

Extensions on twisted crossed products of completely positive invariant projective u -covariant multi-linear maps

Tania-Luminița Costache

*Dedicated to the 70-th anniversary
of Professor Constantin Udriste*

Abstract. In this paper we give the projective covariant representation of a C^* -dynamical system (G, A, α) on a Hilbert C^* -module associated with a completely bounded symmetric and completely positive invariant projective u -covariant multi-linear map, result that generalizes (Theorem 3.1, [5]). We also extend such a map to a completely positive multi-linear map on the twisted crossed product $A \times_{\alpha}^{\omega} G$, giving the projective version of (Theorem 3.2, [5]).

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1 Introduction and preliminaries

The physical significance of completely positive linear maps of C^* -algebras has been observed in a number of papers, in which it was shown that the completely positive linear maps describe the change of states of quantum dynamical systems, produced by quantum measurement or may describe the time development of open quantum dynamical systems.

In [3], Christensen and Sinclair were the first that introduced the notion of completely bounded (respectively, completely positive) multi-linear maps from a C^* -algebra into the algebra $\mathcal{B}(H)$ of linear bounded operators on a Hilbert space H . Stinespring's Theorem gives a useful representation for a completely positive linear operator from a C^* -algebra into $\mathcal{B}(H)$ ([1], [14]). Using this representation with Wittstock's Theorem that decomposes a completely bounded linear operator as a finite linear combination of completely positive linear operators, Christensen and Sinclair obtained a representation of a completely bounded linear operator from a C^* -algebra

into $\mathcal{B}(H)$. The authors gave a representation theorem for completely bounded multi-linear operators from a C^* -algebra into $\mathcal{B}(H)$, which generalizes this representation of completely bounded linear operators. Paulsen and Smith, in [13] proved a representation theorem for completely bounded multi-linear maps extended to the case of subspaces of C^* -algebras using the correspondence between completely bounded multi-linear maps and completely bounded linear maps on tensor products endowed with the Haagerup norm. In [5], Heo defined the invariance of multi-linear maps and introduced invariant multi-linear maps as a generalization of invariant sesquilinear forms. The definition of an invariant k -linear map is slightly different in the k odd and the k even cases, so that these were written separately. He constructed the family of commuting representations associated with a completely bounded and completely positive invariant multi-linear map which extends the Stinespring representation. This representation is different from the multi-linear representation in [3]. The author gave the covariant representation of a covariant completely bounded and completely positive invariant multi-linear map on a Hilbert C^* -module. Using this representation, he proved that such a map is extended to a completely positive multi-linear map on a C^* -dynamical system.

In the present paper we introduce the notions of projective covariant representations of a C^* -dynamical system (G, A, α) on Hilbert C^* -modules and projective u -covariant multi-linear maps. In Theorem 2.1 we give the projective covariant representation on a Hilbert C^* -module associated with a completely bounded symmetric and completely positive invariant projective u -covariant multi-linear map, result that generalizes (Theorem 3.1, [5]). Then we extend this map on the twisted crossed product $A \times_{\alpha}^{\omega} G$ (Theorem 3.1), giving the projective version of (Theorem 3.2, [5]).

We recall the definitions introduced in [3], [5] and [6] of completely bounded symmetric and completely positive invariant multi-linear maps and the notion of a twisted crossed product of A by G under the action α and relative to the multiplier ω ([2], [12]).

Definition 1.1. ([3]) Let A and B be two C^* -algebras and let $\rho: A^k \rightarrow B$ be a k -linear map. The k -linear map ρ_n from $M_n(A)^k$ into $M_n(B)$ is defined by

$$\rho_n(A_1, A_2, \dots, A_k) = \left(\sum_{r,s,\dots,t} \rho(a_{1ir}, a_{2rs}, \dots, a_{ktj}) \right)$$

for all $A_p = (a_{pij}) \in M_n(A)$, $1 \leq p \leq k$. We define the norm of ρ_n by

$$\|\rho_n\| = \sup \{ \|\rho_n(A_1, A_2, \dots, A_k)\| : A_p \in M_n(A) \text{ with } \|A_p\| \leq 1 \text{ for } 1 \leq p \leq k \}$$

and define the *completely bounded norm* of ρ_n by $\|\rho\|_{cb} = \sup \{ \|\rho_n\| : n \in \mathbb{N} \}$.

The k -linear map ρ is called *completely bounded* if $\|\rho\|_{cb} < \infty$.

Definition 1.2. ([3]) The k -linear map ρ^* from A^k into B is defined by

$$\rho^*(a_1, a_2, \dots, a_k) = \rho(a_k^*, \dots, a_2^*, a_1^*)^*$$

for all $a_1, a_2, \dots, a_k \in A$. The k -linear map ρ is called *symmetric* if $\rho = \rho^*$.

Note that if ρ is symmetric, then ρ is completely symmetric in that $\rho_n = (\rho_n)^*$ for all n and that if ρ is completely bounded, then so is ρ^* with $\|\rho^*\|_{cb} = \|\rho\|_{cb}$.

Definition 1.3. ([3]) If $(A_1, \dots, A_k) \in M_n(A)^k$. Let

$\|\rho_n\|_s = \sup \{ \|\rho_n(A_1, \dots, A_k)\|; A_j \in M_n(A) \text{ with } \|A_j\| \leq 1 \text{ for } 1 \leq j \leq k \text{ and } (A_1, \dots, A_k) = (A_k^*, \dots, A_1^*) \}$ for all n . Then the *symmetric completely bounded* norm $\|\cdot\|_{scb}$ is defined by $\|\rho\|_{scb} = \sup \{ \|\rho_n\|_s; n \in \mathbb{N} \}$.

Definition 1.4. ([3]) A k -linear map $\rho: A^k \rightarrow B$ is said to be *completely positive* if $\rho_n(A_1, \dots, A_k) \geq 0$ for all $(A_1, \dots, A_k) = (A_k^*, \dots, A_1^*) \in M_n(A)^k$ with $A_m \geq 0$ if k is odd where $m = \lceil \frac{k+1}{2} \rceil$ and all $n \in \mathbb{N}$.

Definition 1.5. ([5]) A k -linear map $\phi: A^k \rightarrow B$ is called *invariant* if:

(i) $k = 2m - 1$ odd, we have

$$\begin{aligned} \phi(a_1 c_1, \dots, a_{m-1} c_{m-1}, a_m, a_{m+1}, \dots, a_{2m-1}) = \\ \phi(a_1, \dots, a_{m-1}, a_m, c_{m-1} a_{m+1}, \dots, c_1 a_{2m-1}), \end{aligned}$$

for all $a_1, \dots, a_{2m-1}, c_1, \dots, c_{m-1} \in A$.

(ii) $k = 2m$ even, we have

$$\phi(a_1 c_1, \dots, a_m c_m, a_{m+1}, \dots, a_{2m}) = \phi(a_1, \dots, a_m, c_m a_{m+1}, \dots, c_1 a_{2m})$$

for all $a_1, \dots, a_{2m}, c_1, \dots, c_m \in A$.

We denote by $ICB_s(A^k, B)$ the set of all completely bounded symmetric invariant k -linear maps and by $ICB^+(A^k, B)$ the set of all completely positive and completely bounded invariant k -linear maps.

Definition 1.6. ([8]) A *multiplier* ω of a locally compact group G is a Borel function $\omega: G \times G \rightarrow \mathbb{T}$, where \mathbb{T} is the group of complex numbers of modulus one, with the following properties :

- i) $\omega(x, e) = \omega(e, x) = 1$ for all $x \in G$;
- ii) $\omega(x, y)\omega(xy, z) = \omega(x, yz)\omega(y, z)$ for all $x, y, z \in G$.

Definition 1.7. ([8]) A multiplier is called *normalized* if $\omega(x, x^{-1}) = 1$ for all $x \in G$.

Busby and Smith in [2] were the first that introduced the notion of a twisted crossed product of a C^* -algebra A by a locally compact group G under the action α relative to the multiplier ω which is denoted by $A \times_\alpha^\omega G$. They define a convolution and an involution on $L^1(G, A)$ such that

$$\begin{aligned} (f * g)(x) &= \int_G f(y) \alpha_y[h(y^{-1}x)] \omega(y, y^{-1}x) dy \\ f^*(x) &= \omega(x, x^{-1})^* \alpha_x[f(x^{-1})^*] \Delta_G(x^{-1}), \end{aligned}$$

where Δ_G is the modular function on G and $L^1(G, A)$ becomes a Banach $*$ -algebra $L^1(G, A, \alpha, \omega)$ with respect to these operations (Theorem 2.2, [2]). When $L^1(G, A, \alpha, \omega)$ has a bounded left approximate identity, then the representations of (G, A, α, ω) are in one-to-one correspondence with the covariant representations of (G, A, α, ω) (Theorem 3.3, [2]) and the enveloping C^* -algebra of $L^1(G, A, \alpha, \omega)$ is a *twisted crossed product*.

We denote by $\mathcal{L}_B(E)$ the C^* -algebra of all adjointable module homomorphisms from a Hilbert C^* -module E to itself (i.e. T is a bounded module homomorphism such that there is T^* a bounded module homomorphism from E to E satisfying $\langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle$ for all $\xi, \eta \in E$).

2 The projective covariant representation on a Hilbert C^* -module of a completely positive and completely bounded symmetric invariant projective u -covariant multi-linear map

Definition 2.1. Let B be a C^* -algebra and let E be a Hilbert C^* -module over B . A *projective representation* of G on E with the multiplier ω is a map $\pi: G \rightarrow \mathcal{L}_B(E)$ such that

- i) $\pi(xy) = \omega(x, y)\pi(x)\pi(y)$ for all $x, y \in G$;
- ii) $\pi(e) = I_E$, where e is the identity of G and I_E is the identity operator on E .

We say that π is a *projective unitary representation* of G on a Hilbert module E over a C^* -algebra B if, in addition, $\pi(x)$ is a unitary element in $\mathcal{L}_B(E)$ for all $x \in G$.

Definition 2.2. A *projective covariant representation* of a C^* -dynamical system (G, A, α) on a Hilbert B -module E is a triple (Φ, v, E) , where Φ is a representation of A on E and v is a projective unitary representation of G into $\mathcal{L}_B(E)$ with the multiplier ω such that

$$\Phi(\alpha_g(a)) = v_g \Phi(a) v_g^*$$

for all $a \in A$ and $g \in G$.

Definition 2.3. If (G, A, α) is a C^* -dynamical system, then the action $\alpha: G \rightarrow \text{Aut}(A)$ induces an action $\tilde{\alpha}: G \rightarrow \text{Aut}(A^k)$ by

$$\tilde{\alpha}_g(a_1, \dots, a_k) = (\alpha_g(a_1), \dots, \alpha_g(a_k))$$

for all $a_1, \dots, a_k \in A$. Let E be a Hilbert C^* -module over a C^* -algebra B and let $u: G \rightarrow \mathcal{L}_B(E)$ be a projective unitary representation of G with the multiplier ω . The k -linear map $\phi: A^k \rightarrow \mathcal{L}_B(E)$ is called *projective u -covariant* if

$$\phi(\tilde{\alpha}_g(a_1, \dots, a_m)) = \phi(\alpha_g(a_1), \dots, \alpha_g(a_m)) = u_g \phi(a_1, \dots, a_m) u_g^*$$

for all $a_1, \dots, a_m \in A$ and $g \in G$.

Theorem 2.1. Let (G, A, α) be a unital C^* -dynamical system, let B be a unital C^* -algebra and let u be a projective unitary representation $u: G \rightarrow \mathcal{U}(B)$ (where $\mathcal{U}(B)$ is the unitary group of B) with the normalized multiplier ω . Let $\phi \in \text{ICB}_s^+(A^k, B)$ be a projective u -covariant k -linear map ($k \geq 2$) and let $m = \lceil \frac{k+1}{2} \rceil$. Then there are:

- (i) a Hilbert B -module F ;
- (ii) a family of representations $(\pi_p)_{p=\overline{1, m}}$ of A on F ;
- (iii) a family (π_p, v, F) , $p = \overline{1, m}$ of projective covariant representations of (G, A, α) , where v is a projective unitary representation of G on F with the multiplier ω ;
- (iv) $V \in \mathcal{L}_B(B, F)$ such that
 - (1) $V m_{u_g} = v_g V$ for all $g \in G$, where m is a left multiplication operator;
 - (2) $V^* \pi_1(a_1) \dots \pi_m(a_m) V = m_{\phi(1_A, \dots, 1_A, a_1, \dots, a_m)}$, where the number of 1_A 's is $m-1$ in odd case and m in even case.

Proof. To prove (i) and (ii) we follow the proof of (Theorem 2.1, [5]).

The case $k = 2m - 1$ odd. Let $A^{\otimes m}$ be the algebraic tensor product of m copies of A . We define the map $\langle \cdot, \cdot \rangle : (A^{\otimes m} \otimes B) \times (A^{\otimes m} \otimes B) \rightarrow B$ by

$$\left\langle \sum_{i=1}^n a_{1i} \otimes \dots \otimes a_{mi} \otimes b_i, \sum_{j=1}^l c_{1j} \otimes \dots \otimes c_{mj} \otimes d_j \right\rangle = \sum_{i=1}^n \sum_{j=1}^l b_i^* \phi(a_{mi}^*, \dots, a_{2i}^*, a_{1i}^* c_{1j}, c_{2j}, \dots, c_{mj}) d_j$$

for all $a_{1i}, \dots, a_{mi}, c_{1j}, \dots, c_{mj} \in A$ and $b_i, d_j \in B$.

Then the map $\langle \cdot, \cdot \rangle : (A^{\otimes m} \otimes B) \times (A^{\otimes m} \otimes B) \rightarrow B$ is well-defined. It is easy to check that this map is linear in the second variable and conjugate-linear in the first variable. To show that this sesquilinear form is semi-definite, we take an element

$x = \sum_{i=1}^n a_{1i} \otimes \dots \otimes a_{mi} \otimes b_i$ in $A^{\otimes m} \otimes B$. From the definition we get

$$\begin{aligned} \langle x, x \rangle &= \left\langle \sum_{i=1}^n a_{1i} \otimes \dots \otimes a_{mi} \otimes b_i, \sum_{j=1}^n a_{1j} \otimes \dots \otimes a_{mj} \otimes b_j \right\rangle = \\ &= \sum_{i,j=1}^n b_i^* \phi(a_{mi}^*, \dots, a_{2i}^*, a_{1i}^* a_{1j}, a_{2j}, \dots, a_{mj}) b_j = \\ &= [b_1^* \dots b_n^*] [\phi(a_{mi}^*, \dots, a_{2i}^*, a_{1i}^* a_{1j}, a_{2j}, \dots, a_{mj})] [b_1 \dots b_n]^t \geq 0, \end{aligned}$$

where $[\phi(a_{mi}^*, \dots, a_{2i}^*, a_{1i}^* a_{1j}, a_{2j}, \dots, a_{mj})]$ is an $n \times n$ matrix in $M_n(B)$ and $[b_1 \dots b_n]^t$ is the transpose matrix of $[b_1 \dots b_n]$, because, by the definition of ϕ_n , we have

$$[\phi(a_{mi}^*, \dots, a_{2i}^*, a_{1i}^* a_{1j}, a_{2j}, \dots, a_{mj})]_{i,j=1}^n = \phi_n(A_m^*, \dots, A_2^*, A_1^* A_1, A_2, \dots, A_m) \geq 0,$$

where $A_1 \in M_n(A)$ has the first row a_{11}, \dots, a_{1n} and all other rows zeros and $A_p = \text{diag}(a_{p1}, \dots, a_{pn})$ ($2 \leq p \leq m$) is the diagonal $n \times n$ matrix in $M_n(A)$.

For all $x = \sum_{i=1}^n a_{1i} \otimes \dots \otimes a_{mi} \otimes b_i$ and $y = \sum_{j=1}^l c_{1j} \otimes \dots \otimes c_{mj} \otimes d_j$ in $A^{\otimes m} \otimes B$, we have

$$\begin{aligned} \langle x, y \rangle &= \left\langle \sum_{i=1}^n a_{1i} \otimes \dots \otimes a_{mi} \otimes b_i, \sum_{j=1}^l c_{1j} \otimes \dots \otimes c_{mj} \otimes d_j \right\rangle = \\ &= \sum_{i=1}^n \sum_{j=1}^l b_i^* \phi(a_{mi}^*, \dots, a_{2i}^*, a_{1i}^* c_{1j}, c_{2j}, \dots, c_{mj}) d_j = \\ &= \left(\sum_{i=1}^n \sum_{j=1}^l d_j^* \phi(a_{mi}^*, \dots, a_{2i}^*, a_{1i}^* c_{1j}, c_{2j}, \dots, c_{mj})^* b_i \right)^* = \end{aligned}$$

$$\left(\sum_{i=1}^n \sum_{j=1}^l d_j^* \phi(c_{mj}^*, \dots, c_{2j}^*, c_{1j}^* a_{1i}, a_{2i}, \dots, a_{mi}) b_i \right)^* = \left\langle \sum_{j=1}^l c_{1j} \otimes \dots \otimes c_{mj} \otimes d_j, \sum_{i=1}^n a_{1i} \otimes \dots \otimes a_{mi} \otimes b_i \right\rangle^* = \langle y, x \rangle^*$$

where the fourth equality results from the symmetry of ϕ .

For all $x = \sum_{i=1}^n a_{1i} \otimes \dots \otimes a_{mi} \otimes b_i$, we define $x \cdot b = \sum_{i=1}^n a_{1i} \otimes \dots \otimes a_{mi} \otimes b_i b$ ($b \in B$).

Then $A^{\otimes m} \otimes B$ becomes a right B -module and we have

$$\begin{aligned} \langle x, y \cdot b \rangle &= \left\langle \sum_{i=1}^n a_{1i} \otimes \dots \otimes a_{mi} \otimes b_i, \sum_{j=1}^l c_{1j} \otimes \dots \otimes c_{mj} \otimes d_j b \right\rangle = \\ &= \left(\sum_{i=1}^n \sum_{j=1}^l b_i^* \phi(a_{mi}^*, \dots, a_{2i}^*, a_{1i}^* c_{1j}, c_{2j}, \dots, c_{mj}) d_j \right) b = \langle x, y \rangle b \end{aligned}$$

Let $N = \{x \in A^{\otimes m} \otimes B; \langle x, y \rangle = 0\}$. From the Cauchy-Schwarz inequality, we observe that N is a subspace of $A^{\otimes m} \otimes B$. From $\langle x, y \cdot b \rangle = \langle x, y \rangle b$, we have $N \cdot B \subset N$, so N is a B -submodule of $A^{\otimes m} \otimes B$. The induced sesquilinear form on $A^{\otimes m} \otimes B/N$ defined by $\langle x + N, y + N \rangle = \langle x, y \rangle$ will be a B -valued inner product. We denote by F the Hilbert B -module which is the completion of the inner product space $(A^{\otimes m} \otimes B)/N$.

For all $a \in A$ and $p = 1, \dots, m$, we define a linear map $\pi_p(a): A^{\otimes m} \otimes B \rightarrow A^{\otimes m} \otimes B$ by

$$\pi_p(a) \left(\sum_{i=1}^n a_{1i} \otimes \dots \otimes a_{mi} \otimes b_i \right) = \sum_{i=1}^n a_{1i} \otimes \dots \otimes a a_{pi} \otimes \dots \otimes a_{mi} \otimes b_i$$

It follows from the definition that each $\pi_p(a)$ is a B -module map.

If $x = a_1 \otimes \dots \otimes a_m \otimes b \in A^{\otimes m} \otimes B$, then we have

$$\begin{aligned} \langle \pi_p(a), \pi_p(a) \rangle &= \langle a_1 \otimes \dots \otimes a a_p \otimes \dots \otimes a_m \otimes b, a_1 \otimes \dots \otimes a a_p \otimes \dots \otimes a_m \otimes b \rangle = \\ &= b^* \phi(a_m^*, \dots, a_p^* a^*, \dots, a_2^*, a_1^* a_1, a_2, \dots, a a_p, \dots, a_m) b \leq \\ &= \|a^* a\| b^* \phi(a_m^*, \dots, a_p^*, \dots, a_2^*, a_1^* a_1, a_2, \dots, a_p, \dots, a_m) b = \|a\|^2 \langle x, x \rangle. \end{aligned}$$

Hence, each $\pi_p(a), \forall p = 1, \dots, m$ leaves N invariant and defines a linear transformation on $(A^{\otimes m} \otimes B)/N$, that we denote also by $\pi_p(a)$. The previous inequality shows that $\pi_p(a)$ is bounded with $\|\pi_p(a)\| \leq \|a\|$, so $\pi_p(a)$ extends to a bounded linear operator on F , still denoted by $\pi_p(a)$.

By the invariance of ϕ , we obtain

$$\begin{aligned} \langle \pi_p(a) x, y \rangle &= \left\langle \sum_{i=1}^n a_{1i} \otimes \dots \otimes a a_{pi} \otimes \dots \otimes a_{mi} \otimes b_i, \sum_{j=1}^l c_{1j} \otimes \dots \otimes c_{mj} \otimes d_j \right\rangle = \\ &= \sum_{i=1}^n \sum_{j=1}^l b_i^* \phi(a_{mi}^*, \dots, a_{pi}^* a^*, \dots, a_{2i}^*, a_{1i}^* c_{1j}, c_{2j}, \dots, c_{mj}) d_j = \end{aligned}$$

$$\sum_{i=1}^n \sum_{j=1}^l b_i^* \phi(a_{mi}^*, \dots, a_{2i}^*, a_{1i}^*, c_{1j}, c_{2j}, \dots, a^* c_{pj}, \dots, c_{mj}) d_j =$$

$$\left\langle \sum_{i=1}^n a_{1i} \otimes \dots \otimes a_{mi} \otimes b_i, \sum_{j=1}^l c_{1j} \otimes \dots \otimes a^* c_{pj} \otimes \dots \otimes c_{mj} \otimes d_j \right\rangle = \langle x, \pi_p(a^*)y \rangle,$$

and it follows that $\pi_p(a) \in \mathcal{L}_B(F)$ for all $a \in A$. Thus, π_p is a representation of A on F for all $p = 1, \dots, m$.

The case $k = 2m$ even. We define the sesquilinear form $\langle \cdot, \cdot \rangle$ on the algebraic tensor product $A^{\otimes m} \otimes B$ by

$$\left\langle \sum_{i=1}^n a_{1i} \otimes \dots \otimes a_{mi} \otimes b_i, \sum_{j=1}^l c_{1j} \otimes \dots \otimes c_{mj} \otimes d_j \right\rangle =$$

$$\sum_{i=1}^n \sum_{j=1}^l b_i^* \phi(a_{mi}^*, \dots, a_{1i}^*, c_{1j}, \dots, c_{mj}) d_j,$$

for all $a_{1i}, \dots, a_{mi}, c_{1j}, \dots, c_{mj} \in A$ și $b_i, d_j \in B$.

For all $x = \sum_{i=1}^n a_{1i} \otimes \dots \otimes a_{mi} \otimes b_i$ in $A^{\otimes m} \otimes B$, we have

$$\langle x, x \rangle = \left\langle \sum_{i=1}^n a_{1i} \otimes \dots \otimes a_{mi} \otimes b_i, \sum_{j=1}^l a_{1j} \otimes \dots \otimes a_{mj} \otimes b_j \right\rangle =$$

$$\sum_{i,j=1}^n b_i^* \phi(a_{mi}^*, \dots, a_{1i}^*, a_{1j}, \dots, a_{mj}) b_j =$$

$$[b_1^* \dots b_n^*][\phi(a_{mi}^*, \dots, a_{1i}^*, a_{1j}, \dots, a_{mj})][b_1 \dots b_n]^t \geq 0,$$

where $[\phi(a_{mi}^*, \dots, a_{2i}^*, a_{1i}^*, a_{1j}, a_{2j}, \dots, a_{mj})]$ is an $n \times n$ matrix in $M_n(B)$, since

$$[\phi(a_{mi}^*, \dots, a_{2i}^*, a_{1i}^*, a_{1j}, a_{2j}, \dots, a_{mj})]_{i,j=1}^n = \phi_n(A_m^*, \dots, A_2^*, A_1^*, A_1, A_2, \dots, A_m) \geq 0$$

where A_1 and A_p are as in the odd case. The construction of the representation π_p is analogue to that in the odd case.

Now it is sufficient to construct a family (π_p, v, F) , $p = \overline{1, m}$ of projective covariant representations of (G, A, α) on F , where v is a projective unitary representation of G on F with the multiplier ω and $V \in \mathcal{L}_B(B, F)$ satisfying (1) and (2).

For all $g \in G$ și $x = \sum_{i=1}^n a_{1i} \otimes \dots \otimes a_{mi} \otimes b_i$, we define a linear map $v: G \rightarrow \mathcal{L}_B(F)$ by

$$v_g(x) = \sum_{i=1}^n \alpha_g(a_{1i}) \otimes \dots \otimes \alpha_g(a_{mi}) \otimes u_g b_i.$$

It is clear that v_g is bounded. For all $x = a_1 \otimes \dots \otimes a_m \otimes b$ and $y = c_1 \otimes \dots \otimes c_m \otimes d$ in F , we have, using the u -covariance of ϕ ,

$$\langle v_g(x), y \rangle = \langle \alpha_g(a_1) \otimes \dots \otimes \alpha_g(a_m) \otimes u_g b, c_1 \otimes \dots \otimes c_m \otimes d \rangle$$

$$\begin{aligned}
&= \begin{cases} (u_g b)^* \phi(\alpha_g(a_m)^*, \dots, \alpha_g(a_2)^*, \alpha_g(a_1)^* c_1, c_2, \dots, c_m) d, & k \text{ odd} \\ (u_g b)^* \phi(\alpha_g(a_m)^*, \dots, \alpha_g(a_1)^*, c_1, \dots, c_m) d, & k \text{ even} \end{cases} \\
&= \begin{cases} b^* \phi(a_m^*, \dots, a_2^*, a_1^* \alpha_{g-1}(c_1), \alpha_{g-1}(c_2), \dots, \alpha_{g-1}(c_m)) u_g^* d, & k \text{ odd} \\ b^* \phi(a_m^*, \dots, a_1^*, \alpha_{g-1}(c_1), \dots, \alpha_{g-1}(c_m)) u_g^* d, & k \text{ even} \end{cases} \\
&= \langle a_1 \otimes \dots \otimes a_m \otimes b, \alpha_{g-1}(c_1) \otimes \dots \otimes \alpha_{g-1}(c_m) \otimes u_{g-1} d \rangle = \langle x, v_{g-1}(y) \rangle
\end{aligned}$$

Therefore, v_g admits v_{g-1} as adjoint and it is straightforward that the map $g \mapsto v$ is a unitary representation on F .

We prove that v is a projective representation with the multiplier ω . Let $x = a_1 \otimes \dots \otimes a_m \otimes b$ and $y = c_1 \otimes \dots \otimes c_m \otimes d$ in F . We have

$$\begin{aligned}
\langle v_e(x), y \rangle &= \langle \alpha_e(a_1) \otimes \dots \otimes \alpha_e(a_m) \otimes u_e b, c_1 \otimes \dots \otimes c_m \otimes d \rangle \\
&= \begin{cases} (u_e b)^* \phi(\alpha_e(a_m)^*, \dots, \alpha_e(a_2)^*, \alpha_e(a_1)^* c_1, c_2, \dots, c_m) d, & k \text{ odd} \\ (u_e b)^* \phi(\alpha_e(a_m)^*, \dots, \alpha_e(a_1)^*, c_1, \dots, c_m) d, & k \text{ even} \end{cases} \\
&= \begin{cases} b^* \phi(a_m^*, \dots, a_2^*, a_1^* c_1, c_2, \dots, c_m) d, & k \text{ odd} \\ b^* \phi(a_m^*, \dots, a_1^*, c_1, \dots, c_m) d, & k \text{ even} \end{cases} \\
&= \langle a_1 \otimes \dots \otimes a_m \otimes b, c_1 \otimes \dots \otimes c_m \otimes d \rangle = \langle x, y \rangle,
\end{aligned}$$

and follows that $v_e = I_F$.

Let $g_1, g_2 \in G$. Taking into account that u is a projective unitary representation with the normalized multiplier ω (see pg. 216, [7]) and $\langle \cdot, \cdot \rangle$ is conjugate-linear in the first variable, we have

$$\begin{aligned}
&\langle \omega(g_1, g_2) v_{g_1} v_{g_2}(x), y \rangle \\
&= \overline{\omega(g_1, g_2)} \langle v_{g_1}(\alpha_{g_2}(a_1) \otimes \dots \otimes \alpha_{g_2}(a_m) \otimes u_{g_2} b), c_1 \otimes \dots \otimes c_m \otimes d \rangle \\
&= \overline{\omega(g_1, g_2)} \langle \alpha_{g_1}(\alpha_{g_2}(a_1)) \otimes \dots \otimes \alpha_{g_1}(\alpha_{g_2}(a_m)) \otimes u_{g_1} u_{g_2} b, c_1 \otimes \dots \otimes c_m \otimes d \rangle \\
&= \overline{\omega(g_1, g_2)} \langle \alpha_{g_1 g_2}(a_1) \otimes \dots \otimes \alpha_{g_1 g_2}(a_m) \otimes u_{g_1} u_{g_2} b, c_1 \otimes \dots \otimes c_m \otimes d \rangle \\
&= \begin{cases} \overline{\omega(g_1, g_2)} (u_{g_1} u_{g_2} b)^* \phi(\alpha_{g_1 g_2}(a_m)^*, \dots, \alpha_{g_1 g_2}(a_2)^*, \alpha_{g_1 g_2}(a_1)^* c_1, c_2, \dots, c_m) d, & k \text{ odd} \\ \overline{\omega(g_1, g_2)} (u_{g_1} u_{g_2} b)^* \phi(\alpha_{g_1 g_2}(a_m)^*, \dots, \alpha_{g_1 g_2}(a_1)^*, c_1, \dots, c_m) d, & k \text{ even} \end{cases} \\
&= \begin{cases} \overline{\omega(g_1, g_2)} b^* u_{g_2}^{-1} u_{g_1}^{-1} \phi(\alpha_{g_1 g_2}(a_m)^*, \dots, \alpha_{g_1 g_2}(a_2)^*, \alpha_{g_1 g_2}(a_1)^* c_1, c_2, \dots, c_m) d, & k \text{ odd} \\ \overline{\omega(g_1, g_2)} b^* u_{g_2}^{-1} u_{g_1}^{-1} \phi(\alpha_{g_1 g_2}(a_m)^*, \dots, \alpha_{g_1 g_2}(a_1)^*, c_1, \dots, c_m) d, & k \text{ even} \end{cases} \\
&= \begin{cases} \overline{\omega(g_1, g_2) \omega(g_2^{-1}, g_1^{-1})} b^* u_{(g_1 g_2)^{-1}} \phi(\alpha_{g_1 g_2}(a_m)^*, \dots, \\ \alpha_{g_1 g_2}(a_2)^*, \alpha_{g_1 g_2}(a_1)^* c_1, c_2, \dots, c_m) d, & k \text{ odd} \\ \overline{\omega(g_1, g_2) \omega(g_2^{-1}, g_1^{-1})} b^* u_{(g_1 g_2)^{-1}} \phi(\alpha_{g_1 g_2}(a_m)^*, \dots, \\ \alpha_{g_1 g_2}(a_1)^*, c_1, \dots, c_m) d, & k \text{ even} \end{cases} \\
&= \begin{cases} (u_{g_1 g_2} b)^* \phi(\alpha_{g_1 g_2}(a_m)^*, \dots, \alpha_{g_1 g_2}(a_2)^*, \\ \alpha_{g_1 g_2}(a_1)^* c_1, c_2, \dots, c_m) d, & k \text{ odd} \\ (u_{g_1 g_2} b)^* \phi(\alpha_{g_1 g_2}(a_m)^*, \dots, \\ \alpha_{g_1 g_2}(a_1)^*, c_1, \dots, c_m) d, & k \text{ even} \end{cases} \\
&= \langle \alpha_{g_1 g_2}(a_1) \otimes \dots \otimes \alpha_{g_1 g_2}(a_m) \otimes u_{g_1 g_2} b, c_1 \otimes \dots \otimes c_m \otimes d \rangle = \langle v_{g_1 g_2}(x), y \rangle,
\end{aligned}$$

so $\omega(g_1, g_2) v_{g_1} v_{g_2} = v_{g_1 g_2}$.

For all $x = \sum_{i=1}^n a_{1i} \otimes \dots \otimes a_{mi} \otimes b_i$ și $a_i \in A$, we have

$$v_g \pi_1(a_1) \dots \pi_m(a_m) v_g^*(x) = v_g \pi_1(a_1) \dots \pi_m(a_m) v_{g-1}(x) =$$

$$\begin{aligned}
& v_g \pi_1(a_1) \dots \pi_m(a_m) \left(\sum_{i=1}^n \alpha_{g^{-1}}(a_{1i}) \otimes \dots \otimes \alpha_{g^{-1}}(a_{mi}) \otimes u_{g^{-1}} b_i \right) = \\
& v_g \left(\sum_{i=1}^n a_1 \alpha_{g^{-1}}(a_{1i}) \otimes \dots \otimes a_m \alpha_{g^{-1}}(a_{mi}) \otimes u_{g^{-1}} b_i \right) = \sum_{i=1}^n \alpha_g(a_1) a_{1i} \otimes \dots \otimes \alpha_g(a_m) a_{mi} \otimes b_i = \\
& \pi_1(\alpha_g(a_1)) \dots \pi_m(\alpha_g(a_m)) \left(\sum_{i=1}^n a_{1i} \otimes \dots \otimes a_{mi} \otimes b_i \right) = \pi_1(\alpha_g(a_1)) \dots \pi_m(\alpha_g(a_m))(x)
\end{aligned}$$

Hence, $v_g \pi_p(a) v_g^* = \pi_p(\alpha_g(a))$ for all $a \in A$ and $p = 1, \dots, m$. Thus, $(\pi_p, v, F), p = \overline{1, m}$ is a family of projective covariant representations of (G, A, α) .

We define $V: B \rightarrow F$ by $V(b) = 1_A \otimes \dots \otimes 1_A \otimes b$ for all $b \in B$. Clearly, V is a B -module map (considering the C^* -algebra B as a Hilbert B -module with the inner-product $\langle a, b \rangle_B = a^* b, a, b \in B$).

For all $b \in B$ and $\sum_{i=1}^n a_{1i} \otimes \dots \otimes a_{mi} \otimes b_i \in F$, we have

$$\begin{aligned}
\left\langle V(b), \sum_{i=1}^n a_{1i} \otimes \dots \otimes a_{mi} \otimes b_i \right\rangle_F &= \left\langle 1_A \otimes \dots \otimes 1_A \otimes b, \sum_{i=1}^n a_{1i} \otimes \dots \otimes a_{mi} \otimes b_i \right\rangle_F = \\
& \sum_{i=1}^n b^* \phi(1_A, \dots, 1_A, a_{1i}, \dots, a_{mi}) b_i = \left\langle b, \sum_{i=1}^n \phi(1_A, \dots, 1_A, a_{1i}, \dots, a_{mi}) b_i \right\rangle_B
\end{aligned}$$

$$\text{Therefore, } V^* \left(\sum_{i=1}^n a_{1i} \otimes \dots \otimes a_{mi} \otimes b_i \right) = \sum_{i=1}^n \phi(1_A, \dots, 1_A, a_{1i}, \dots, a_{mi}) b_i.$$

For all $a_i \in A$ and $b \in B$,

$$\begin{aligned}
V^* \pi_1(a_1) \dots \pi_m(a_m) V(b) &= V^* \pi_1(a_1) \dots \pi_m(a_m) (1_A \otimes \dots \otimes 1_A \otimes b) = \\
& V^* (a_1 \otimes \dots \otimes a_m \otimes b) = \phi(1_A, \dots, 1_A, a_1, \dots, a_m) b
\end{aligned}$$

Hence, $V^* \pi_1(a_1) \dots \pi_m(a_m) V = m_{\phi(1_A, \dots, 1_A, a_1, \dots, a_m)}$.

For all $b \in B$, we have

$$V m_{u_g}(b) = V(u_g b) = 1_A \otimes \dots \otimes 1_A \otimes u_g b = v_g(1_A \otimes \dots \otimes 1_A \otimes b) = v_g V(b),$$

and it results that $V m_{u_g} = v_g V$ for all $g \in G$. \square

3 The extension of a completely positive and completely bounded symmetric invariant projective u -covariant multi-linear map on the twisted crossed product $(A \times_{\alpha}^{\omega} G)^k$

Theorem 3.1. *Let (G, A, α) be a unital C^* -dynamical system, let B be a unital C^* -algebra and let u be a projective unitary representation $u: G \rightarrow \mathcal{U}(B)$ with the normalized multiplier ω . If $\phi \in ICB_s^+(A^k, B)$ ($k \geq 2$) is a projective u -covariant k -linear map, then there is a completely positive k -linear map $\psi: (A \times_{\alpha}^{\omega} G)^k \rightarrow B$ given by*

$$\begin{aligned}
& \psi(f_1, \dots, f_k) \\
&= \int_{G^k} \phi(f_1(s_1), \alpha_{s_1}(f_2(s_2)), \dots, \alpha_{s_1 \dots s_{k-1}}(f_k(s_k))) u_{s_1} \dots u_{s_k} d\mu(s_1) \dots d\mu(s_k),
\end{aligned}$$

for all $f_1, \dots, f_k \in L^1(G, A)$.

Proof. From Theorem 2.1, there are a family of projective covariant representations $(\pi_p, v, F), p = \overline{1, m}$ of (G, A, α) and $V \in \mathcal{L}_B(B, F)$ such that $Vm_{u_g} = v_g V$ for all $g \in G$, where m is the left multiplication operator and $V^* \pi_1(a_1) \dots \pi_m(a_m) V = m_{\phi(1_A, \dots, 1_A, a_1, \dots, a_m)}$, where the number of 1_A 's is $m - 1$ in odd case and m in even case.

We denote $\pi_p \times v, 1 \leq p \leq m$ by

$$(\pi_p \times v)(f) = \int_G \pi_p(f(s)) v_s d\mu(s),$$

for all $f \in L^1(G, A)$. By Theorem 3.3, [2], $\pi_p \times v$ extends to a representation on the twisted crossed product $A \times_\alpha^\omega G$ in $\mathcal{L}_B(F)$, denoted also by $\pi_p \times v$.

We define a k -linear map $\varphi: (A \times_\alpha^\omega G)^k \rightarrow \mathcal{L}_B(B)$ by

$$\varphi(f_1, \dots, f_k) = \begin{cases} V^*(\pi_m \times v)(f_1) \dots (\pi_2 \times v)(f_{m-1})(\pi_1 \times v)(f_m)(\pi_2 \times v)(f_{m+1}) \dots \\ \quad (\pi_m \times v)(f_k)V, & k \text{ odd} \\ V^*(\pi_m \times v)(f_1) \dots (\pi_1 \times v)(f_m)(\pi_1 \times v)(f_{m+1}) \dots \\ \quad (\pi_m \times v)(f_k)V, & k \text{ even} \end{cases}$$

(i) k even. We consider the case $k = 2$, because the general case is similar.

For all $f_1, f_2 \in L^1(G, A)$ and $b \in B$, by the projective covariance of the representations $(\pi_i, v, F), i = 1, 2$ and by Theorem 2.1, the conditions (1) and (2), we have:

$$\begin{aligned} \varphi(f_1, f_2)(b) &= [V^*(\pi_1 \times v)(f_1)(\pi_2 \times v)(f_2)V](b) = \\ &= \int_G \int_G V^* \pi_1(f_1(s_1)) v_{s_1} \pi_2(f_2(s_2)) v_{s_2} V(b) d\mu(s_1) d\mu(s_2) = \\ &= \int_G \int_G V^* \pi_1(f_1(s_1)) \pi_2(\alpha_{s_1}(f_2(s_2))) v_{s_1} v_{s_2} V(b) d\mu(s_1) d\mu(s_2) = \\ &= \int_G \int_G V^* \pi_1(f_1(s_1)) \pi_2(\alpha_{s_1}(f_2(s_2))) \overline{\omega(s_1, s_2)} v_{s_1 s_2} V(b) d\mu(s_1) d\mu(s_2) = \\ &= \int_G \int_G \overline{\omega(s_1, s_2)} V^* \pi_1(f_1(s_1)) \pi_2(\alpha_{s_1}(f_2(s_2))) V(b) u_{s_1 s_2} d\mu(s_1) d\mu(s_2) = \\ &= \int_G \int_G \overline{\omega(s_1, s_2)} m_{\phi(1_A, 1_A, f_1(s_1), \alpha_{s_1}(f_2(s_2)))}(b) u_{s_1 s_2} d\mu(s_1) d\mu(s_2) = \\ &= \int_G \int_G \overline{\omega(s_1, s_2)} \phi(f_1(s_1), \alpha_{s_1}(f_2(s_2))) u_{s_1 s_2} b d\mu(s_1) d\mu(s_2) = \\ &= \int_G \int_G \phi(f_1(s_1), \alpha_{s_1}(f_2(s_2))) u_{s_1} u_{s_2} b d\mu(s_1) d\mu(s_2) \end{aligned}$$

(ii) k odd. We consider the case $k = 3$, because we can pass inductively to the general case like in the odd case.

For all $f_1, f_2, f_3 \in L^1(G, A)$ and $b \in B$, by the projective covariance of the representations $(\pi_i, v, F), i = \overline{1, 3}$ and by Theorem 2.1, the conditions (1) and (2), we

have:

$$\begin{aligned}
\varphi(f_1, f_2, f_3)(b) &= [V^*(\pi_3 \times v)(f_1)(\pi_2 \times v)(f_2)(\pi_1 \times v)(f_3)V](b) = \\
&\int_{G^3} V^* \pi_3(f_1(s_1))v_{s_1} \pi_2(f_2(s_2))v_{s_2} \pi_1(f_3(s_3))v_{s_3} V(b) d\mu(s_1)d\mu(s_2)d\mu(s_3) = \\
&\int_{G^3} V^* \pi_3(f_1(s_1))\pi_2(\alpha_{s_1}(f_2(s_2)))v_{s_1} v_{s_2} \pi_1(f_3(s_3))v_{s_3} V(b) d\mu(s_1)d\mu(s_2)d\mu(s_3) = \\
&\int_{G^3} \overline{\omega(s_1, s_2)} V^* \pi_3(f_1(s_1))\pi_2(\alpha_{s_1}(f_2(s_2)))v_{s_1 s_2} \pi_1(f_3(s_3))v_{s_3} V(b) d\mu(s_1)d\mu(s_2)d\mu(s_3) = \\
&\int_{G^3} \overline{\omega(s_1, s_2)} V^* \pi_3(f_1(s_1))\pi_2(\alpha_{s_1}(f_2(s_2)))\pi_1(\alpha_{s_1 s_2}(f_3(s_3)))v_{s_1 s_2} v_{s_3} \cdot \\
&\qquad\qquad\qquad V(b) d\mu(s_1)d\mu(s_2)d\mu(s_3) = \\
&\int_{G^3} \overline{\omega(s_1, s_2)\omega(s_1 s_2, s_3)} V^* \pi_3(f_1(s_1))\pi_2(\alpha_{s_1}(f_2(s_2)))\pi_1(\alpha_{s_1 s_2}(f_3(s_3)))v_{s_1 s_2 s_3} \cdot \\
&\qquad\qquad\qquad V(b) d\mu(s_1)d\mu(s_2)d\mu(s_3) = \\
&\int_{G^3} \overline{\omega(s_1, s_2)\omega(s_1 s_2, s_3)} V^* \pi_3(f_1(s_1))\pi_2(\alpha_{s_1}(f_2(s_2)))\pi_1(\alpha_{s_1 s_2}(f_3(s_3))) \cdot \\
&\qquad\qquad\qquad V u_{s_1 s_2 s_3} b d\mu(s_1)d\mu(s_2)d\mu(s_3) = \\
&\int_{G^3} \overline{\omega(s_1, s_2)\omega(s_1 s_2, s_3)} \phi(f_1(s_1), \alpha_{s_1}(f_2(s_2)), \alpha_{s_1 s_2}(f_3(s_3)))u_{(s_1 s_2) s_3} b d\mu(s_1)d\mu(s_2)d\mu(s_3) = \\
&\int_{G^3} \overline{\omega(s_1, s_2)} \phi(f_1(s_1), \alpha_{s_1}(f_2(s_2)), \alpha_{s_1 s_2}(f_3(s_3)))u_{s_1 s_2} u_{s_3} b d\mu(s_1)d\mu(s_2)d\mu(s_3) = \\
&\int_{G^3} \phi(f_1(s_1), \alpha_{s_1}(f_2(s_2)), \alpha_{s_1 s_2}(f_3(s_3)))u_{s_1} u_{s_2} u_{s_3} b d\mu(s_1)d\mu(s_2)d\mu(s_3).
\end{aligned}$$

Inductively we obtain

$$\begin{aligned}
\varphi(f_1, \dots, f_k)(b) &= \\
&\int_{G^k} \phi(f_1(s_1), \alpha_{s_1}(f_2(s_2)), \dots, \alpha_{s_1 \dots s_{k-1}}(f_k(s_k)))u_{s_1} \dots u_{s_k} b d\mu(s_1) \dots d\mu(s_k),
\end{aligned}$$

for all $f_1, \dots, f_k \in L^1(G, A)$ și $b \in B$.

Since $\pi_p \times v$ is a representation from $A \times_{\alpha}^{\omega} G$ into $\mathcal{L}_B(F)$, for all $p = 1, \dots, m$, φ is a completely positive linear map from $A \times_{\alpha}^{\omega} G$ into $\mathcal{L}_B(B)$. Let $\tau: \mathcal{L}_B(B) \rightarrow B$ be a natural isomorphism. Taking $\psi = \tau \circ \varphi$ we obtain the required completely positive k -linear map ψ . \square

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Author's address:

Tania-Luminița Costache
Faculty of Applied Sciences, University "Politehnica" of Bucharest,
Splaiul Independenței 313, Bucharest, Romania.
E-mail: lumycos1@yahoo.com