M^{2n+1}.

Corollary 3.8 *Let ψ be a C^∞ $(\varphi, J)-$ holomorphic semiconformal submersion with dilation λ, $\psi : (M^3, \xi, \eta, \varphi, g) \to (N^2, h, J)$ where M is a Sasakian manifold of dimension 3 and N is a Kählerian manifold of dimension 2. Then M is a η-Einstein manifold.*

Theorem 3.9 (Weitzenböck formula on Riemannian manifolds)(see [2]) *Let $\psi : (M^m, g) \to (N^n, h), n \geq 1$ be a submersive harmonic morphism and X an horizontal vector field . Then*

$$i)\Delta \ln \lambda g(X,X) = Ric^M(X,X) - Ric^N(d\psi(X), d\psi(X)) + (n-2)\{X(\ln\lambda)\}^2 +$$

$$+\sum_r \left|B^*_{e_r}X\right|^2 + \frac{1}{2}\sum_a |I(X,e_a)|^2$$

$$ii)n\Delta \ln \lambda = Tr^h Ric^M - \lambda^2 Scal^N + (n-2)\left|\mathcal{H}(grad \ln\lambda)\right|^2 + \|B\|^2 + \frac{1}{2}\|I\|^2$$

Proposition 3.10 (The Weitzenböck formula on Sasakian manifold of dimension three) *Let ψ be a C^∞ $(\varphi, J)-$ holomorphic semiconformal submersion with dilation λ, $\psi : (M^3, \varphi, \xi, \eta, g) \to (N^2, h, J)$ where M is a Sasakian manifold of dimension 3 and N is a Kählerian manifold of dimension 2. Let $\{e_1, e_2, e_3\} = \{e, \varphi e, \xi\}$ an orthonormal local frame of manifold M. Let X be an horizontal vector field. Then*

$i)\Delta \ln \lambda g(X,X) = Ric^M(X,X) - Ric^N(d\psi(X), d\psi(X)) + 2g(X,X)$
$i^{'})\Delta \ln \lambda g(X,X) = (K^h+1)g(X,X) - \lambda^2 K^N g(X,X) + 2g(X,X)$
$ii)\Delta \ln \lambda = K^h - \lambda^2 K^N + 3.$

Proof. Computing the components of Theorem 3.9 we obtain:

$$\begin{aligned}
\left|B^*_\xi X\right|^2 &= g(B^*_\xi X, B^*_\xi X)\\
&= g(-v(\nabla_\xi X), -v(\nabla_\xi X))\\
&= g(v(\nabla_\xi X), \xi)g(\xi, v(\nabla_\xi X))\\
&= g(\nabla_\xi X, \xi)g(\xi, \nabla_\xi X)\\
&= g(\nabla_\xi \xi, X)g(\nabla_\xi \xi, X)\\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}\sum_a |I(X,e_a)|^2 &= \frac{1}{2}\sum_a g(I(X,e_a), I(X,e_a))\\
&= \frac{1}{2}\{g(I(X,e), I(X,e)) + g(I(X,\varphi e), I(X,\varphi e))\}\\
&= \frac{1}{2}\{4g(e,\varphi X)g(e,\varphi X) + 4g(e,X)g(e,X)\}\\
&= \frac{1}{2}\{4g(\varphi e,X)g(\varphi e,X) + 4g(e,X)g(e,X)\}\\
&= 2\sum_a g(e_a,X)g(e_a,X)\\
&= 2g(X,X).
\end{aligned}$$

and the theorem follows. □

Remark 3.11 As before, in the same conditions we have:
i) $\|I\|^2 = 8$
ii) $\xi(\ln\lambda) = 0$.

Proof. For ii), we have $g(I(e,e), I(e,e)) = 0$. On the other side,

$$\begin{aligned} g(I(e,e), I(e,e)) &= g(v[e,e], v[e,e]) \\ &= g(2v\nabla_e e - 2v grad \ln\lambda, 2v\nabla_e e - 2v grad \ln\lambda) \\ &= 4\{g(v\nabla_e e, v\nabla_e e) - 2g(v\nabla_e e, v grad \ln\lambda) + \\ &+ g(v grad \ln\lambda, v grad \ln\lambda)\} \\ &= 4\{g(v\nabla_e e, \xi)g(\xi, v\nabla_e e) - 2g(v\nabla_e e, \xi)g(\xi, v grad \ln\lambda) + \\ &+ g(v grad \ln\lambda, \xi)g(\xi, v grad \ln\lambda)\} \\ &= 4\{g(e, \nabla_e\xi)g(\nabla_e\xi, e) - 2g(e, \nabla_e\xi)\xi(\ln\lambda) + (\xi(\ln\lambda))^2\} \\ &= 4\{g(e, -\varphi e)g(-\varphi e, e) - 2g(e, -\varphi e)\xi(\ln\lambda) + (\xi(\ln\lambda))^2\} \\ &= 4\{g(e, \varphi e)g(\varphi e, e) + 2g(e, \varphi e)\xi(\ln\lambda) + (\xi(\ln\lambda))^2\} \\ &= 4(\xi(\ln\lambda))^2 \end{aligned}$$

which ends the proof. □

Corollary 3.12 ([13, 14]) *Let $M(\varphi, \xi, \eta, g)$ be a Sasakian manifold and $N(J, h)$ a Kählerian manifold. Then any $(\varphi, J)-$ holomorphic map $\psi : M \to N$ is harmonic.*

Remark 3.13 *From [2] we know that any semiconformal harmonic map is a harmonic morphism. Then, Proposition 3.6, Corollary 3.8, and Proposition 3.10 are true for ψ be a harmonic morphism.*

The Boothby-Wang fibration is an example of a $(\varphi, J)-$holomorphic map from a Sasakian manifold into a Kählerian manifold. The well known particular case of a such fibration is the Hopf fibration $S^1 \to S^{2n+1} \to P^n(C)$.

Example 3.14 Let $S^3 \to S^2(1/2) \cong CP^1$, $(z_0, z_1) \longrightarrow (|z_0|^2 - |z_1|^2, 2\bar{z_0} z_1)$, $z_i \in C, |z_0|^2 + |z_1|^2 = 1$. The Hopf map is a Riemannian submersion and semiconformal map with dilation $\lambda = 1$ and satisfy the Weitzenböck formula of Proposition 3.10.

4 Stability on 3-dimensional Sasakian manifolds

Let (M, g) be a compact Riemannian manifold and let $\psi : (M, g) \to (N, h)$ be a harmonic map. We take a smooth variation $\psi_{s,t}$, with parameters $s, t \in (-\varepsilon, -\varepsilon)$, and with $\psi_{0,0} = \psi$. The corresponding variation fields are denoted by V and W. The Hessian of a harmonic map ψ is defined as follows

$$Hess_\psi(V, W) = \frac{\partial^2}{\partial s \partial t}|_{(s,t)=(0,0)} E(\psi_{s,t})$$

where $E(\psi_{s,t}) = \frac{1}{2}\int_M |d\psi_{s,t}|^2 \vartheta_g$ is the energy of ψ over M([2]).

Proposition 4.1(The second variation of the energy) ([2]) *Let $\psi : (M, g) \to (N, h)$ be a harmonic map and M compact. Then the Hessian of ψ for any V and W vector fields along ψ is given by:*

$$Hess_\psi(V, W) = \int_M h(J_\psi(V), W)\vartheta_g$$

$$= \int_M h(-Tr(\nabla^\psi)^2 V - TrR^N(V, d\psi)d\psi, W)\vartheta_g.$$

Here J_ψ is a second order selfadjoint elliptic differential operator acting on the space of variation vector fields along ψ, $\Gamma(\psi^{-1}(TN))$ of the form

$$J_\psi(V) := -\sum_{i=1}^{m}(\nabla^\psi_{e_i}\nabla^\psi_{e_i} - \nabla^\psi_{\nabla_{e_i}e_i})V - \sum_{i=1}^{m} R^N(V, d\psi(e_i))d\psi(e_i)$$

for any $V \in \Gamma(\psi^{-1}(TN))$ and $\{e_i\}_{i=1}^m$ a local orthonormal frame on M. The operator $J_\psi := \overline{\Delta}_\psi - \mathcal{R}_\psi$ is called the Jacobi operator. The operator $\overline{\Delta}_\psi$ defined by

$$\overline{\Delta}_\psi V := -\sum_{i=1}^{m}(\nabla^\psi_{X_i}\nabla^\psi_{X_i} - \nabla^\psi_{\nabla_{X_i}X_i})V, \qquad V \in \Gamma(\psi^{-1}TN)),$$

is called *the rough Laplacian.*

By definition (see [2]) we have that a harmonic map defined on a compact manifold is energy-stable if the Hessian for the energy is positive semi-definite, i.e.we have $Hess_\psi(V, V) \geq 0$ for $V \in \Gamma(\psi^{-1}TN)$. Otherwise, it is called unstable. One the other hand we can define this using the index definition. The index of a harmonic map $\psi : (M, g) \to (N, h)$ is defined as the dimension of the largest subspace of $\Gamma(\psi^{-1}(TN))$ on which the Hessian $Hess_\psi$ is negative definite. A harmonic map ψ is said to be stable if the index of ψ is zero and otherwise, is said to be unstable. Now let M^3 be a compact Sasakian manifold of dimension 3 and let $\psi : (M, g) \to (N, h, J)$ be a $(\varphi, J)-$ holomorphic and semiconformal map (i.e. harmonic morphism) with dilation λ, where (N, h, J) is a Kählerian manifold of dimension 2. Then the Ricci curvature is given by (Proposition 3.6) :

$$Ric(g) = \{\lambda^2 K^N + \Delta ln\lambda - 2\}(g - \eta^2) + 2\eta^2$$

Let $V = d\psi(X)$ where $X \in \Gamma(TM)$. From [12, 13] we have that

$$Hess_\psi(d\psi(X), d\psi(X)) = \int_M h(d\psi(\sum(R^M(e_i, X)e_i)), d\psi(X))\vartheta_g +$$

$$+ \int_M h(d\psi(X), d\psi(X))\vartheta_g.$$

In our case

$$\sum_i R^M(e_i, X)e_i = \{\lambda^2 K^N + \Delta ln\lambda - 4\}\eta(X)\xi - \{\lambda^2 K^N + \Delta ln\lambda - 2\}X$$

How ξ is vertical then $d\psi(\xi) = 0$. It follow that

$$Hess_\psi(d\psi(X), d\psi(X)) = -\{\lambda^2 K^N + \Delta ln\lambda - 3\} \int_M h(d\psi(X), d\psi(X))\vartheta_g.$$

We obtain the following theorem:

Theorem 4.2 *Let M^3 be a Sasakian compact manifold. Let $\psi : (M, g) \to (N, h, J)$ be a harmonic morphism where (N, h, J) is a Kählerian manifold of dimension 2. If $\lambda^2 K^N + \Delta ln\lambda > 3$, then ψ is unstable.*

If we consider in the last theorem that M^3 is the standard sphere of constant curvature 1, ψ Riemannian submersion with $\lambda = 1$ we obtain the following corollary:

Corollary 4.3 *Any harmonic morphism with $\lambda = 1$ from sphere $S^3 \to CP^1$ is unstable (see also [2]).*

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