Quadratic and homogeneous Hamilton-Poisson systems on the 13th Lie algebra from Bianchi's classification

Anania Aron, Camelia Pop

Dedicated to the 70-th anniversary of Professor Constantin Udriste

Abstract. The quadratic and homogeneous Hamilton-Poisson systems on the 13th Lie algebra from Bianchi's classification (see [5]) are discussed and some of their geometrical and dynamical properties are pointed out.

M.S.C. 2000: 53D17.
Key words: Hamilton-Poisson system; Kahan integrator; Runge-Kutta integrator; spectral stability; nonlinear stability.

1 Introduction

Starting with Bianchi's classification of all 3-dimensional Lie algebras and using their minus Lie-Poisson structures which are presented in [9], we study the quadratic and homogeneous Hamilton-Poisson systems on the real Lie algebra of type XIII. Similar problems have been studied in [1], [2] and [3].

The material is divided up as follows. In Section 2 we present the real Lie algebra of type XIII from Bianchi's classification, its plus Lie-Poisson structure and the quadratic Hamilton-Poisson systems on it. Numerical integration via Kahan integrator and numerical simulation are the subjects of the third paragraph. We are interested especially to find the conditions for which Kahan integrator is energy [resp. Casimir, resp. Poisson structure] preserving. A comparison between the results obtained via Kahan's integrator and the results obtained Runge-Kutta 4 steps integrator is presented, too. Stability problems are studied in the last section.

2 The geometrical picture of the problem

Let (e_1, e_2, e_3) be the canonical basis for $\mathbb{R}^3$, i.e.

$$e_1 = [1, 0, 0]^t, \ \ e_2 = [0, 1, 0]^t, \ \ e_3 = [0, 0, 1]^t.$$

BSG Proceedings 17. The International Conference "Differential Geometry-Dynamical Systems 2009" (DGDS-2009), October 8-11, 2009, Bucharest-Romania, pp. 12-20.

Definition 2.1. The 13th Lie algebra from Bianchi's classification (see [5]) is $\mathbb{R}^3$ with the bracket operation given by:

[.,.]	e_1	e_2	e_3
e_1	0	$-e_3$	$-e_2$
e_2	e_3	0	0
e_3	e_2	0	0

Then the plus Lie-Poisson structure on the dual g^* of our Lie algebra is generated by the matrix:

$$\Pi_- = \begin{bmatrix} 0 & -x_3 & -x_2 \\ x_3 & 0 & 0 \\ x_2 & 0 & 0 \end{bmatrix}$$

Definition 2.2. A quadratic and homogeneous Hamilton-Poisson system on $(g^*, \Pi_-) \simeq (\mathbb{R}^3, \Pi_-)$ is the triple

$$(\mathbb{R}^3, \Pi_-, H),$$

where $H \in C^\infty(\mathbb{R}^3, \mathbb{R})$ is given by:

$$\begin{aligned} H(x_1, x_2, x_3) = {} & \frac{1}{2}(b_1x_1^2 + b_2x_2^2 + b_3x_3^2) \\ & + d_1x_2x_3 + d_2x_1x_3 + d_3x_1x_2 + a_1x_1 + a_2x_2 + a_3x_3. \end{aligned}$$

Its dynamics are described by the following set of differential equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \Pi_- \cdot \nabla H,$$

or equivalent:

$$(2.1) \qquad \begin{cases} \dot{x}_1 = -x_2(a_3 + d_2x_1 + d_1x_2 + b_3x_3) - x_3(a_2 + d_3x_1 + b_2x_2 + d_1x_3) \\ \dot{x}_2 = x_3(a_1 + b_1x_1 + d_3x_2 + d_2x_3) \\ \dot{x}_3 = x_2(a_1 + b_1x_1 + d_3x_2 + d_2x_3). \end{cases}$$

It is not hard to see that the smooth function $C \in C^\infty(\mathbb{R}^3, \mathbb{R})$ given by

$$C(x_1, x_2, x_3) = \frac{1}{2}\left(x_2^2 - x_3^2\right)$$

is a Casimir of our Poisson configuration $(\mathbb{R}^3, \Pi_-, H)$.

The phase curves of the dynamics (2.1) are the intersections of the surfaces:

$$H(x_1, x_2, x_3) = \text{constant}$$

and

$$C(x_1, x_2, x_3) = \text{constant},$$

see the Figures 3.1, 3.2, 3.3, 3.6. and 3.7.

3 Numerical integration of the dynamics (2.1) via Kahan's integrator

It is well known that Kahan's integrator (see [7]) for the dynamics (2.1) can be written in the following way:

$$(3.1)\quad \begin{cases} x_1^{n+1}-x_1^n=\dfrac{h}{2}(-a_3(x_2^{n+1}+x_2^n)-d_2(x_1^{n+1}x_2^n+x_2^{n+1}x_1^n)+2d_1(-x_2^nx_2^{n+1} \\ \qquad -x_3^nx_3^{n+1})-a_2(x_3^{n+1}+x_3^n)-d_3(x_1^{n+1}x_3^n+x_3^{n+1}x_1^n) \\ \qquad +(-b_3-b_2)(x_2^{n+1}x_3^n+x_3^{n+1}x_2^n)) \\ x_2^{n+1}-x_2^n=\dfrac{h}{2}(a_1(x_3^{n+1}+x_3^n)+b_1(x_1^{n+1}x_3^n+x_3^{n+1}x_1^n) \\ \qquad +d_3(x_2^{n+1}x_3^n+x_3^{n+1}x_2^n)+2d_2x_3^nx_3^{n+1}) \\ x_3^{n+1}-x_3^n=\dfrac{h}{2}(a_1(x_2^{n+1}+x_2^n)+b_1(x_1^{n+1}x_2^n+x_2^{n+1}x_1^n)+2d_3x_2^nx_2^{n+1} \\ \qquad +d_2(x_2^{n+1}x_3^n+x_3^{n+1}x_2^n)). \end{cases}$$

Now we can prove:

Proposition 3.1. *Kahan's integrator (3.1) is Poisson preserving if and only if at least one of the following conditions holds:*

(i) $b_1=0;\ d_2=0;\ d_3=0;$

(ii) $a_2=\dfrac{a_1d_3}{b_1};\ a_3=\dfrac{a_1d_2}{b_1};\ \ b_2=\dfrac{-b_1b_3+{d_2}^2+{d_3}^2}{b_1};d_1=\dfrac{d_2d_3}{b_1}.$

Proof. The proof can be obtained using MATHEMATICA 7. □

Proposition 3.2. *Kahan's integrator (3.1) is Casimir preserving if and only if at least one of the following conditions holds:*

(i) $b_1=0;\ d_2=0;\ d_3=0;$

(ii) $a_2=\dfrac{a_1d_3}{b_1};\ a_3=\dfrac{a_1d_2}{b_1};\ b_2=\dfrac{-b_1b_3+{d_2}^2+{d_3}^2}{b_1};d_1=\dfrac{d_2d_3}{b_1}.$

Proof. The proof can be obtained using MATHEMATICA 7. □

Proposition 3.3. *Kahan's integrator (3.1) is energy preserving if and only if one of the following conditions holds:*

(i) $a_1=0;\ b_1=0;\ d_2=0;\ d_3=0;$

(ii) $a_1 = 0;\ a_2 = 0;\ a_3 = 0;\ b_1 = 0;\ b_2 = 0;\ d_1 = 0;\ d_3 = 0;$

(iii) $b_1 = 0;\ b_3 = -b_2;\ d_1 = 0;\ d_2 = 0;\ d_3 = 0;$

(vi) $a_2 = \dfrac{a_1 d_3}{b_1};\ a_3 = \dfrac{a_1 d_2}{b_1};\quad b_2 = \dfrac{-b_1 b_3 + {d_2}^2 + {d_3}^2}{b_1}; d_1 = \dfrac{d_2 d_3}{b_1}.$

Proof. The proof can be obtained using MATHEMATICA 7. □

In every cases mentioned in the above propositions, namely:

Case 1: $b_1 = 0;\ d_2 = 0;\ d_3 = 0;$

Case 2: $a_1 = 0;\ b_1 = 0;\ d_2 = 0;\ d_3 = 0;$

Case 3: $a_1 = 0;\ a_2 = 0;\ a_3 = 0;\ b_1 = 0;\ b_2 = 0;\ d_1 = 0;\ d_3 = 0;$

Case 4: $b_1 = 0;\ b_3 = -b_2;\ d_1 = 0;\ d_2 = 0;\ d_3 = 0;$

Case 5: $a_2 = \dfrac{a_1 d_3}{b_1};\ a_3 = \dfrac{a_1 d_2}{b_1};\quad b_2 = \dfrac{-b_1 b_3 + {d_2}^2 + {d_3}^2}{b_1}; d_1 = \dfrac{d_2 d_3}{b_1}$,pasn we have sketched the phase curves, the 4th order Runge-Kutta integrator and Kahan's integrator (see the Figures 3.4. and 3.5.)

Case 1: $b_1 = 0,\ d_2 = 0,\ d_3 = 0;$

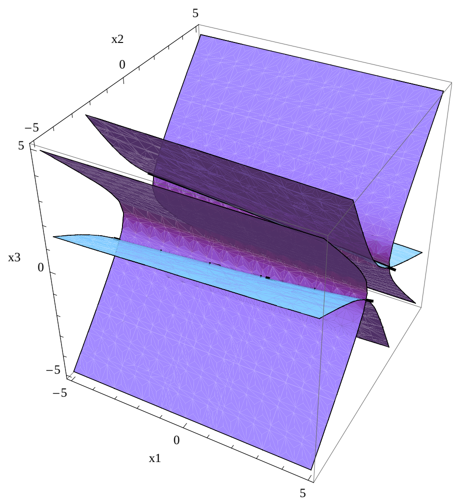

Fig. 3.1. The phase curves of the system (2.1)

In this case the Kahan integrator does not provide any relevant results.

Case 2: $a_1 = 0,\ b_1 = 0,\ d_2 = 0,\ d_3 = 0;$

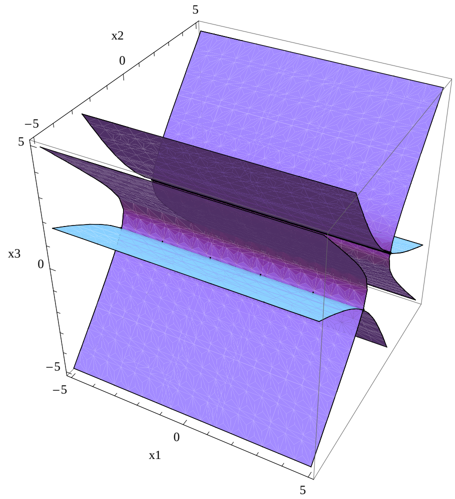

Fig. 3.2. The phase curves of the system (2.1)

In this case the Kahan integrator does not provide any relevant results.

<u>Case 3</u>: $a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ b_1 = 0,\ b_2 = 0,\ d_1 = 0,\ d_3 = 0;$

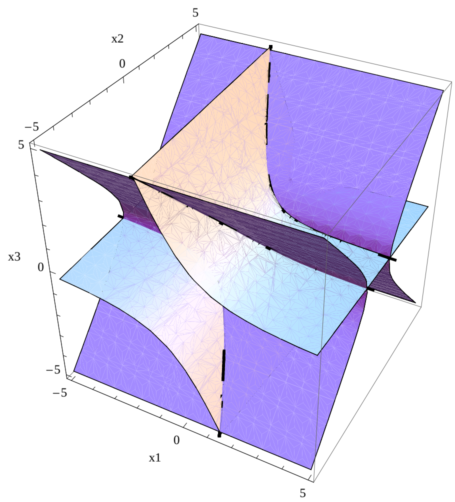

Fig. 3.3. The phase curves of the system (2.1)

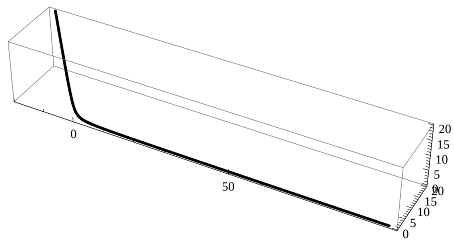

Fig. 3.4. 4th order Runge-Kutta integrator

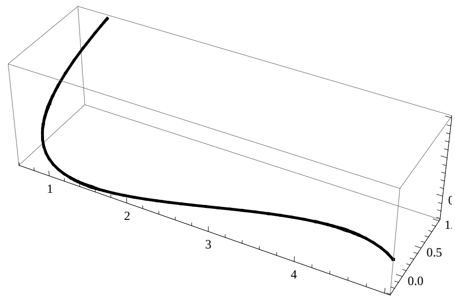

Fig. 3.5. Kahan's integrator

Case 4: $b_1 = 0,\ b_3 = -b_2,\ d_1 = 0,\ d_2 = 0,\ d_3 = 0;$

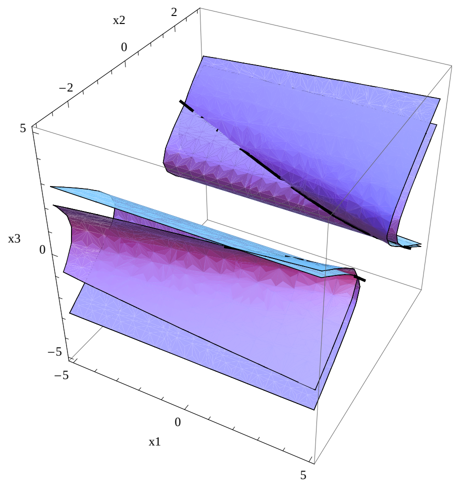

Fig. 3.6. The phase curves of the system (2.1)

In this case the Kahan integrator does not provide any relevant results.

Case 5:

$$a_2 = \frac{a_1 d_3}{b_1},\ a_3 = \frac{a_1 d_2}{b_1},\ \ b_2 = \frac{-b_1 b_3 + {d_2}^2 + {d_3}^2}{b_1}, d_1 = \frac{d_2 d_3}{b_1};$$

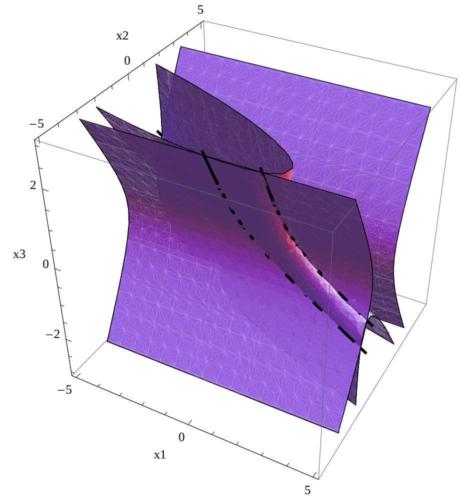

Fig. 3.7. The phase curves of the system (2.1)

In this case Kahan's integrator does not provide any relevant results.

If we make a comparison between 4th order Runge-Kutta integrator and Kahan's integrator we can see that we obtain almost the same results. However, Kahan's integrator has the advantage to be easier implemented.

4 Stability

We shall discuss now the nonlinear stability of the equilibrium states of the system (2.1) in each of the cases 1-5 mentioned in the above section.

Case 1: $b_1 = 0,\ d_2 = 0,\ d_3 = 0.$

The equilibrium states of the dynamics (2.1) have the form: $e^M = (M, 0, 0),\ M \in \mathbb{R}$.

Proposition 4.1. *If $a_1 \neq 0$, then the equilibrium states e^M, $M \in \mathbb{R}^*$ are unstable; if $a_1 = 0$ then the equilibrium states e^M, $M \in \mathbb{R}^*$ are spectrally stable.*

Case 2: $a_1 = 0,\ b_1 = 0,\ d_2 = 0,\ d_3 = 0$. Then the equilibrium states of the dynamics (2.1) have the following form:

$$e_1^{MP} = \left(M, \frac{-a_3 - b_2P - b_3P - \sqrt{(a_3 + b_2P + b_3P)^2 - 4d_1P(a_2 + d_1P)}}{2d_1}, P\right),$$

$$e_2^{MP} = \left(M, \frac{-a_3 - b_2P - b_3P + \sqrt{(a_3 + b_2P + b_3P)^2 - 4d_1P(a_2 + d_1P)}}{2d_1}, P\right),$$

for $M, P \in \mathbb{R}$.

Proposition 4.2. *If $d_1 \neq 0$ then the equilibrium states e_1^{MP}, e_2^{MP}, $M, P \in \mathbb{R}^*$ are unstable; if $d_1 = 0$ then the equilibrium states e_1^{MP}, e_2^{MP}, $M, P \in \mathbb{R}^*$ are spectrally stable.*

Case 3: $a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ b_1 = 0,\ b_2 = 0,\ d_1 = 0,\ d_3 = 0$. Then the equilibrium states of the dynamics (2.1) have the following form:

$$e_1^M = (0, M, 0),\ M \in \mathbb{R};$$
$$e_2^M = (M, 0, 0),\ M \in \mathbb{R}.$$

Proposition 4.3. *If $d_2 \neq 0$ then the equilibrium state e_1^M, $M \in \mathbb{R}^*$ is unstable; if $d_2 = 0$ then the equilibrium state e_1^M, $M \in \mathbb{R}^*$ is spectrally stable.*

Proposition 4.4. *The equilibrium state e_2^M, $M \in \mathbb{R}^*$ is spectrally stable.*

Case 4: $b_1 = 0,\ b_3 = -b_2,\ d_1 = 0,\ d_2 = 0,\ d_3 = 0$. Then the equilibrium states of the dynamics (2.1) have the following form:

$$e^M = (M, 0, 0),\ M \in \mathbb{R}.$$

Proposition 4.5. *The equilibrium state e^M, $M \in \mathbb{R}^*$ is spectrally stable.*

Case 5: $a_2 = \dfrac{a_1 d_3}{b_1}$, $a_3 = \dfrac{a_1 d_2}{b_1}$, $b_2 = \dfrac{-b_1 b_3 + {d_2}^2 + {d_3}^2}{b_1}$, $d_1 = \dfrac{d_2 d_3}{b_1}$, $b_1 \neq 0$. Then the equilibrium states of the dynamics (2.1) have the following form:

$$e_1^M = (M, 0, 0),\ M \in \mathbb{R};$$
$$e_2^{MP} = \left(-\frac{a_1 + d_3 M + d_2 P}{b_1}, M, P\right), M, P \in \mathbb{R}.$$

Proposition 4.6. *The equilibrium states e_1^M, $M \in \mathbb{R} \setminus \{-\dfrac{a_1}{b_1}\}$ are unstable. If $M = -\dfrac{a_1}{b_1}$ then the equilibrium states e_1^M are spectrally stable.*

Proposition 4.7. *The equilibrium state e_2^{MP}, $M, P \in \mathbb{R}$ is spectrally stable.*

It can be verified that for each of these equilibrium states Arnold's method (see [4]) does not work and then via [6] any other energy methods are inconclusive. The nonlinear stability of the all equilibrium states remains an open problem.

5 Conclusions

In this paper we present the quadratic Hamilton-Poisson systems on the real Lie algebra of type XIII. We have defined the quadratic and homogeneous Hamilton-Poisson systems on the dual of the 13th Lie algebra from Bianchi's classification. Starting with a quadratic and homogeneous general form for the Hamiltonian, we find that the dynamics of our system is generated by a set of differential equations; for these equations we have sketched numerical integration via Kahan's integrator and Runge-Kutta 4th steps integrator. In this case we can see that both integrators give us a good approximation of the trajectory movement. The stability problem of the equilibrium states of our dynamics is discussed, too. This time, we have not find any nonlinear stable equilibrium points like in other cases (see [1], [2] and [3].) As a direct consequence, we were not able to find the periodical orbits of this equilibrium states. This problem remains an open one. Another open problem is the extension to Poisson-Lie algebroids (see [8]).

References

[1] A. Aron, M. Craioveanu, C. Pop, M. Puta, *Quadratic and homogeneous Hamilton-Poisson systems on* $A^*_{3,,6,-1}$, Balkan J. Geom. Appl. 15, 1 (2010), 1-7.

[2] A. Aron, C. Dănăiasă, M. Puta, *Quadratic and homogeneous Hamilton-Poisson systems on* $(so(3))^*$, International Journal of Geometric Methods in Modern Physics 4, 7(2007), 1173 - 1186.

[3] A. Aron, C. Pop, M. Puta, *Quadratic and homogeneous Hamilton-Poisson systems on (sl (2,R))* and Kahan's integrator*, In (Eds.: D. Andrica, S. Moroianu), "Contemporary Geometry, Topology and Related Topics", Cluj-Napoca, 19-25 August 2007, 63-72.

[4] V. Arnold, *Conditions for nonlinear stability of stationary plane curvilinear flows of an ideal fluid*, Doklady 162, 5 (1965), 773-777.

[5] L. Bianchi, *Lezioni sulla teoria dei gruppi continui finite di transformazioni*, Pisa, 1918, 550-777, Princeton Fine Hall Math-Phys Library SM 81134.177.11.

[6] P. Birtea and M. Puta, *Equivalence of energy methods in stability theory*, Journ. of Math. Physics, vol. 48, no. 4 (2007), 81-99.

[7] W. Kahan, *Unconventional numerical methods for trajectory calculation*, Unpublished Lecture Notes, 1993.

[8] L. Popescu, *A note on Poisson-Lie algebroids (I)*, Balkan J. Geom. Appl. 14, 2 (2009), 79-89.

[9] M. Puta, C. Georgescu, *3-Dimensional Poisson manifolds. Casimirs*, Annals of Univ. Timisoara, Ser. Mat-Inform. 34, 1 (1996), 151-159.

Authors' addresses:

Anania Aron, Camelia Pop
Mathematics Department, "Politehnica" University of Timişoara
Piaţa Victoriei nr. 2, 300006-Timişoara, Romania.
e-mail: anania.girban@gmail.com, cariesanu@yahoo.com