On entropy production in relativistic thermodynamics

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Dedicated to the 70-th anniversary of Professor Constantin Udriste

Abstract. After to recall the energy-momentum tensor we will introduce the four vector current entropy density in which the entropy density is regarded as function of the energy density and of the symmetric strain tensor. By considering the concept of relativistic equilibrium state it will be defined temperature in a generic inertial frame (no in proper frame) obtaining the transformation equations introduced by H. Ott for the temperature. Moreover, further considerations on the introduction of relativistic stress tensor allows us to obtain the transformation equations for the strain tensor. Finally the time component of the derivative with respect to the coordinate of the four vector current entropy density will express the entropy balance and consequently the entropy production.

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1 Introduction

We consider a continuum medium in motion with respect to an arbitrary inertial frame of reference Σ and we introduce Galilean coordinates defined in terms of spatial and temporal variables (x, y, z, t) by the equations:

$$x_0 = ct; \;\; x_1 = x; \;\; x_2 = y; \;\; x_3 = z \tag{1.1}$$

where "c" is the scalar velocity of light in vacuum. In four dimensional space we will use the following metric:

$$ds^2 = dx_0^2 - dx_i^2 \tag{1.2}$$

in which Einstein convection is used. It is well known that the coordinate transformation relating two inertial frames Σ and Σ' in relative general configuration are the Lorentz transformation which can be written:

$$\begin{cases} x_i = x'_i + \frac{\alpha v_i}{c} x'_0 + (\alpha - 1) \frac{v_i v_k}{v^2} x'_k \\ x_0 = \alpha \Big(x'_0 + \frac{v_i x'_i}{c} \Big) \end{cases} \tag{1.3}$$

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where $\mathbf{v} \equiv (v_1, v_2, v_3)$ is the uniform velocity of σ with respect to Σ' and

$$\alpha = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{1.4}$$

Now, if we consider an infinitesimal element of medium, we can introduce the so called "*proper reference*" as the inertial frame of reference which results instantaneously at the rest with respect to it. Moreover a function referred to this reference will be indicated with the symbol "^{0}" over the function.

2 Energy momentum tensor and first law of thermodynamics

Suppose the medium in motion with respect to Σ and indicating with Σ_0 the proper reference of an element of fluid it is well known that the study of the motion of such a medium implies to consider different forms of density of energy flow. We will limit our considerations only on three forms of density of energy flow, don't taking into account electrodynamics phenomena.

In such a context we consider the following:

i) E_i is the vector representing density of energy flow of not mechanical nature (as the heat),

ii) $\rho c^2 v_i$ is the density of energy flow due only to the motion of the medium where ρ is the mass density,

iii) $v^j \phi_{ji}$ as the density of energy energy flow due to the action of the forces of stress flowing in the positive x_i direction, where ϕ_{ji} is the relativistic (no symmetric) stress tensor.

Therefore the total density of energy flow, which we indicate L_i, will be

$$L_i = E_i + \rho c^2 v_i + v^j \phi_{ji} \tag{2.1}$$

according to Einstein relation between mass and energy, it is possible associate to this quantity the following total momentum density

$$H_i = \frac{L_i}{c^2} = \rho v_i + \frac{E_i}{c^2} + \frac{v^j \phi_{ji}}{c^2} \tag{2.2}$$

Now we are able to introduce the so called *energy momentum tensor* $T_{\alpha\beta}$:

$$T_{\alpha\beta} = \begin{cases} T_{ik} = H_i v_k + \phi_{ik} \\ T_{i0} = T_{0i} = cH_i \\ T_{00} = \rho c^2 \end{cases} \tag{2.3}$$

in which latin index assumes the values $1, 2, 3$ and greek index assumes the values $0, 1, 2, 3$. If ρF_i is the unitary volume force, the introduction of a four vector W_α defines as

$$W_\alpha \equiv \left(\frac{\rho v_i F_i}{c}, \rho F_i\right) \tag{2.4}$$

allows us to write the tensorial equation

$$\frac{\partial T_{\alpha\beta}}{\partial x^{\beta}} = W_{\alpha} \tag{2.5}$$

in which the "temporal component ("$(^0)$" index) represents the balance equation for energy density and the spatial components ("1,2,3" index) represent the balance equation for momentum density. It is well known tensorial nature of equation (2.5). By introducing the four vector V^{α} defined as

$$V^{\alpha} \equiv \left(\alpha, \alpha\frac{v_i}{c}\right) \tag{2.6}$$

it is easy to obtain the first law of thermodynamics combining equations (2.5) and (2.6):

$$\frac{\partial T_{\alpha\beta}}{\partial x^{\beta}} V^{\alpha} = W_{\alpha} V^{\alpha} \tag{2.7}$$

From this we obtain the following expression utilized in the next section

$$\frac{\partial T_{00}}{\partial x^0} = \frac{2\rho F_i v_i}{c} - \frac{\partial T_{i0}}{\partial x^0} V^i - \frac{\partial T_{i0}}{\partial x^i} - \frac{\partial T_{ik}}{\partial x^k} V^i \tag{2.8}$$

It is important to remark that all the solutions obtained are relative to generic inertial frame Σ.

3 Transformed tensors

In the following we will use the quantities introduced in the last sections referred to a proper reference Σ_0 and it is useful to determine the relations between the same quantities expressed in Σ. For this reason it is very important underline the tensorial character of the quantities $T_{\alpha\beta}$. As it is well known a tensor of order two will transforms according the following law:

$$T_{\alpha\beta} = \frac{\partial x^{\mu}}{\partial x^{\alpha'}} \frac{\partial x^{\nu}}{\partial x^{\beta'}} T^{(0)}_{\mu\nu} \tag{3.1}$$

where $T^{(0)}_{\mu\nu}$ is the energy momentum tensor in a proper reference and the law of transformation is related to Lorentz transformation (1.3_1) and (1.3_2).
It is important to observe that the velocity v_i which appears in (1.3) describes the distribution of velocity of medium in the particular instant in which the initial reference has the same velocity of the generic infinitesimal portion of medium. Taking into account (1.3_1), (1.3_2), (2.3) and (3.1) it is easy to obtain the following relations:

$$T_{00} = \alpha^2\rho_0 c^2 + 2\frac{\alpha^2 v^i}{c^2} E_i{}^{(0)} + \frac{\alpha^2}{c^2} v^i v^k \phi^{(0)}_{ik} = \rho c^2 \tag{3.2}$$

$$\begin{aligned} T_{i0} = {} & \alpha^2 v_i \rho_0 c + \frac{\alpha^2}{c^3} v_i v^k E^{(0)}_k + \frac{\alpha}{c} E^(0)_i + \frac{\alpha(\alpha-1)}{v^2 c} v^i v^k E^{(0)}_k + \frac{\alpha v^h}{c} \phi^{(0)}_{ih} + \\ & + \frac{\alpha(\alpha-1)}{v^2 c} v_i v^h v^k \phi^{(0)}_{hk} = cH_i \end{aligned} \tag{3.3}$$

$$T_{ik} = \alpha^2 v_i v_k \rho_0 + \phi^{(0)}_{ik} + \frac{v^l v_k}{v^2}(\alpha-1)\phi^{(0)}_{il} + \frac{v_i v^h}{v^2}(\alpha-1)\phi^{(0)}_{hk} + +v_i v_k \frac{v^h v^l}{v^4}(\alpha-1)^2\phi^{(0)}_{hl} + \frac{\alpha v_i E^{(0)}_k}{c^2} + \frac{\alpha v_k E^{(0)}_i}{c^2} + 2\frac{\alpha(\alpha-1)}{v^2c^2} v_i v^h v_k E^{(0)}_h \tag{3.4}$$

where ρ_0 and $\phi^{(0)}_{ik}$ are respectively the mass density and the symmetric Cauchy stress tensor in a proper reference. By taking into account $(2.3)_1$, $(2.3)_2$ and (3.3) it is easy to obtain:

$$\phi_{ik} = \phi^{(0)}_{ik} + \frac{v^l v_k}{v^2}(\alpha-1)\phi^{(0)}_{il} + \frac{v_i v^h}{v^2}(\alpha-1)\phi^{(0)}_{hk} + v_i v_k \frac{v^h v^l}{v^2}(\alpha-1)^2 + -\alpha\frac{v^h v_k}{c^2}\phi^{(0)}_{ih} - \frac{\alpha(\alpha-1)}{v^2c^2} v_i v_k v^h v^l \phi^{(0)}_{hl} + \frac{\alpha v_i}{c^2}E^{(0)}_k + \frac{\alpha(\alpha-1)}{v^2c^2} v_i v^h v_k E^{(0)}_h - \frac{\alpha^2 v_i v_k v^h}{c^4}E^{(0)}_h. \tag{3.5}$$

4 Relativistic temperature and strain tensor

Let ϕ_0 the entropy density in Σ_0 such that

$$d\sigma_0 = \phi_0 dV_0 \tag{4.1}$$

represents the entropy of an element of medium which in Σ_0 has the volume dV_0. It is well known that entropy density ϕ_0 will depends by density of internal energy and strain tensor:

$$\phi_0 = \phi_0(T^{(0)}_{00}, \gamma^{(0)}_{ik}) \tag{4.2}$$

moreover, since $d\sigma_0$ is a relativistic invariant the entropy density will transform as:

$$\phi = \phi_0 \alpha \tag{4.3}$$

Now, if we will preserve that in Σ_0 the entropy density depends on density of internal energy and stress tensor it is necessary that density entropy ϕ in Σ will depends on quantity which reduce to relation (4.2) as Σ approach to Σ_0. This is true if:

$$\phi = \phi(T_{00}, \gamma_{ik}) \tag{4.4}$$

where T_{00} is expressed by (3.2) and γ_{ik} is the transformed tensor of $\gamma^{(0)}_{ik}$ which it is calculated in the following. Let us remark that the temperature is defined in an equilibrium state and this state in Σ, for the principle of relativity, is preserved. Moreover in Σ_0 it follows:

$$\frac{\partial\phi_0}{\partial T^{(0)}_{00}} = \frac{1}{T_0} \tag{4.5}$$

By equation (4.4), in Σ we have

$$\frac{\partial\phi}{\partial T_{00}} = \alpha\frac{\partial\phi_0}{\partial T^{(0)}_{00}}\frac{\partial T^{(0)}_{00}}{\partial T_{00}} \tag{4.6}$$

By taking into account relation (3.2) it follows:

$$\frac{\partial T_{00}^{(0)}}{\partial T_{00}} = \frac{1}{\alpha^2} \tag{4.7}$$

and the equation(4.6) becomes:

$$\frac{\partial \phi}{\partial T_{00}} = \frac{1}{T_0}\frac{1}{\alpha} \tag{4.8}$$

This last equation allows us to define the temperature in Σ as follows:

$$\frac{1}{T} = \frac{\partial \phi}{\partial T_{00}} = \frac{1}{T_0}\frac{1}{\alpha} \tag{4.9}$$

or

$$T = \alpha T_0 \tag{4.10}$$

in agreement with Ott's transformation formula. Our proposal is now to obtain the transformation law for strain tensor. Then, from equation (4.6) it obtains:

$$\frac{\partial \phi}{\partial \gamma_{rs}} = \frac{\partial \phi_0}{\partial \gamma_{ik}^{(0)}}\frac{\partial \gamma_{ik}^{(0)}}{\gamma_{rs}}\alpha \tag{4.11}$$

By virtue of the relation

$$\frac{\partial \phi_0}{\partial \gamma_{ik}^{(0)}} = \frac{1}{T_0}\phi_{ik}^{(}0) \tag{4.12}$$

the equation (4.11) becomes:

$$\frac{\partial \phi}{\partial \gamma_{rs}} = \frac{1}{T_0}\phi_{ik}^{(0)}\frac{\partial \gamma_{ik}^{(0)}}{\partial \gamma_{rs}}\alpha = \alpha^2\frac{1}{T}\phi^{(0)}ik\frac{\partial \gamma_{ik}^{(0)}}{\partial \gamma_{rs}} \tag{4.13}$$

from which

$$T\frac{\partial \phi}{\partial \gamma_{rs}} = \alpha^2\phi_{ik}^{(0)}\frac{\partial \gamma_{ik}^{(0)}}{\partial \gamma_{rs}} \tag{4.14}$$

Analogously to the equation(4.12) the stress tensor in Σ is defined as follows:

$$\phi_{rs} = T\frac{\partial \phi}{\partial \gamma_{rs}} \tag{4.15}$$

and therefore

$$\phi_{rs} = \alpha^2\phi_{ik}^{(0)}\frac{\partial \gamma_{ik}^{(0)}}{\partial \gamma_{rs}} \tag{4.16}$$

Let us assume that our original system of coordinates Σ is oriented so that the material, at the point of interest in the medium, will be moving with respect to this system with the velocity, **v** parallel to the x-axis. Moreover, the system Σ_0 moves with the same velocity **v** respect to the system Σ. By virtue of these considerations, the transformation law of the stress tensor are the following:

$$\phi_{11} = \phi_{11}^{(0)} = \alpha^2 \phi_{rs}^{(0)} \frac{\partial \gamma_{rs}^{(0)}}{\partial \gamma_{11}} \tag{4.17}$$

$$\phi_{12} = \phi_{12}^{(0)} \alpha = \alpha^2 \phi_{rs}^{(0)} \frac{\partial \gamma_{rs}^{(0)}}{\partial \gamma_{12}} \tag{4.18}$$

$$\phi_{13} = \phi_{13}^{(0)} \alpha = \alpha^2 \phi_{rs}^{(0)} \frac{\partial \gamma_{rs}^{(0)}}{\partial \gamma_{13}} \tag{4.19}$$

$$\phi_{21} = \phi_{21}^{(0)} = \alpha^2 \phi_{rs}^{(0)} \frac{\partial \gamma_{rs}^{(0)}}{\partial \gamma_{21}} \tag{4.20}$$

$$\phi_{22} = \phi_{22}^{(0)} = \alpha^2 \phi_{rs}^{(0)} \frac{\partial \gamma_{rs}^{(0)}}{\partial \gamma_{22}} \tag{4.21}$$

$$\phi_{23} = \phi_{23}^{(0)} = \alpha^2 \phi_{rs}^{(0)} \frac{\partial \gamma_{rs}^{(0)}}{\partial \gamma_{23}} \tag{4.22}$$

$$\phi_{31} = \frac{\phi_{31}^{(0)}}{\alpha} = \alpha^2 \phi_{rs}^{(0)} \frac{\partial \gamma_{rs}^{(0)}}{\partial \gamma_{31}} \tag{4.23}$$

$$\phi_{32} = \phi_{32}^{(0)} = \alpha^2 \phi_{rs}^{(0)} \frac{\partial \gamma_{rs}^{(0)}}{\partial \gamma_{32}} \tag{4.24}$$

$$\phi_{33} = \phi_{33}^{(0)} = \alpha^2 \phi_{rs}^{(0)} \frac{\partial \gamma_{rs}^{(0)}}{\partial \gamma_{33}} \tag{4.25}$$

Let us observe that the stress tensor is not symmetric, then by considering the relation (4.16) it results:

$$\alpha^2 \phi_{ik}^{(0)} \frac{\partial \gamma_{ik}^{(0)}}{\partial \gamma_{rs}} \neq \alpha^2 \phi_{ik}^{(0)} \frac{\partial \gamma_{ik}^{(0)}}{\partial \gamma_{sr}} \tag{4.26}$$

therefore it follows the non symmetry of the strain tensor, i.e.:

$$\gamma_{rs} \neq \gamma_{sr} \tag{4.27}$$

The transformation law of the strain tensor follow by development of the equations (4.17) and after to observe that the components in Σ of the stress tensor depend only on the corresponding components in Σ_0, this implies that

$$\frac{\partial \gamma_{rs}^{(0)}}{\gamma_{ik}} = 0 \quad \text{if} \quad i \neq r, \;\; k \neq s \tag{4.28}$$

this implies that even components in Σ of strain tensor depend only on the corresponding components in Σ_0. Thus the transformation law of the strain tensor is expressed in the following form:

$$\gamma_{11} = \alpha^2 \gamma_{(11)}^{(0)}, \;\; \gamma_{12} = \alpha \gamma_{(12)}^{(0)}, \;\; \gamma_{13} = \alpha \gamma_{(13)}^{(0)} \tag{4.29}$$

$$\gamma_{21} = \alpha^3 \gamma_{(21)}^{(0)}, \;\; \gamma_{22} = \alpha^2 \gamma_{(22)}^{(0)}, \;\; \gamma_{23} = \alpha^2 \gamma_{(23)}^{(0)} \tag{4.30}$$

$$\gamma_{31} = \alpha^3 \gamma_{(31)}^{(0)}, \;\; \gamma_{32} = \alpha^2 \gamma_{(32)}^{(0)}, \;\; \gamma_{11} = \alpha^2 \gamma_{(33)}^{(0)} \tag{4.31}$$

From relation (4.4) we obtain:

$$\frac{\partial \phi}{\partial x^0} = \frac{\partial \phi}{\partial T_{(00)}} \frac{\partial T_{00}}{\partial x_0} + \frac{\partial \phi}{\partial \gamma_{ik}} \frac{\partial \gamma_{ik}}{\partial x^0} \tag{4.32}$$

and equation (4.32) becomes:

$$\frac{\partial \phi}{\partial x^0} = \frac{1}{T} \frac{T_{00}}{\partial x^0} + \frac{1}{T} \phi_{ik} \frac{\partial \gamma_{ik}}{\partial x^0} \tag{4.33}$$

By substituting the relation (2.8) in this last one obtains:

$$\frac{\partial \phi}{\partial x_0} = \frac{1}{T} \Big[- \frac{\partial T_{0i}}{\partial x_i} + \frac{2\rho v_i F_i}{c} - \frac{\partial T_{i0}}{\partial x_0} V^i - \frac{\partial T_{ik}}{\partial x_k} V^i \Big] + \frac{1}{T} \phi_{ik} \frac{\partial \gamma_{ik}}{\partial x_0} \tag{4.34}$$

But from equation (2.5) we have:

$$\Big(\frac{\partial T_{i0}}{\partial x_0} + \frac{\partial T_{ik}}{\partial x_k} \Big) V^i = \frac{\rho F_i v_i}{c} \tag{4.35}$$

By substituting this relation into (4.34):

$$\frac{\partial \phi}{\partial x_0} = - \frac{\partial J^{(\sigma)}}{\partial x_i} + \Omega^{(\sigma)} \tag{4.36}$$

where

$$J^{(\sigma)} = \frac{T_{0i}}{T}$$

and

$$\Omega^{(\sigma)} = - \frac{T_{0i}}{T^2} \frac{\partial T}{\partial x_i} + \frac{\rho F_i v_i}{Tc} + \frac{\phi_{ik}}{T} \frac{\partial \gamma_{ik}}{\partial x_0} \tag{4.37}$$

This relation is the relativistic entropy density production. it is useful to observe that in Σ_0 the equation (4.36) and (4.37) become respectively:

$$\frac{\partial \phi_0}{\partial t_0} = - \frac{\partial \Big(\frac{E_i^{(0)}}{T_0} \Big)}{\partial x_i^{(0)}} + \Omega^{(0)}(\sigma) \tag{4.38}$$

$$\Omega^{(0)}(\sigma) = - \frac{E_i}{c} \frac{1}{T_0^2} \frac{\partial T_0}{\partial x_i^{(0)}} + \frac{\phi_{ik}}{T_0 c} \frac{\partial \gamma_{ik}^{(0)}}{\partial t_0} \tag{4.39}$$

which if we remember the meaning of $E_i^{(0)}$ represent well known balance and production of entropy density equations.

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