On some generalizations of the Palais-Smale condition

G. Cicortaş

Dedicated to the 70-th anniversary of Professor Constantin Udriste

Abstract. In this paper we study the weak Palais-Smale condition introduced in [C. K. Zhong-*A generalization of Ekeland's variational principle and application to the study of the relation between the weak PS condition and coercivity*, Nonlinear Anal., Theory Methods Appl. 29 (1997) 1421–1431], the generalized Cerami condition of [A. R. El Amrouss-*Critical point theorems and applications to differential equations*, Acta Math. Sin., Engl. Ser. 21 (2005) 129–142] and the smooth version of the compactness condition used in [A. Kristaly, V. V. Motreanu, Cs. Varga-*A minimax principle with a general Palais-Smale condition*, Commun. Appl. Anal. 9 (2005) 285–297]. Then a generalization of a previous results of the author is given.

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1 Introduction

In a series of papers published in 1960's (see [16]-[18], [20]), R. Palais and S. Smale introduced a compactness condition, now called the Palais-Smale condition, which provides the existence of a critical point for many variational problems.

Various relevant functionals originating in physics and in differential geometry only satisfy this condition for certain levels (see [10], [11]); a local condition, introduced in [5], is successfully applied in many problems.

A compactness condition of the Palais-Smale type was introduced by G. Cerami in [6]. See also [7]. This is slightly weaker than the Palais-Smale condition, while the most important implications are retained; see [4]. Generalizations of both Palais-Smale and Cerami conditions appears in [1], [13], [24]. Applications are given in [2], [3], [19]. See also [9], [15], [22].

In [8] the author proved that for a function F which is a finite sum of functions bounded from below, multiplied by positive real numbers, such that any such function has bounded derivative on sets on which it is bounded, a family of functions

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which have the same critical set as F at some level can be obtained. If F satisfies the Palais-Smale condition at the level c, then any function of the family satisfies the Palais-Smale condition at some level. Conversely, if there exist a function of this family which satisfies the Palais-Smale condition at some level, then F will satisfy the Palais-Smale condition at level c. We mention that the idea of considering such functions goes back to a perturbation problem studied in [23].

In this paper we study the relationship between the weak Palais-Smale condition introduced by C. K. Zhong in [24], the generalized Cerami condition defined by A. R. El Amrouss in [1] and the smooth version of the compactness condition used by A. Kristaly, V. V. Motreanu, Cs. Varga in [13]. Then we prove that, in suitable hypotheses, all the results of [8] remain true if we replace the Palais-Smale condition with the smooth version of the compactness condition of [13].

2 Preliminaries

Let M be a C^1 Banach Finsler manifold (see [16], [17]) and $f \in C^1(M, \mathbf{R})$. A point $p \in M$ is critical for f if $df(p) = 0$ and $c \in \mathbf{R}$ is a critical value of f if there exists $p \in M$ such that $df(p) = 0$ and $f(p) = c$. The critical set of f is the set of all critical points of f and it is denoted by $C[f]$. The critical set of level c of f is $C_c[f] = C[f] \cap f^{-1}(c)$.

We recall the following definitions.

Definition 2.1. (i) We say that f satisfies the Palais-Smale condition, denoted by (PS), if any sequence (x_n) in M such that $(f(x_n))$ is bounded and $\|df(x_n)\| \to 0$ has a convergent subsequence.
(ii) We say that f satisfies the Palais-Smale condition at level c, denoted by $(PS)_c$, if any sequence (x_n) in M such that $f(x_n) \to c$ and $\|df(x_n)\| \to 0$ has a convergent subsequence.

When (PS) is satisfied, we can verify immediately that the local condition $(PS)_c$ holds for all $c \in \mathbf{R}$, while the converse is not true.

Definition 2.2. (i) We say that f satisfies the Cerami condition, denoted by (C), if any sequence (x_n) in M such that $(f(x_n))$ is bounded and $(1 + \|x_n\|)\|df(x_n)\| \to 0$ has a convergent subsequence.
(ii) We say that f satisfies the Cerami condition at level c, denoted by $(C)_c$, if any sequence (x_n) in M such that $f(x_n) \to c$ and $(1+\|x_n\|)\|df(x_n)\| \to 0$ has a convergent subsequence.

It is easy to see that the condition (C) implies $(C)_c$. Moreover, (PS) is stronger than (C) and $(PS)_c$ is stronger than $(C)_c$. See [12] for examples and comments.

3 Generalized Palais-Smale conditions

Let M be a C^1 Banach Finsler manifold and $f \in C^1(M, \mathbf{R})$. In this section we discuss the conditions $(PS)^h_c$, $(C)^\alpha_c$ and $(\varphi C)_c$ introduced in [24], [1] and [13] respectively.

Let $h : [0, \infty) \to [0, \infty)$ be a continuous function with the following property:

$$\int_0^{+\infty} \frac{1}{1 + h(t)} dt = +\infty.$$

Definition 3.1. We say that f satisfies the weak Palais-Smale condition at the level c, denoted by $(PS)_c^h$, if any sequence (x_n) in M such that $f(x_n) \to c$ and $(1 + h(\|x_n\|))\|df(x_n)\| \to 0$ has a convergent subsequence.

Remark 3.1. If h is a constant, then $(PS)_c^h$ is exactly the $(PS)_c$ condition; if $h(t) = t$ we recover the local Cerami condition $(C)_c$.

Remark 3.2. Under the assumption that h is nondecreasing, the Definition 3.1 was introduced by C. K. Zhong in [24]. See also [21], [14].

Let $\alpha : [0, \infty) \to (0, 1]$ be a continuous function such that

$$\int_0^{+\infty} \alpha(t) dt = +\infty.$$

Definition 3.2. We say that f satisfies the generalized Cerami condition at level c, denoted by $(C)_c^\alpha$, if the following statements are true:
(i) any bounded sequence (x_n) in M such that $f(x_n) \to c$ and $\|df(x_n)\| \to 0$ has a convergent subsequence;
(ii) there exist $r > 0$ and $\sigma > 0$ such that

$$\|df(x)\| \geq \alpha(\|x\|),\ \forall\ x \in f^{-1}([c - \sigma, c + \sigma]) \setminus B(0, r).$$

Remark 3.3. The condition $(C)_c^\alpha$, in a slightly different form, was introduced by A. R. El Amrouss in [1]. The above definition is inspired by the non-differentiable case, studied in [14]. Moreover, the function f satisfies the conditions $(PS)_c^h$ if and only if it satisfies the $(C)_c^\alpha$ condition. See [14].

We prove now the following proposition.

Proposition 3.1. *Assume that f satisfies the $(PS)_c^h$ condition. Then for any $\delta > 0$ there exist $\gamma > 0$ and $\sigma > 0$ such that the following property holds:*

$$(1 + h(\|x\|))\, \|df(x)\| \geq \gamma,\ \forall\ x \in f^{-1}([c - \sigma, c + \sigma]) \setminus U_\delta,$$

where U_δ is a δ-neighborhood of the critical set of level c of f, $C_c[f]$.

Proof. By contradiction, suppose that there exists $\delta > 0$ such that for any $\gamma > 0$ and any $\sigma > 0$ there exists $x \in M \setminus U_\delta$ such that $c - \sigma \leq f(x) \leq c + \sigma$ and $(1 + h(\|x\|))\, \|df(x)\| < \gamma$. By choosing $\sigma = \dfrac{1}{n}$, $\gamma = \dfrac{1}{n}$ for arbitrary n, we obtain a sequence (x_n) in M such that:

$$dist(x_n, C_c[f]) \geq \delta;$$

$$c - \frac{1}{n} \leq f(x) \leq c + \frac{1}{n};$$

$$(1 + h(\|x_n\|))\,\|df(x_n)\| < \frac{1}{n}.$$

It follows that $f(x_n) \to c$ and $(1 + h(\|x_n\|))\|df(x_n)\| \to 0$. Because f satisfies the $(PS)_c^h$ condition, there exists a subsequence (x_{n_k}) such that $x_{n_k} \to x^0$. It is obvious the fact that $x^0 \notin U_\delta$. On the other hand, the continuity of f respectively df implies that $f(x^0) = c$ and $df(x^0) = 0$. Then $x^0 \in C_c[f]$. □

Let $\varphi : M \to \mathbf{R}$ be a continuous function such that $\varphi(x) \geq 1,\ \forall\, x \in M$.

Definition 3.3. We say that f satisfies the general compactness condition at level c with respect to φ, denoted by $(\varphi C)_c$, if any sequence (x_n) in M such that $f(x_n) \to c$ and $\varphi(x_n)\|df(x_n)\| \to 0$ has a convergent subsequence.

Remark 3.4. The condition $(\varphi C)_c$, for locally Lipschitz functions, was introduced by A. Kristaly, V. V. Motreanu and Cs. Varga in [13]. A Deformation Lemma and a general version of Mountain Pass Theorem in this context were established.

We prove the following property of functions satisfying the $(\varphi C)_c$ condition.

Proposition 3.2. *Let f be a function satisfying the $(\varphi C)_c$ condition. Then for any $\delta > 0$ there exist $\gamma > 0$ and $\sigma > 0$ such that*

$$\varphi(x)\|df(x)\| \geq \gamma,\ \forall\, x \in f^{-1}([c - \sigma, c + \sigma]) \setminus U_\delta,$$

where U_δ is a δ-neighborhood of the critical set of level c of f, $C_c[f]$.

Proof. By contradiction, suppose that there exists $\delta > 0$ such that for any $\gamma > 0$ and any $\sigma > 0$ there exists $x \in M \setminus U_\delta$ with the properties:

$$c - \sigma \leq f(x) \leq c + \sigma; \quad \varphi(x)\|df(x)\| < \gamma.$$

For $\sigma = \gamma = \dfrac{1}{n}$, with arbitrary n, we obtain the sequence (x_n) such that

$$dist(x_n, C_c[f]) \geq \delta;$$

$$c - \frac{1}{n} \leq f(x_n) \leq c + \frac{1}{n};$$

$$\varphi(x_n)\|df(x_n)\| < \frac{1}{n}.$$

Then $f(x_n) \to c$ and $\varphi(x_n)\|df(x_n)\| \to 0$. By using the fact that f satisfies the $(\varphi C)_c$ condition, it follows that (x_n) has a convergent subsequence $x_{n_k} \to x^0$. Remark that $x^0 \in C_c[f]$, which contradicts the fact that $dist(x_n, C_c[f]) \geq \delta > 0$. □

Remark 3.5. The general compactness condition $(\varphi C)_c$ contains all the compactness conditions mentioned above:
(i) if φ is constant, we obtain the $(PS)_c$ condition;
(ii) if $\varphi(x) = 1 + \|x\|$, then $(\varphi C)_c$ coincides with the $(C)_c$ condition;
(iii) if $\varphi(x) = 1 + h(\|x\|)$, where $h : [0, \infty) \to [0, \infty)$ is a continuous function with the property $\displaystyle\int_0^{+\infty} \frac{1}{1 + h(t)}\,dt = +\infty$, we obtain the $(PS)_c^h$ condition;

(iv) if $\varphi(x) = \dfrac{1}{\alpha(\|x\|)}$, where $\alpha : [0,\infty) \to (0,1]$ is a continuous function with the property $\int_0^{+\infty} \alpha(t)dt = +\infty$, then $(\varphi C)_c$ coincides with the $(C)_c^\alpha$ condition.

4 A class of functions with $(\varphi C)_c$ property

Let M be a C^2-Finsler manifold and $f_k : M \to \mathbf{R}$, $k = \overline{1,n}$, of C^1 class. Define $F = \sum_{k=1}^{n} a_k f_k$, where $a_k > 0, k = \overline{1,n}$, are fixed. In [8] we obtained a family of functions which have the same critical set as F at some level. More precisely, for $c \in \mathbf{R}$ fixed and $l = \overline{1,n}$ we define

$$g_l = \frac{f_l}{c - \sum\limits_{k=\overline{1,n},\ k\neq l} a_k f_k}.$$

Then $C_c[F] = C_{\frac{1}{a_l}}[g_l], l = \overline{1,n}$. See [8], Theorem 3.1.

Moreover, if $\inf\limits_M f_k > 0$ and df_k is bounded on sets on which f_k is bounded, $k = \overline{1,n}$, then the following assertions are true:

1. If F satisfies the $(PS)_c$ condition, then any g_l satisfies the $(PS)_{\frac{1}{a_l}}$ condition.

2. If one of the mappings g_l above defined satisfies the $(PS)_{\frac{1}{a_l}}$ condition, then F satisfies the $(PS)_c$ condition. See [8], Theorems 3.2. and 3.3.

In this section we show that, in suitable hypotheses, all the results of [8] remains true if we replace the local Palais-Smale condition with the general compactness condition with respect to φ, at some level.

From now, we assume that $\inf\limits_M f_k > 0$, $\varphi : M \to \mathbf{R}$ is a continuous mapping such that $\varphi \geq 1$ on M and $\varphi(\cdot)\|df_k(\cdot)\|$ is bounded on sets on which f_k is bounded, $k = \overline{1,n}$.

Theorem 4.1. *Suppose that F satisfies the $(\varphi C)_c$ condition. Then any mapping g_l satisfies the $(\varphi C)_{\frac{1}{a_l}}$ condition.*

Proof. Let $(x_p) \subset M$ be such that $g_l(x_p) \to \dfrac{1}{a_l}$ and $\varphi(x_p)\|dg_l(x_p)\| \to 0$ $(p \to \infty)$. By standard computation we obtain the following relations:

$$F(x_p) = a_l f_l(x_p) + \sum_{k=\overline{1,n},k\neq l} a_k f_k(x_p) = c + f_l(x_p)\left(a_l - \frac{1}{g_l(x_p)}\right)$$

$$\varphi(x_p)\|dF(x_p)\| = \varphi(x_p)\left\|a_l df_l(x_p) + \frac{f_l(x_p)}{(g_l(x_p))^2} dg_l(x_p) - \frac{df_l(x_p)}{g_l(x_p)}\right\| \leq$$

$$\leq \varphi(x_p)\|dg_l(x_p)\| \frac{|f_l(x_p)|}{(g_l(x_p))^2} + \varphi(x_p)\|df_l(x_p)\| \left|a_l - \frac{1}{g_l(x_p)}\right|.$$

For any $\varepsilon > 0$ we have $\left| g_l(x_p) - \dfrac{1}{a_l} \right| < \varepsilon$. This inequality combined with the assumption $\inf\limits_M f_k > 0$, $k = \overline{1,n}$, implies that

$$f_l(x_p) = g_l(x_p) \left(c - \sum_{k=\overline{1,n}, k \neq l} a_k f_k(x_p) \right) < c \left(\frac{1}{a_l} + \varepsilon \right).$$

We conclude that $F(x_p) \to c$ $(p \to \infty)$. On the other hand, because $\varphi(\cdot) \|df_l(\cdot)\|$ is bounded on sets on which f_l is bounded, it follows that $\varphi(x_p) \|dF(x_p)\| \to 0$ $(p \to \infty)$. We use the $(\varphi C)_c$ condition for F and we find some convergent subsequence $(x_{p_s})_s$. Then g_l satisfies $(\varphi C)_{\frac{1}{a_l}}$. □

Theorem 4.2. *If one of the mappings g_l defined above satisfies the $(\varphi C)_{\frac{1}{a_l}}$ condition, then F satisfies the $(\varphi C)_c$ condition.*

Proof. Let $(x_p) \subset M$ be such that $F(x_p) \to c$ and $\varphi(x_p) \|dF(x_p)\| \to 0$ $(p \to \infty)$. We can write $g_l(x_p) - \dfrac{1}{a_l} = \dfrac{1}{a_l} \cdot \dfrac{F(x_p) - c}{a_l f_l(x_p) + c - F(x_p)}$ and we obtain $g_l(x_p) \to \dfrac{1}{a_l}$ $(p \to \infty)$. On the other hand, we have:

$$\varphi(x_p) \|dg_l(x_p)\| = \varphi(x_p) \left\| \frac{df_l(x_p)(c - F(x_p)) + f_l(x_p) dF(x_p)}{(c - F(x_p) + a_l f_l(x_p))^2} \right\| \leq$$

$$\leq \frac{|c - F(x_p)|}{(c - F(x_p) + a_l f_l(x_p))^2} \varphi(x_p) \|df_l(x_p)\| + \frac{|f_l(x_p)|}{(c - F(x_p) + a_l f_l(x_p))^2} \varphi(x_p) \|dF(x_p)\|.$$

From $g_l(x_p) \to \dfrac{1}{a_l}$ $(p \to \infty)$ we deduce that $(f_l(x_p))$ is bounded. Because $\varphi(\cdot) \|df_l(\cdot)\|$ is bounded on sets on which f_l is bounded, we deduce that $dg_l(x_p) \to 0$ $(p \to \infty)$. Then the $(\varphi C)_{\frac{1}{a_l}}$ condition for g_l ensures the existence of a convergent subsequence $(x_{p_s})_s$ and it follows that F satisfies the $(\varphi C)_c$ condition. □

Remark 4.1. If one of the mappings g_l above defined satisfies the $(\varphi C)_{\frac{1}{a_l}}$ condition, then any mapping g_k satisfies the $(\varphi C)_{\frac{1}{a_k}}$ condition.

References

[1] A. R. El Amrouss, *Critical point theorems and applications to differential equations*, Acta Math. Sin., Engl. Ser. 21 (2005), 129–142.

[2] A. R. El Amrouss, *Multiplicity results for semilinear elliptic problems with resonance*, Nonlinear Anal. 65 (2006), 634–646.

[3] A. R. El Amrouss, *Nontrivial solutions of semilinear equations at resonance*, J. Math. Anal. Appl. 325 (2007), 19–35.

[4] P. Bartolo, V. Benci and D. Fortunato, *Abstract critical point theorems and applications to some nonlinear problems with "strong" resonance at infinity*, Nonlinear Anal. 7 (1983), 981–1012.

[5] H. Brezis, J. M. Coron and L. Nirenberg, *Free vibrations for a nonlinear wave equation and a theorem of P. Rabinowitz*, Commun. Pure Appl. Math. 33 (1980) 667–684.

[6] G. Cerami, *An existence criterion for the critical points on unbounded manifolds (Italian)*, J. Rend. Sci. Mat. Fis. Chim. Geol. 112 (1978), 332–336.

[7] G. Cerami, *On the existence of eigenvalues for a nonlinear boundary value problem. (Italian)*, Ann. Mat. Pura Appl. (4) 124 (1980), 161–179.

[8] G. Cicortaş, *Note on a function satisfying Palais-Smale condition*, Nonlinear Anal., Theory Methods Appl. 71 (2009), 5090–5092.

[9] I. Duca, *Palais-Smale condition for multi-time actions that produce Poisson-gradient PDEs*, BSG Proc. 14, Geometry Balkan Press 2007, 63–67.

[10] D. Fortunato and G. Palmieri, *Remarks on the Yamabe problem and the Palais-Smale condition*, Rend. Semin. Mat. Univ. Padova 75 (1986,) 47–65.

[11] W. Gordon, *Physical variational principles which satisfy the Palais-Smale condition*, Bull. Am. Math. Soc. 78 (1972), 712–716.

[12] Y. Jahbri, *The Mountain Pass theorem. Variants, Generalizations and Some Applications*, Cambridge University Press, 2003.

[13] A. Kristaly, V. V. Motreanu and Cs. Varga, *A minimax principle with a general Palais-Smale condition*, Comm. Appl. Anal. 9 (2005), 285–297.

[14] R. Livrea and S. A. Marano, *A min-max principle for non-differentiable functions with a weak compactness condition*, Comm.Pure Appl.Anal. 8 (2009), 1019–1029.

[15] I. Meghea, *Some results obtained in dynamical systems using a variational calculus theorem*, BSG Proc. 16, Geometry Balkan Press 2009, 91–98.

[16] R. S. Palais, *Morse theory on Hilbert manifolds*, Topology 2 (1963), 299–340.

[17] R. S. Palais, *Lusternik-Schnirelmann theory on Banach manifolds*, Topology 5 (1966), 115–132.

[18] R. S. Palais and S. Smale, *A generalized Morse theory*, Bull. Am. Math. Soc. 70 (1964), 165–172.

[19] M. Schechter, *The Use of Cerami Sequences in Critical Point Theory*, Abstr. Appl. Anal. 2007, Art. ID 58948, 28 pp.

[20] S. Smale, *Morse theory and a non-linear generalization of the Dirichlet problem*, Ann. of Math. 80 (1964), 382–396.

[21] T. Suzuki, *On the relations between the weak Palais-Smale condition and coercivity by Zhong*, Nonlin. Analysis, Theory Meth. Appl. 68 (2008), 2471–2478.

[22] C. Udrişte, *Simplified multitime maximum principle*, Balkan J. Geom. Appl. 14 (2009), 102–119.

[23] K. Uhlenbeck, *Morse theory by perturbation methods with applications to harmonic maps*, Trans. Am. Math. Soc. 267 (1981), 569–583.

[24] C. K. Zhong, *A generalization of Ekeland's variational principle and application to the study of the relation between the weak PS condition and coercivity*, Nonlinear Anal., Theory, Methods Appl. 29 (1997), 1421–1431.

Author's address:

G. Cicortaş
University of Oradea, Faculty of Science,
University Street 1, 410087 Oradea, Romania.
E-mail: cicortas@uoradea.ro