Notable submanifolds in Berwald-Moór spaces

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Dedicated to the 70-th anniversary
of Professor Constantin Udriste

Abstract. After proving that the Berwald-Moór structures are pseudo-Finsler of Lorentz type, for co-isotropic submanifolds of Berwald-Moór spaces ([4]), the Gauss-Weingarten and Gauss-Codazzi, Peterson-Mainardi and Ricci-Kühne equations of the such submanifolds are explicitly presented and discussed, taking into account the locally-Minkowski character of the Finslerian ambient metric.

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1 The Berwald-Moór pseudo-Finsler structure

The Berwald-Moór framework was initiated by P.K. Rashevski ([23], [24],[25]) and further fundamented by D. Pavlov, G. Garasko and S. Kokarev ([8], [9], [20], [10], [21], [22]). The main geometric structure is here provided by the structure $\mathcal{H}_n = (\tilde{M} = \mathbb{R}^n, F)$, with

$$F(y) = \begin{cases} \sqrt[n]{y^1 y^2 \cdot \ldots \cdot y^n}, & \text{for } n \text{ odd} \\ \sqrt[n]{|y^1 y^2 \cdot \ldots \cdot y^n|}, & \text{for } n \text{ even} \end{cases} = \sqrt[n]{\varepsilon_y \cdot y^1 y^2 \cdot \ldots \cdot y^n},$$

with $\varepsilon_y = (sign(y^1 \cdot \ldots \cdot y^n))^{n+1}$. This is a Finsler metric function of locally Minkowski type, i.e., F does not effectively depend on the points of M.

Remarks. The Berwald-Moór structure (as M. Matsumoto remarked in [16]), is peculiar, and has specific features which make it very special. In this respect, we note the following:

- though F is defined and continuous on the whole TM, the structure is $\mathcal{C}^\infty$ not on the whole slit tangent space $\widetilde{T\tilde{M}} = T\tilde{M}\backslash\{0\}$, but on the open subset:

  $$\widehat{T\tilde{M}} = \cup_{x\in\tilde{M}} \widehat{T_x\tilde{M}} \subset \widetilde{T\tilde{M}}, \;\; \widehat{T_x\tilde{M}} = \{y \in \mathbb{R}^n \equiv T_x M | y^1 \cdot \ldots \cdot y^n \neq 0\},$$

  i.e. on the intersection of the main open subsets of the slit tangent space which project to the natural open charts of the projectivized space PTM;

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- for n even F is (only) positive homogeneous of first order, while for n odd, it is (completely) homogeneous;
- in the odd case, F might have negative values as well.

Regarding the associated Finsler tensor field, we have the following result:

Proposition 1.1 ([16]). *The fundamental tensor field $g_{ij}dx^i \otimes dx_j$ of H^n defined by $g_{ij}(y) = \dfrac{1}{2}\dfrac{\partial^2 F^2}{\partial y^i \partial y^j}$ has the explicit expression*

$$g_{ij} = \frac{F^2}{ny^i y^j}\left(\frac{2}{n} - \delta_{ij}\right). \tag{1.1}$$

Its determinant is $g \equiv \det(g_{ij})_{i,j\in\overline{1,n}} = \dfrac{(-1)^{n+1}}{n^n}$, and its inverse (the dual $(2,0)-$tensor field g^{ij} given by $g^{is}g_{sj} = \delta^i_j$) has the components

$$g^{ij} = \frac{ny^i y^j}{F^2}\left(\frac{2}{n} - \delta^{ij}\right).$$

The proof is mainly computational, and relies on the properties of the determinant and on the following classical result from linear algebra:

Lemma 1.2. *If the matrix $[a] = (a_{ij})_{i,j=\overline{1,n}}$ is invertible, $b_i \in \mathbb{R}, i \in \overline{1,n}$, and $\varepsilon \in \{\pm 1\}$, then $[\tilde{a}] = (\tilde{a}_{ij} = a_{ij} + \varepsilon b_i b_j)_{i,j=\overline{1,n}}$ is invertible and*

a) $\det[\tilde{a}] = a_*(1 + B_*)$, where $a_* = \det[a]$, $b_* = a^{ij}b_i b_j$, with $[a]^{-1} = (a^{ij})_{i,j=\overline{1,n}}$;

b) $[\tilde{a}]^{-1}$ has the coefficients $\tilde{a}^{ij} = a^{ij} - \dfrac{\varepsilon}{1+\varepsilon b_*}b^i b^j$, where $b^i = a^{is}b_s$.

As consequence, the metric tensor g_{ij} is non-degenerate. Regarding its signature we prove the following

Theorem 1.3. *The Berwald-Moór metric tensor of $\mathcal{H}_n$ having the coefficients* (1.1) *is of Minkowski type, hence provides in each fiber of the tangent bundle a pseudo-Riemannian structure.*

Proof. We further denote $[g] = (g_{ij})_{i,j=\overline{1,n}}$. Then the spectrum of $[g]$ is given by

$$\sigma([g]) = \{\lambda \in \mathbb{R} | \det([g] - \lambda I_n) = 0\}.$$

We note that

$$[g] = \frac{\rho^{2/n}}{n^2}\begin{pmatrix} \mu(a_1)^2 & 2a_1a_2 & \dots & 2a_1a_n \\ 2a_2a_1 & \mu(a_2)^2 & \dots & 2a_2a_n \\ \vdots & \vdots & \ddots & \vdots \\ 2a_na_1 & 2a_na_2 & \dots & \mu(a_n)^2 \end{pmatrix},$$

where $\rho = y^1y^2 \cdot \ldots \cdot y^n = F^n$, $\mu = 2 - n$, $a_k = 1/y^k$. We perform a change of coordinates given by the matrix $C = diag\left(\dfrac{n}{\rho^{1/n}\sqrt{2}} \cdot (y^1, \ldots, y^n)\right)$, and then the matrix of the bilinear symmetric form changes to

$$[g]' = C^t[g]C = \begin{pmatrix} \alpha & 1 & \dots & 1 \\ 1 & \alpha & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & \alpha \end{pmatrix}, \text{ where } \alpha = \frac{\mu}{2} = 1 - \frac{n}{2}.$$

Hence $[g]' = U - \frac{n}{2}I_n$, where U is the $n \times n$ matrix with all entries equal to 1. Then by Sylvester's Theorem, $\sigma([g]) = \sigma([g]')$ and the characteristic polynomial is

$$P_{[g]'}(\lambda) = \det([g]' - \lambda I_n) = \det\left(U - \underbrace{(\lambda + \frac{n}{2})}_{\lambda'}I_n\right) = \det(U - \lambda' I_n).$$

But the eigenvalues λ' of U are n (simple eigenvalue), and 0 (with multiplicity $n-1$). Hence $\lambda = \lambda' - \frac{n}{2} \in \left\{\frac{n}{2}, -\frac{n}{2}, \dots, -\frac{n}{2}\right\} = \sigma([g]')$.

We conclude that $[g]$ is *not* positive definite, its signature is $(+, -, \dots, -)$, and the space $(\mathbb{R}^n, [g])$ is of Minkowski type. Hence the Berwald-Moór structure is pseudo-Finslerian of Lorentz type. □

There exist numerous notable submanifolds which naturally emerge from the algebraic and geometric richness of the Berwald-Moór structure of $\mathcal{H}_n$. We can mention the *indicatrix* of $\mathcal{H}_n$, $\Sigma_{\mathcal{H}_n} : y^1 \cdot \ldots \cdot y^n = 1$, which is a *Tzitzeica hypersurface* (i.e., the ratio between the curvature at a point $P \in \Sigma_{\mathcal{H}_n}$ and the $(n+1)$−th power of the distance from origin to the tangent plane to $\Sigma_{\mathcal{H}_n}$ at P, is constant, briefly $K(p) \cdot d(O, T_P\Sigma_{\mathcal{H}_n})^{-(n+1)} = const.$, see [26]), and whose study is essential, due to the fact that its isometries which commute with homotheties prove to be isometries of $\mathcal{H}_n$. The Tzitzeica hypersurface is tightly related the geometry of angles on $\mathcal{H}_n$ ([21], [22]). As well, it leads to natural PDEs of soliton type. The study of the symmetries of the PDEs which admit $\Sigma_{\mathcal{H}_n}$ as solution produces mappings which preserve the hypersurface, and which may contain subclasses of isometries of $\mathcal{H}_n$.

Another significant submanifold is the implicitly defined *simultaneity hypersurface*, with relevance in the applications of the Berwald-Moór structures in Relativity theory (see e.g., [15]).

Since the geometry of pseudo-Finsler spaces is a very recent field of research and $\mathcal{H}_n$ is an illustrative example of such space, we shall describe further a specific class of pseudo-Finsler submanifolds (the lightlike, or co-isotropic sumbanifolds), and point out the differences to the proper Finsler approach, illustrated for the case of particular ambient space $\mathcal{H}_n$.

2 Co-isotropic submanifolds in Berwald-Moór spaces

The pseudo-Riemann lightlike (co-isotropic) submanifolds are intensive subject of recent research, since they produce mathematical horizons in General Relativity (e.g., Cauchy or Kruskal; see [1], [6], [11], [7], [14], [13]) and appear in the theory of electromagnetism ([12]). Very recent advances regarding the ruled lightlike surfaces in

Minkowski 3-space have been obtained in [12]. The essential tool in studying the induced objects on a lightlike submanifold is the construction of a certain transversal bundle (see e.g. [6]). There exist at present few works which address such submnifolds in pseudo-Finsler spaces (e.g., [4]).

For convenience, we shall further replace the pseudo-Finslerian metric g_{ij}, whose signature is $(+,-,\dots,-)$ with the opposite metric, g_{ij}, of signature $(-,+,\dots,+)$. Then we note that the metric has index 1, is non-degenerate and has constant signature.

Consider now a submanifold $M \subset \tilde{M} \equiv \mathbb{R}^n$, $\dim M = m$ $(2 \leq m < n)$, locally parametrized by $x^i = x^i(u), u = (u^1, \dots u^m), i = \overline{1,n}$. We denote the local coordinates of TM by (u,v), where $v = (v^1, \dots, v^m) \in \mathbb{R}^m \equiv T_u M$. Then TM is immersed in $T\tilde{M}$, by $2n-$uples of the form $x(u), y(u,v)) \in R^{2n} \equiv T\tilde{M}$, with ([17], [4]) $y^i = B^i_*(u)$, where $B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$ and by $*$ we denote transvection with v (i.e., $B^i_* = B^i_\alpha v^\alpha$, and we use the indices $\alpha, \beta, \dots$ within the submanifold index range $\overline{1,m}$. The vector fields tangent to TM have their canonic basis related to the one of the ambient space $T\tilde{M}$ via

$$\frac{\partial}{\partial u^\alpha} = B^i_\alpha(u)\frac{\partial}{\partial x^i} + B^i_{\alpha *}(u)\frac{\partial}{\partial y^i}, \quad \frac{\partial}{\partial v^\alpha} = B^i_\alpha(u)\frac{\partial}{\partial y^i},$$

where $B^i_{\alpha\beta}(u) du^\alpha \otimes du^\beta$ is the flat Hessian of $x^i(u)$, $B^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}$. We note that g_{ij} induces on the submanifold M the $(0,2)-$symmetric tensor field

$$g^M_{\alpha\beta} = B^i_\alpha(u) B^j_\beta(u) g_{ij}(x(u), y(u,v)).$$

Let now TM^0 be the normal space to M in $\tilde{M}$,

$$TM^0 = \bigcup_{p \in TM} TM^0_p, \qquad \text{where } TM^0_p = \{X \in VTM_p \mid X \perp_g Y, \forall Y \in VTM_p\}.$$

In the proper Finslerian case (i.e., for g_{ij} positive definite), we have $TM^0 \cap VTM = \{0\}$. But in the pseudo-Finsler space $\mathcal{H}_n$ this property might fail, and we distinguish the typical case of co-isotropic submanifolds ([6], [4]):

Definition 2.1. The submanifold $M \subset \tilde{M}$ is called *co-isotropic* (or *lightlike, null*) if $TM^0 \subset VTM$.

Example 2.2. The hyperquadrics

$$K_y = \{w \in \mathbb{R}^n | w^t [g]_y w = 0\} \subset \mathbb{R}^n, \ \ \forall y \in \mathbb{R}^n$$

and their submanifolds are co-isotropic. We have denoted $[g]_y = (g_{ij}(y))_{i,j=\overline{1,n}}$.

In the following, in order to study the induced geometry of the co-isotropic submanifolds M of $\mathcal{H}_n$, we remark the need of a proper transversal to M bundle, for canonically obtaining the shape extrinsic objects. This is performed by constructing several total spaces for subbundles in $VT\tilde{M}$, determined by the following decompositions

$$\text{(2.1)} \qquad \begin{cases} VTM = S(VTM) \perp TM^0 \\ VT\tilde{M} = S(VTM) \perp S(VTM)^\perp \\ S(VTM)^\perp = TM^0 \oplus tr(VTM), \end{cases}$$

where $\perp$ and $\oplus$ denote the orthogonal direct, and simple direct sums, respectively. Then (2.1) lead to the decompositions ([4]):

$$VT\tilde{M} = S(VTM) \perp \underbrace{(TM^0 \oplus tr(VTM))}_{S(VTM)^\perp} = VTM \oplus tr(VTM). \tag{2.2}$$

The defined by (2.1) subspaces are

- $S(VTM)$ - called *the screen distribution* of VTM;
- $S(VTM)^\perp$ - the orthogonal complement of the screen distribution;
- $tr(VTM)$ - called *the transversal vector bundle* of M.

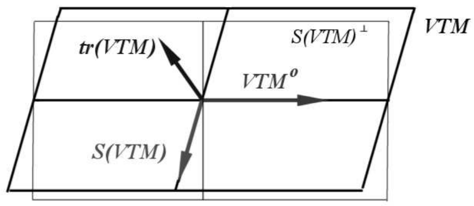

Fig.1. Building the transversal bundle in co-isotropic submanifolds.

The proof of existence of a local basis $B' = \{N_a | a \in \overline{m+1, n}\}$ of the $\mathcal{F}(TM)-$module of sections $\Gamma(tr(VTM))$ relies on imposing the requirements:

$$g(N_a, \xi_b) = \delta_{ab}, \ \ B' \perp_g B', \ \ B' \perp_g S(VTM),$$

i.e., the new basis has to be dual to a pre-existing basis $B = \{\xi_a\} | a \in \overline{m+1, n}\}$ of TM^0, has to be $g-$orthogonal and isotropic, and $g-$orthogonal to the screen distribution. The new local basis is still lightlike, but slant, transversal to M, hence replacing the tangent to M orthogonal basis B. Then one may consider the classic Gauss-Weingarten and Gauss-Codazzi, Peterson-Mainardi and Ricci-Kühne equations of the co-isotropic submanifold M, as follows.

3 Nonlinear connections on co-isotropic submanifolds of $\mathcal{H}_n$

We further use the common for general Finsler spaces abbreviations:

$$\partial_i = \frac{\partial}{\partial x^i}, \dot{\partial}_i = \frac{\partial}{\partial y^i}, \delta_i \equiv \frac{\delta}{\delta x^i} = \partial_i - N_i^j \dot{\partial}_j,$$

where $N_j^i = \dot{\partial}_j \gamma_{00}^i$ are the coefficients of *the nonlinear Cartan canonic connection* of $\tilde{M}$, the null index denotes transvection with y and $\gamma_{jk}^i = \frac{1}{2} g^{is}(\partial_j g_{sk} + \partial_k g_{sj} - \partial_s g_{jk})$ are the usual Christoffel symbols of second kind. We note that since $\mathcal{H}_n$ is locally Minkowski, the nonlinear connection has locally null coefficients. Assuming that H_n admits an affine atlas, the null nonlinear connection can be regarded as a global geometric object. This connection still locally induces a generally non-trivial non-linear connection $\bar{N}^\beta_\alpha$ on $M \subset \mathcal{H}_n$, given by

$$\bar{N}^\beta_\alpha(u,v) = \tilde{B}^\beta_i (B^j_\alpha N^i_j + B^i_{\alpha\beta} v^\beta) = \tilde{B}^\beta_i B^i_{\alpha\beta} v^\beta, \tag{3.1}$$

which is practically provided by the flat Hessian of the immersion components, where we denoted by $[\tilde{B}^\alpha_i, \tilde{N}^a_j]$ the inverse matrix of $[B^i_\alpha, N^i_a]$, and assume that the indices $a, b, \ldots$ take the complementary index values $\overline{m+1, n}$. The components of this inverse matrix practically link the fields of frames $\{\dot{\partial}_1, \ldots \dot{\partial}_n\}$ to the local adapted frame $\{\frac{\partial}{\partial v^1}, \ldots, \frac{\partial}{\partial v^m}, N_{m+1}, \ldots, N_n\} \subset \Gamma_{loc}(VT\tilde{M})$. As well, for M co-isotropic, $M \subset \tilde{M} = \mathcal{H}_n$ admits a local adapted basis of the sections of $TTM = HTM \oplus VTM$, with the horizontal fields

$$\begin{aligned} \frac{\delta}{\delta u^\alpha} &= \frac{\partial}{\partial u^\alpha} - \tilde{N}^\beta_\alpha \frac{\partial}{\partial v^\beta} = \\ &= B^i_\alpha \frac{\delta}{\delta x^i} + \tilde{N}^a_i (B^j_\alpha N^i_j + B^i_{\alpha *}) B^i_a \frac{\partial}{\partial y^i} = \\ &= B^i_\alpha \frac{\partial}{\partial x^i} + \tilde{N}^a_i B^i_{\alpha *} N_a \in \Gamma(HT\tilde{M} \oplus tr(VTM)). \end{aligned} \tag{3.2}$$

We note that $\{\frac{\delta}{\delta u^\alpha}\}$ locally span $\Gamma(HTM)$, where HTM is a subbundle of $HT\tilde{M} \oplus tr(VTM)$.

4 The co-isotropic Gauss and Weingarten equations in $\mathcal{H}_n$

For a given Finsler linear connection $\tilde{\nabla}$ on $\tilde{M}$, we infer the Gauss formula

$$\underbrace{\tilde{\nabla}_X Y}_{\in\Gamma(VT\tilde{M})} = \underbrace{\nabla_X Y}_{\in\Gamma(VTM)} + \underbrace{B(X,Y)}_{\in\Gamma(tr(VTM))}, \tag{4.1}$$

for all $X \in \Gamma(TTM), Y \in \Gamma(VTM)$. Here B is the *second fundamental form* on M and ∇ is *the induced by* $\tilde{\nabla}$ *on* M *linear connection*; locally we have $B(X,Y) = B^a(X,Y)N_a$. Then, due to a result in [4, Prop. 2.1, p. 65], we have $B^a(X,Y) = g(\tilde{\nabla}_X Y, \xi_a)$ and hence the second fundamental form does not depend on the choice of the screen distribution; this form vanishes for $X, Y \in \Gamma(TM^0)$. Moreover, the induced linear connection is metrical, if and only if it vanishes for $Y \in \Gamma(S(VTM))$. Hence we can state the following

Theorem 4.1. *The induced tangent connection on a co-isotropic submanifold* $M \subset \mathcal{H}_n$ *is metrical if and only if the second fundamental form is zero outside the sections*

$$\Gamma(S(VTM)) \times \Gamma(TM^0) \ni (X,Y).$$

The Weingarten formula has the form

$$\underbrace{\tilde{\nabla}_X V}_{\in\Gamma(VT\tilde{M})} = \underbrace{-A_V(X)}_{\in\Gamma(VTM)} + \underbrace{\nabla^{tr}_X V}_{\in\Gamma(tr(VTM))}, \tag{4.2}$$

for all $X \in \Gamma(TTM), V \in \Gamma(tr(VTM))$. Here ∇^{tr} is the *vertical linear connection* on $tr(VTM)$ and A_V is the Weingarten (shape) morphism of $\mathcal{F}(TM)$ modules A_V :

$\Gamma(TTM) \to \Gamma(VTM)$. We note that in $\mathcal{H}_n$ the components (L^i_{jk}, C^i_{jk}) of the Cartan linear connection are given by

$$(4.3)\qquad \begin{aligned} L^i_{jk} &= \tfrac{1}{2} g^{is}(\delta_j g_{sk} + \delta_k g_{sj} - \delta_s g_{jk}) \equiv 0 \\ C^i_{jk} &= g^i s C_{jsk} = \tfrac{y^i}{y^j y^k}\left(-\tfrac{2}{n^2} + \tfrac{\delta_{jk}+\delta^i_j+\delta^i_k}{n} - \delta^i_j \delta^i_k\right). \end{aligned}$$

where $\tilde{\nabla}_{\delta_k}\dot{\partial}_j = L^i_{jk}\dot{\partial}_i, \tilde{\nabla}_{\dot{\partial}_k}\dot{\partial}_j = C^i_{jk}\dot{\partial}_i$. Hence for $\tilde{M} = \mathcal{H}_n$, $L^i_{jk} \equiv 0$ and the Gauss-Weingarten equations provide on M trivial tangent linear and normal connections, Weingarten operator and second fundamental form when $X \in \Gamma(HTM)$, and effectively provides nontrivial induced geometric objects only for $X \in \Gamma(tr(VTM))$.

5 The co-isotropic Gauss, Codazzi and Ricci equations in $\mathcal{H}_n$

The three *curvature tensors* of a Finsler space have the coefficients:

$$(5.1)\qquad \begin{cases} \delta_{[l} L^i_{jk]} + L^h_{j[k} L^i_{hl]} + C^i_{ja} R^a_{kl} \\ P^i_{jkc} = \dot{\partial}_C L^i_{jk} - C^i_{jc|k} + C^i_{jb} P^b_{kc} \\ S^a_{bcd} = \dot{\partial}_{[d} C^a_{bc]} + C^r_{b[c} C^a_{rd]}, \end{cases}$$

where $_{|k}$ denotes the h-covariant derivation associated to the Cartan connection, i.e.,

$$C^i_{jc|k} = \delta_k C^i_{jc} + C^s_{jc} L^i_{sk} - C^i_{sc} L^s_{jk} - C^i_{js} L^s_{ck},$$

and we considered *the torsion fields* , $R^a_{kl} = \delta_{[l} N^a_{k]}$ $P^b_{kc} = \dot{\partial}_c N^b_k - L^b_{ck}$ (which vanish for Berwald-Moór spaces), *the Cartan tensor* $C_{ikj} = \frac{1}{2}\dot{\partial}^3_{ijk} F^2$, which for H^n has the form ([16]):

$$C_{ijk} = \frac{F^2}{ny^i y^j y^k}\left(\frac{2}{n^2} - \frac{\delta_{ij} + \delta_{jk} + \delta_{ki}}{n} + \delta_{ij}\delta_{jk}\delta_{ki}\right),$$

and we denoted by square braces the skew-symmetrization (e.g., $\tau_{[i\ldots j]} = \tau i \ldots j - \tau_{j\ldots i}$). The *covariant curvature tensors* are given by

$$R_{ijkl} = g_{js} R^s_{ikl}, \quad P_{ijkl} = g_{js} P^s_{ikl}, \quad S_{ijkl} = g_{js} S^s_{ikl}.$$

Since H^n is locally Minkowski, its horizontal and mixed curvatures identically vanish; moreover, it is known that ([16])

Theorem 5.1. *The Berwald-Moór space $\mathcal{H}_n$ has the following properties:*

a) $A_i \equiv F \cdot \dot{\partial}_i(\log g^{1/2}) = 0,;$

b) The covariant vertical curvature satisfies the equality

$$S_{ijkl} = F^{-2} S \cdot (g_{i[k} g_{jl]}),$$

with $S = -1 = const$, i.e., $\mathcal{H}_n$ is $S3-$like.

c) The $T-$tensor $T_{ijkl} = FC_{ijk|l} + \Sigma_{ijkl} l_i C_{jkl}$ identically vanishes, where $l_i = \frac{F}{ny^i}$, and Σ_{ijkl} denotes the cyclic sum over the lower indices.

It is known as well ([16]), that $S_{ijkl} \equiv 0$ for $n = 2$, and that the $S3-$ property always holds true for 3-dimensional Finsler spaces. The $T-$property shows that the Finsler space is locally symmetric, like in (pseudo-)Riemannian geometry, and that the space becomes (pseudo-)Riemannian under some weak assumptions. The conformally deformed space $\tilde{F}(x,y) = e^{\sigma(x)}\sqrt[n]{y^1 y^2 \cdot \ldots \cdot y^n}$ satisfies the Theorem as well, and $P_{ijkl} \equiv 0$.

For deriving the Gauss, Codazzi and Ricci equations, we build the linear connection ∇' on TM ([4]) via

$$\nabla'_X Y = \tilde{\nabla}_X v(Y) + \varphi(\tilde{\nabla}_X \varphi(hY)), \forall X, Y \in \Gamma(TTM),$$

where h and v are the horizontal, respectively vertical projectors for the decomposition $\Gamma(TTM) = \Gamma(HTM) \oplus \Gamma(VTM)$, and $\varphi \in End(\Gamma(TTM)$ is the almost product $\mathcal{F}(TM)-$module endomorphism given on the local basis via

$$\varphi\left(\frac{\delta}{\delta u^\alpha}\right) = \frac{\partial}{\partial v^\alpha},\ \varphi\left(\frac{\partial}{\partial v^\alpha}\right) = \frac{\delta}{\delta u^\alpha}, \forall \alpha \in \overline{1,m}.$$

Denoting by $\tilde{R}, R, R^{tr}$ the curvatures of $\tilde{\nabla}, \nabla, \nabla^{tr}$ one can explicitly derive the Gauss, Codazzi and Ricci equations of $M \subset \tilde{M} = \mathcal{H}_n$. Namely, denoting by $+(X/Y)$ the addition of the term which has X and Y interchanged, and by $-(X/Y)$ its substraction, $A(V,X) = A_V(X)$, considering the torsion T' of ∇' and the objects:

$$\begin{cases} (\nabla^{tr}_X B)(Y,Z) = \nabla^{tr}_X(B(Y,Z)) - [B(\nabla'_x Y) + X/Y], \\ (\nabla_X A)(V,Y) = \nabla_X(A(V,Y) - [A(\nabla^{tr}_X V, Y) + X/Y], \end{cases} \tag{5.2}$$

we infer the announced three sets of equations ([4, Sec. 3.6]):

(5.3)

$$\begin{cases} \tilde{R}(X,Y)Z & = R(X,Y)Z + [A(B(X,Z),Y) + (\nabla^{tr}_X B)(Y,Z) - (X/Y)] + \\ & +B(T'(X,Y)Z) \\ \tilde{R}(X,Y)V & = R^{tr}(X,Y)V + [B(Y, A_V(X) + (\nabla_Y A)(V,X) - (X/Y)] + \\ & +A_V(T'(X,Y), \end{cases}$$

$$\begin{cases} g(\tilde{R}(X,Y)Z,U) & = g(R(X,Y)Z,U) + [\delta_{ab}B^a(X,Z)g(A_{N_b}(Y),U) - (X/Y)] \\ g(\tilde{R}(X,Y)Z,\xi) & = [g(\nabla^{tr}_Y B)(Y,Z),\xi) - (X/Y)] + g(B(T'(X,Y),Z),\xi) \\ g(\tilde{R}(X,Y)Z,N) & = g(R(X,Y)Z,N), \end{cases}$$

$$\begin{cases} g(\tilde{R}((X,Y)N_a,U) & = [g((\nabla_Y A)(N_a,X) - (X/Y)] - g(A_{N_a}(T'(X,Y),U) = \\ & = -g(R(X,Y)U,N_a) \\ g(\tilde{R}((X,Y)N_a,\xi_b) & = g(R^{tr}(X,Y)N_a,\xi_b + [B^b(Y,A_{N_a}(x)) - (X/Y)] = \\ & = -g(R(X,Y)\xi_b,N_a) \\ g(\tilde{R}((X,Y)N_a,N_b) & = g(A_{N_a}(T'(X,Y),N_b) + [(\nabla_Y A)(N_a,X) - (X/Y)], \end{cases}$$

for all $X, Y \in \Gamma(TTM)$, $Z \in \Gamma(VTM)$, $V \in \Gamma(tr(VTM))$, $U \in \Gamma(S(VTM))$, $\xi \in \Gamma(TM^0)$, $N \in \Gamma(tr(VTM))$. In $\tilde{M} = \mathcal{H}_n$, the relations (5.3) are nontrivial only when $X, Y \in \Gamma(VTM)$, due to the splitting $\Gamma(VTM) = \Gamma(TM^0) \perp \Gamma(S(VTM))$ and considering the skew-symmetry in X, Y, these are refined to $3^{(2+3+3)}$ equations.

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