Stochastic generalized fractional HP equations and applications

I.D. Albu, M. Neamţu, D. Opriş

Dedicated to the 70-th anniversary of Professor Constantin Udriste

Abstract. In this paper we established the condition for a curve to satisfy stochastic generalized fractional HP (Hamilton-Pontryagin) equations. These equations are described using Itō integral. We have also considered the case of stochastic generalized fractional Hamiltonian equations, for a hyperregular Lagrange function. From the stochastic generalized fractional Hamiltonian equations, Langevin generalized fractional equations were found and numerical simulations were done.

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Key words: HP equations; stochastic generalized fractional Hamilton equations; hyperregular Lagrange function; generalized fractional Langevin equations; Euler scheme.

1 Introduction

J.M. Bismut was the first one who introduced concepts of stochastic geometric mechanics, in his work from 1981, when he defined the notion of "stochastic Hamiltonian system". He showed that the stochastic flow of certain randomly perturbed Hamiltonian systems on flat spaces extremizes a stochastic action and using this property, he proved symplecticity and the Noether theorem for stochastic Hamiltonian systems. Since then, there has been a need in to find out tools and algorithms for the study of this kind of systems with uncertainty. Bismut's work was continued by Lazaro-Cami and Ortega ([11], [12]), in the sense that his work was generalized to manifolds. Stochastic Hamiltonian systems on manifolds extremize a stochastic action on the space of manifold-valued semimartingales, the reduction of stochastic Hamiltonian system on the cotangent bundle of a Lie group, a counter example for the converse of Bismut's original theorem.

Very important in many scientific domains is the fractional calculus: fractional derivatives, fractional integrals, of any real or complex order. Fractional calculus is used when fractional integration is needed. It is used for studying simple dynamical systems, but it also describes complex physical systems. For example, applications of

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the fractional calculus can be found in chaotic dynamics, control theory, stochastic modelling, but also in finance, hydrology, biophysics, physics, astrophysics, cosmology, economics and so on ([2], [4], [5], [9], [10]). But some other fields have just started to study problems from fractional point of view. It is very fashionable to study the fractional problems of the calculus of variations and Euler-Lagrange type equations. The most famous fractional integrals are Riemann-Liouville, Caputo, Grunwald-Letnikov and the most frequently used is the Riemann-Liouville fractional integral. The study of Euler-Lagrange fractional equations was continued by Agrawal ([1], [6], [8]) which obtained these equations by using the left, respectively right fractional derivatives in the Riemann-Liouville sense. The standard multi-variable variational calculus also has some limitations. But in [13], C.Udriste and D. Opris showed that these limitations can by broken using the multi-linear control theory. In [7] the novel concepts of fractional action-like variational approach (FALVA) with time-dependent fractional exponent and exponential time-dependent term is introduced.

In this paper, we restrict our attention to stochastic generalized fractional Hamiltonian systems given by Wiener processes and assume that the space of admissible curves in configuration space is of class C^1. Random effects appear in the balance of momentum equations, as white noise, that is why we may consider randomly perturbed mechanical systems. It should be mentioned that the ideas in this paper can be readily extended to stochastic Hamiltonian systems driven by semimartingales, but for the sake of clarity we restrict ourselves to Wiener processes. In this paper we use the generalized left fractional Riemann-Liouville integral defined as a mixture of the fractal action from physics and the discounted action at rate ρ, given in [7]. Within this context, the results of the paper are as follows:

1. The paper presents the results from [3] which show that almost surely that a curve satisfies stochastic HP equations if and only if it extremizes a stochastic action. Suggestive examples and numerical simulations are done.

2. Generalized fractional HP equations are described using the generalized fractional Riemann-Liouville integral and the fractional Itō integral;

3. Langevin type stochastic generalized fractional equations are obtained in the case of a hyperregular Lagrange function. Relevant examples and numerical simulations are presented.

The paper is organized as follows: In Section 2, Hamilton-Pontryagin (HP) principle is given to the stochastic setting to prove that a class of mechanical systems with multiplicative noise appearing as forces and torques possess a variational structure. For a hyperregular Lagrange function, we get the stochastic Hamiltonian equations that lead to Langevin equations. Examples and numerical simulations for the Lagrangian describing the Samuelson model from economics ([5]) are given. In Section 3, we extend the generalized fractional Hamilton-Pontryagin (HP) principle to the stochastic setting to prove that a class of mechanical systems with multiplicative noise appearing as forces and torques possesses a variational structure. For a hyperregular Lagrange function, we obtain the stochastic generalized fractional Hamiltonian equations that lead to Langevin generalized fractional equations. For a Lagrange function, defined on $\mathbb{R}^2$, the corresponding generalized fractional Langevin equations are simulated. The generalized fractional Hamiltonian and the Lagrangian description are joined together to get the generalized fractional HP system.

2 Stochastic HP mechanics

In this section, a variational principle is introduced for a class of stochastic Hamilton systems on manifolds. The stochastic action is a sum of the classical action and a stochastic integral. The key feature of this principle is that one can recover stochastic Hamilton equations for these systems. Roughly speaking, this is accomplished by means of taking variations of this action within the space of curves only (not the probability space) and imposing the condition that this partial differential of the action must be zero.

Let be a paracompact, configuration manifold Q and $J^1(\mathbb{R}, Q) = \mathbb{R} \times TQ$, T^*Q, the associated bundles of Q. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(w(t), \mathcal{F}_t)_{t\in[a,b]}$, where $[a, b] \subset \mathbb{R}$, $w(t)$ is a real-valued Wiener process and $\mathcal{F}_t$ is the filtration generated by the Wiener process ([3]).

The paper adopts an HP viewpoint to develop a Lagrangian description of stochastic Hamiltonian systems ([3]). The HP principle unifies the Hamiltonian and Lagrangian descriptions of mechanical system. The classical HP action integral will be perturbed by using deterministic function $\gamma : Q \to \mathbb{R}$.

We consider the Lagrangian $\mathcal{L} : J^1(\mathbb{R}, Q) \to \mathbb{R}$. In the stochastic context the HP principle states the following critical point condition on $\mathcal{P}Q = J^1(\mathbb{R}, Q) \oplus T^*Q$ for stochastic HP action integral given by ([3]):

$$\begin{aligned}
&\mathcal{A} : \Omega \times C(\mathcal{P}Q) \to \mathbb{R}, \\
&\mathcal{A}(q, v, p) = \int_a^b [\mathcal{L}(s, q(s), v(s)) + <p(s), \frac{dq}{ds} - v(s)>]ds + \int_a^b \gamma(q(s))dw(s)
\end{aligned} \tag{2.1}$$

where

$$C(\mathcal{P}Q) = \{(q, v, p) \in C^0([a, b], \mathcal{P}Q) | q \in C^1([a, b], Q), q(a) = q_a, q(b) = q_b\},$$

$[a, b] \subset \mathbb{R}$, $q_a, q_b \in Q$.

The action integral in the above principle consists of a Lebesgue integral with respect to s and an Itō stochastic integral with respect to w. The action is random; i.e. for every sample point $\omega \in \Omega$ we will obtain a different time-dependent Lagrangian system. We will use the following notation: $q(\omega, s) = q(s)$, $v(\omega, s) = v(s)$, $p(\omega, s) = p(s)$. The HP path space is a smooth infinite dimensional manifold. One can show that its tangent space at $c = (q, v, p) \in C([a, b], q_a, q_b)$ consists of maps $w = (q, v, p, \delta q, \delta v, \delta p) \in C^0([a, b], T(\mathcal{P}Q))$ such that $\delta q(a) = \delta q(b) = 0$ and $q, \delta q$ are of class C^1. Let $(q, v, p)(\cdot, \varepsilon) \in C(\mathcal{P}Q)$ denote a one-parameter family of curves in C, that is differentiable with respect to ε. Define the differential of $\mathcal{A}$ as

$$d\mathcal{A}(\delta q, \delta v, \delta p) = \frac{\partial}{\partial \varepsilon}\mathcal{A}(q(s, \varepsilon), v(s, \varepsilon), p(s, \varepsilon))|_{\varepsilon=0}$$

where

$$\begin{aligned}
\delta q(s) &= \frac{\partial}{\partial \varepsilon} q(s, \varepsilon)|_{\varepsilon=0}, \delta q(a) = \delta q(b) = 0, \\
\delta v(s) &= \frac{\partial}{\partial \varepsilon} v(s, \varepsilon)|_{\varepsilon=0}, \delta p(s) = \frac{\partial}{\partial \varepsilon} p(s, \varepsilon)|_{\varepsilon=0}.
\end{aligned}$$

In terms of this differential one can state the following critical point condition:

Theorem 2.1. **([3])** *Let $\mathcal{L} : J^1(\mathbb{R}, Q) \to \mathbb{R}$ be a Lagrangian on $J^1(\mathbb{R}, Q)$ of class C^2 with respect to t, q, v and with the globally Lipschitz first derivative with respect to t, q and v. Let $\gamma : Q \to \mathbb{R}$ be a class C^2 function and with the globally Lipschitz first derivative. Then almost certainly a curve $c = (q, v, p) \in C(\mathcal{P}Q)$ satisfies the stochastic HP equations:*

$$\begin{aligned} dq^i &= v^i ds, \\ dp_i &= \frac{\partial \mathcal{L}}{\partial q^i} ds + \frac{\partial \gamma}{\partial q^i} dw(s), \\ p_i &= \frac{\partial \mathcal{L}}{\partial v^i}, \quad i = 1, ..., n, \end{aligned} \tag{2.2}$$

if and only if it is a critical point of the function $\mathcal{A} : \Omega \times C(\mathcal{P}Q) \to \mathbb{R}$, i.e. $dA(c)=0$.

Let $\mathcal{L} : J^1(\mathbb{R}, Q) \to \mathbb{R}$ be a Lagrangian on $J^1(\mathbb{R}, Q)$, hyperregular, that means $det \left(\frac{\partial^2 \mathcal{L}}{\partial v^i \partial v^j} \right) \neq 0$.

From (2.2) the following propositions are obtained.

Proposition 2.2. **(Stochastic Hamilton equations).** *The equations (2.2) are equivalent to the following equations:*

$$\begin{aligned} dq^i &= \frac{\partial H}{\partial p_i} ds, \\ dp_i &= -\frac{\partial H}{\partial q^i} ds + \frac{\partial \gamma}{\partial q^i} dw(s), \quad i = 1, ..., n, \end{aligned} \tag{2.3}$$

where $H(t, p, q) = p_i v^i - L(t, q, v)$.

The equations (2.3) represent the Lagevin equations.

Proposition 2.3. *If $\mathcal{L} = \frac{1}{2} g_{ij} v^i v^j$, where g_{ij} are the components of a metric on the manifold Q, equations (2.2) take the form:*

$$\begin{aligned} dq^i &= v^i dt, dp_i = -g_{ij} v^j ds + \frac{\partial \gamma}{\partial q^i} dw(s), \\ dv^i &= -\Gamma^i_{jk} v^j v^k ds + g^{ij} \frac{\partial \gamma}{\partial q^j} dw(s), \quad i = 1, ..., n, \end{aligned} \tag{2.4}$$

where Γ^i_{jk} are Christoffel coefficients associated to the considered metric.

The equations (2.3) become:

$$\begin{aligned} dq^i &= g^{ij} p_j ds, \\ dp_i &= \frac{1}{2} \frac{\partial g_{kl}}{\partial q^i} p^k p^l ds + \frac{\partial \gamma}{\partial q^i} dw(s), \quad i = 1, ..., n. \end{aligned} \tag{2.5}$$

Proposition 2.4. *If $\mathcal{L} : J^1(\mathbb{R}, \mathbb{R}^n) \to \mathbb{R}$ is given by:*

$$\mathcal{L}(q, v) = \frac{1}{2} \delta_{ij} v^i v^j - V(q), q \in \mathbb{R}^n, \tag{2.6}$$

then the equations (2.2) take the form:

$$dq^i = v^i ds, \quad dp_i = -\frac{\partial V}{\partial q^i} ds + \frac{\partial \gamma}{\partial q^i} dw(s), p_i = \delta_{ij} v^j, \quad i = 1, ..., n. \tag{2.7}$$

The equations (2.3) become:

$$dq^i = \delta^{ij} p_j ds, \quad dp_i = -\frac{\partial V}{\partial q^i} ds + \frac{\partial \gamma}{\partial q^i} dw(s), \quad i = 1, ..., n. \tag{2.8}$$

Proposition 2.5. *If* $\mathcal{L} : J^1(\mathbb{R}, \mathbb{R}^n) \to \mathbb{R}$ *is given by:*

$$\mathcal{L}(s, q, v) = e^{-\rho s} L(q, v), q \in \mathbb{R}^n \tag{2.9}$$

then the equations (2.2) take the form:

$$\begin{aligned} dq^i &= v^i ds, \\ dp_i &= e^{-\rho s} \frac{\partial L}{\partial q^i} ds + \frac{\partial \gamma}{\partial q^i} dw(s), \\ p_i &= e^{-\rho s} \frac{\partial L}{\partial v^i}, \quad i = 1, .., n. \end{aligned} \tag{2.10}$$

Proposition 2.6. (Samuelson, [5]) *If* $\mathcal{L} : J^1(\mathbb{R}, \mathbb{R}^2) \to \mathbb{R}$ *is given by:*

$$\mathcal{L}(s, q, v) = -\frac{1}{2} e^{-\rho s} (v^2 + 2avq + q^2), q \in \mathbb{R}, \tag{2.11}$$

then the equations (2.2) take the form:

$$\begin{aligned} dq &= v ds, \\ dp &= -e^{-\rho s} (av + q) ds + \frac{\partial \gamma}{\partial q} dw(s), \\ p &= -e^{-\rho s} (v + aq). \end{aligned} \tag{2.12}$$

From the equations (2.12) we obtain:

$$\begin{aligned} dq &= -(aq + e^{\rho s} p) ds, \\ dp &= ((a^2 - 1) e^{-\rho s} q + ap) ds + \frac{d\gamma}{dq} dw(s). \end{aligned} \tag{2.13}$$

If $\gamma(q) = q^2/2$, the Euler scheme for (2.13) is:

$$\begin{aligned} q(n+1) &= q(n) - h(aq(n) + e^{\rho n} p(n)), \\ p(n+1) &= p(n) + h((a^2-1) e^{-\rho n} q(n) + ap(n)) + q(n) G(n), n = 0, ..., N-1, \end{aligned} \tag{2.14}$$

where $T > 0$, $h = \dfrac{T}{N}$, $N > 0$, $G(n) = w((n+1)h) - w(nh)$ and $q(n) = q(\omega, nh)$, $p(n) = p(\omega, nh)$, $\rho \in (0, 1)$, $a \in (-1, 1)$.

For $\rho = 0.003$, $a = 0.03$, $h = 0.001$ using Maple 13, the orbit $(n, q(nh))$ is represented in Fig 1 and the orbit $(n, q(\omega, nh))$ in Fig 2:

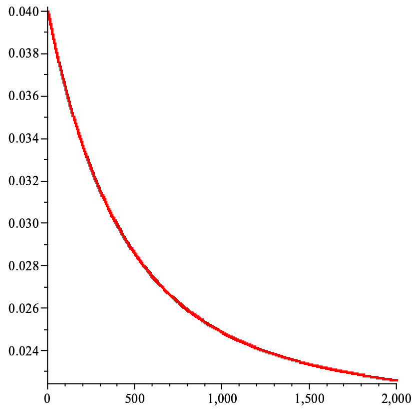

Fig 1. $(n, q(nh))$.

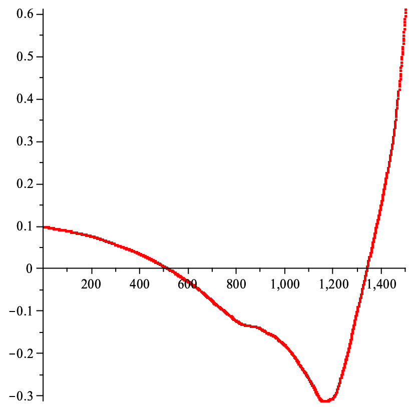

Fig 2. $(n, q(\omega, nh))$.

In Figures 3 and 4 we can visualize the orbits $(n, p(nh))$, $(n, p(\omega, nh))$:

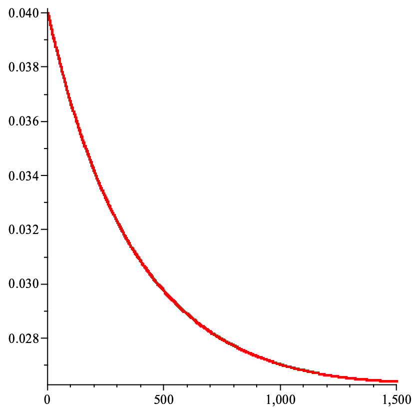

Fig 3. $(n, p(nh))$.

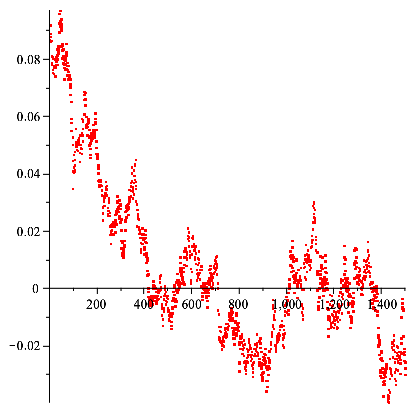

Fig 4. $(n, p(\omega, nh))$.

Figures 5 and 6 represent the orbits $(q(nh), p(nh))$ and $(q(\omega, nh), p(\omega, nh))$:

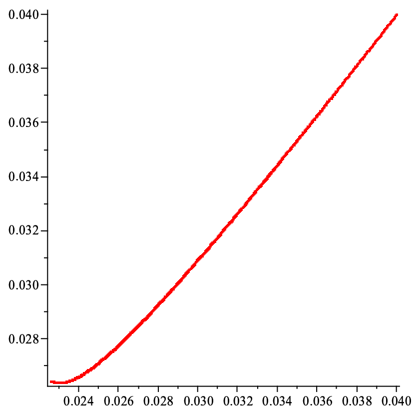

Fig 5. $(q(nh), p(nh))$.

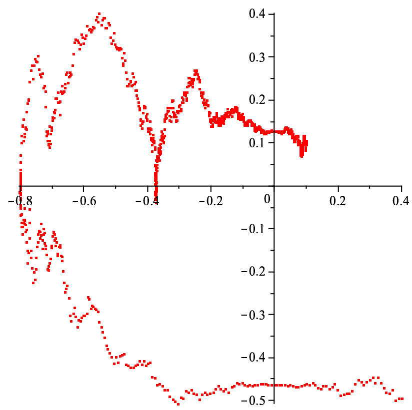

Fig 6. $(q(\omega, nh), p(\omega, nh))$.

3 Stochastic generalized fractional HP principle

In this section a generalized variational principle is introduced for a class of stochastic generalized fractional Hamiltonian systems on manifold. We use the generalized left fractional Riemann-Liouville integral ([7]) defined as a mixture of the fractal action from physics and the discounted action at rate ρ. Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function, $\alpha : \mathbb{R} \to \mathbb{R}$ a function of class C^1. The generalized left fractional Riemman-Liouville integral is given by:

$$ {}_{t_0}I_t^{\alpha} f(t) = \int_{t_0}^{t} \frac{1}{\Gamma_1(\alpha(s-t))} f(s)(t-s)^{\alpha(s-t)-1} e^{\rho(s-t)} ds \tag{3.1} $$

where

$$ \Gamma_1(\alpha(s-t)) = \Gamma(\alpha(z))|z = s-t, \Gamma(\alpha(z)) = \int_0^{\infty} (s-t)^{\alpha(z)-1} e^{-(s-t)} ds, $$

is the modified Euler Gamma function and t is fixed with $t \neq s$.

If $\alpha(z) = a = const.$, $0 < a \leq 1$, $\rho = 0$, from (3.1) we obtain the left fractional Riemman-Liouville integral ([6], [7], [8]). In fact, the generalized left fractional Riemann-Liouville integral is a generalization of the single time Stieltjes integral ([13]).

In (3.1), s is the intrinsic time and t is the observer time, $t \neq s$. Let $g : \mathbb{R} \to \mathbb{R}$ be the function:

$$ g_t(s) = \frac{1}{\Gamma_1(\alpha(s-t))} e^{(\alpha(s-t)-1)ln|t-s|+\rho(s-t)}, t \neq s. \tag{3.2} $$

We consider $\mathcal{L} : J^1(\mathbb{R}, Q) \to \mathbb{R}$ and $\gamma : Q \to \mathbb{R}$. In the stochastic context the HP principle states the following critical point condition on $\mathcal{P}Q = J^1(\mathbb{R}, Q) \oplus T^*(Q)$ for the stochastic HP generalized fractional action given by:

$$ \begin{aligned} \mathcal{A}^{\alpha}(q,v,p,t) = & \int_{t_0}^{t} [\mathcal{L}(s,q(s),v(s)) + < p(s), \frac{dq}{ds} - v(s) >] g_t(s) ds + \\ & + \int_{t_0}^{t} \gamma(q(s)) g_t(s) dw(s). \end{aligned} \tag{3.3} $$

The first integral in (3.3) is a Lebesgue integral with respect to s and the second one is an Itō integral.

Using Theorem 2.1, we get:

Theorem 3.1. *If $\mathcal{L} : J^1(\mathbb{R}, Q) \to \mathbb{R}$ and $\gamma : Q \to \mathbb{R}$ satisfy the hypothesis from Theorem 2.1, then almost certainly a curve $c = (q, v, p) \in C(\mathcal{P}Q)$ satisfies the stochastic HP equations with intrinsic and observer times:*

$$ \begin{aligned} & dq^i = v^i ds, \\ & dp_i = [\frac{\partial \mathcal{L}}{\partial q^i} - p_i(\frac{d\alpha(s-t)}{ds} ln|t-s| + \frac{\alpha(s-t)-1}{s-t}) + \rho - \\ & - \frac{1}{\Gamma_1(\alpha(s-t))} \frac{d\Gamma_1(\alpha(s-t))}{ds}] ds + \frac{\partial \gamma}{\partial q^i} dw(s), \\ & p_i = \frac{\partial \mathcal{L}}{\partial v^i}, i = 1, ..., n, t \neq s. \end{aligned} \tag{3.4} $$

From (3.4) we obtain:

(i) If $\alpha(z) = 1$ and $\rho = 0$ then equations (2.2) are obtained;

(ii) If $\alpha(z) = a = const.$, $0 < a \leq 1$, $\rho = 0$, then the following relations are deduced from (3.4):

$$
\begin{aligned}
dq^i &= v^i ds, \\
dp_i &= (\frac{\partial \mathcal{L}}{\partial q^i} - p_i \frac{a-1}{s-t})ds + \frac{\partial \gamma}{\partial q^i} dw(s), \\
p_i &= \frac{\partial \mathcal{L}}{\partial v^i}, \quad i = 1, ..., n, t \neq s.
\end{aligned} \tag{3.5}
$$

The equations (3.5) represent the stochastic fractional equations.

If $\mathcal{L} : J^1(\mathbb{R}, Q) \to \mathbb{R}$ is hyperregular and autonomous, then by using (3.4) the following propositions hold:

Proposition 3.2. (Stochastic generalized fractional Hamilton equations.) *The equations (3.4) are equivalent with the equations:*

$$
\begin{aligned}
dq^i &= \frac{\partial H}{\partial p_i} ds, \\
dp_i &= (-\frac{\partial H}{\partial q^i} - p_i h(s,t))ds + \frac{\partial \gamma}{\partial q^i} dw(s), \quad i = 1, ..., n,
\end{aligned} \tag{3.6}
$$

where

$$
\begin{aligned}
&H(q,p) = p_i v^i - \mathcal{L}(q,v), \\
&h(s,t) = \frac{d\alpha(s-t)}{ds} ln|t-s| + \frac{\alpha(s-t)-1}{s-t} + \rho - \frac{1}{\Gamma_1(\alpha(s-t))} \frac{d\Gamma_1(\alpha(s-t))}{ds}.
\end{aligned} \tag{3.7}
$$

The equations (3.6) represent the generalized fractional Langevin equations.

Proposition 3.3. *If* $\mathcal{L}(q,v) = \frac{1}{2} g_{ij} v^i v^j$, *where* g_{ij} *are the components of a metric on the manifold Q, then the equations (3.4) take the form:*

$$
\begin{aligned}
dq^i &= v^i ds, \\
dv^i &= -(\Gamma^i_{jk} v^j v^k - h(s,t) v^i) ds + g^{ij} \frac{\partial \gamma}{\partial q^j} dw(s), \quad i = 1, ..., n,
\end{aligned} \tag{3.8}
$$

where Γ^i_{jk} *are the Christoffel coefficients associated to the considered metric and* $h(s,t)$ *is given by (3.7). The equations (3.6) become:*

$$
\begin{aligned}
dq^i &= g^{ij} p_j ds, \\
dp_i &= (\frac{1}{2} \frac{\partial g_{kl}}{\partial q^i} p^l p^k - h(s,t) p_i) ds + \frac{\partial \gamma}{\partial q^i} dw(s), \quad i = 1, ..., n,
\end{aligned} \tag{3.9}
$$

where $h(s,t)$ *is given by (3.7).*

The equations (3.9) can be used for the study of the generalized fractional motion of relativistic particle with noise ([7]).

Proposition 3.4. *If* $\mathcal{L} : J^1(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$ *is given by:*

$$\mathcal{L}(q, v) = \frac{1}{2}v^2 - V(q) \tag{3.10}$$

where $V : \mathbb{R} \to \mathbb{R}$ *and* $\gamma : \mathbb{R} \to \mathbb{R}$*, then equations (3.9) are given by:*

$$\begin{aligned} dq &= pds, \\ dp &= (-\frac{dV}{dq} - h(s,t)p)ds + \frac{d\gamma}{dq}dw(s). \end{aligned} \tag{3.11}$$

If $V(q) = \cos q$, $\gamma(q) = \sin q$, the Euler scheme for (3.11) is given by:

$$\begin{aligned} &q(n+1) = q(n) + kp(n)) \\ &p(n+1) = p(n) + k(sin(q(n)) - h(nk,t)) + cos(q(n))G(n), n = 0, ..., N-1, \end{aligned}$$

where $T > 0$, $k = \frac{T}{N}$, $N > 0$, $G(n) = w((n+1)h) - w(nh)$ and

$$\begin{aligned} &h(nk,t) = \alpha'(nk,t)ln|t - nk| + \frac{\alpha(nk-t) - 1}{nk - t} + \rho - \\ &- \frac{1}{\Gamma_1(\alpha(nk-t))}\Gamma'(\alpha(nk-t)), \\ &\alpha'(s-t) = \frac{d\alpha(s-t)}{ds}, \Gamma'(\alpha(s-t)) = \frac{d\Gamma_1(\alpha(s-t))}{ds}. \end{aligned}$$

For $\alpha(s-t) = a$, $a = 0.6$, $k = 0.001$, $t = 0.8$ using Maple 13 the orbit $(n, p(nk))$ is represented in Figure 7 and the orbit $(n, p(\omega, nk))$ is represented in Figure 8:

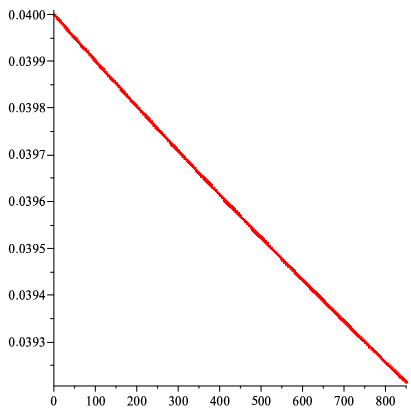

Fig 7. $(n, p(nk))$.

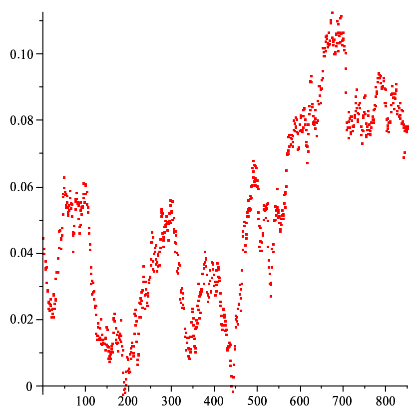

Fig 8. $(n, p(\omega, nk))$.

Figure 9 displays the orbit $(q(nk), p(nk))$ and Figure 10 displays the orbit $(q(\omega, nk), p(\omega, nk))$

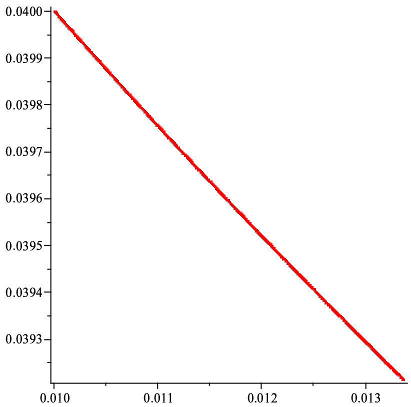

Fig 9. $(q(nk), p(nk))$.

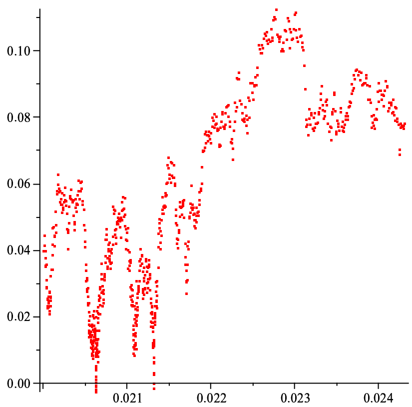

Fig 10. $(q(\omega, nk), p(\omega, nk))$.

4 Conclusions

In this paper we have described the stochastic generalized fractional HP principle, using the classical stochastic HP principle ([3]). Using a hyperregular Lagrange function, Langevin-type generalized fractional equations were illustrated. We have done the numerical simulations for the case of a Lagrange function defined on $\mathbb{R}^2$. In a future paper we will study the stochastic stability of the obtained equations as well as the description of the hybrid generalized fractional HP principle.

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