A characterization of H-strictly convex hypersurfaces in hyperbolic space by the Ricci curvatures

Mehmet Erdoğan, Jeta Alo

Dedicated to the 70-th anniversary of Professor Constantin Udriste

Abstract. In previous papers the first author obtained some upper bound estimations for the Ricci curvatures of the hypersurfaces in a sphere ([6]) and in a hyperbolic manifold ([4], [5]) by the extremum principle. In the present paper, we introduce an H-strictly convex hypersurface in the hyperbolic space $N^{(n+1)}$ and using a result given by B.Y.Chen in [2] we give a lower bound approximation for the Ricci curvature of a H-strictly convex hypersurface in $N^{(n+1)}$.

M.S.C. 2000: 53C21, 53C42.
Key words: Ricci curvature; H-convex hypersurface; hyperbolic space.

1 Introduction

Let M be an n-dimensional oriented hypersurface in an (n+1)-dimensional Riemannian manifold N, that is, M is an n-dimensional Riemannian submanifold embedded in N with the induced Riemannian structure. We will denote by g resp ($\tilde{g}$) the Riemannian metric tensor of M (resp. N). Let $T_xM^{\perp}$ be the normal space to M at x. We denote by $\triangledown$ (resp.$\tilde{\triangledown}$) the covariant differentiation on M (resp. N). Then, for tangent vector fields X, Y and the unit normal field ζ on M, as is well known, the formulas of Gauss and Weingarten are

$$\tilde{\triangledown}_X Y = \triangledown_X Y + \sigma(X, Y) \tag{1.1}$$

$$\tilde{\triangledown}_X \zeta = -A_\zeta(X) \tag{1.2}$$

where σ is the second fundamental form of M and satisfies $\sigma(X, Y) = \sigma(Y.X)$ and A_ζ is the symmetric linear transformation on each tangent space to M, which is called the shape operator. Since M is a hypersurface we may write

$$\sigma(X, Y) = h(X, Y)\zeta. \tag{1.3}$$

BSG Proceedings 17. The International Conference "Differential Geometry-Dynamical Systems 2009" (DGDS-2009), October 8-11, 2009, Bucharest-Romania, pp. 99-103.

Then we easily see that

$$h(X,Y) = \tilde{g}(\sigma(X,Y),\zeta) = g(A_\zeta(X),Y) \tag{1.4}$$

The eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ of the shape operator A_ζ are called *principal curvatures* of M, and the vectors of an orthonormal basis $\{e_1, e_2, ..., e_n\}$ such that

$$A_\zeta e_i = \lambda_i e_i$$

are called *principal vectors* on M. In this case,

$$\lambda_i = h(e_i, e_i), i = 1, 2, ...n,$$

Furthermore, the mean curvature vector of the hypersurface M is defined by $H = \frac{1}{n} trace\sigma$ and $K_n = \lambda_1\lambda_2...\lambda_n$ is called the *Gaussian curvature* of M.

The second fundamental form σ is said to be semidefinite at $x \in M$ if $\sigma(X,X) \geq 0$ or $\sigma(X,X) \leq 0$ for all nonzero vectors $X \in T_xM$, that is, h is either positive semidefinite or negative semidefinite. It is well known that if M is convex at $x \in M$, then the h is semidefinite at the point x. The second fundamental form σ is said to be definite at $x \in M$ if $\sigma(X,X) \neq 0$ for all nonzero vectors $X \in T_xM$, that is, h is either positive definite or negative definite. In this case the hypersurface M is said to be strictly convex at the point x. σ is said to be non-degenerate at x if h is non-degenerate at x. B.Y. Chen defined a convex submanifold in a Riemannian space form in [2], also see [7], [8] and [6].

Definition 1.1. *A Riemannian submanifold is said to be H-Strictly convex submanifold if the shape operator A_H is positive definite at each point of the submanifold.*

Denote by R the Riemannian curvature tensor of M. Then the equation of Gauss is given by

$$\begin{aligned} R(X,Y;Z,W) &= c(g(X,W)g(Y,Z) - g(X,z)g(Y,W)) \\ &+ g(\sigma(X,W),\sigma(Y,Z)) - g(\sigma(X,Z),\sigma(Y,W)). \end{aligned} \tag{1.5}$$

for vector fields X,Y,Z,W tangent to M, where c is the constant sectional curvature of the Riemannian space form N. For the hypersurface M, denote by K(π) the sectional curvature of a 2-plane section $\pi \subset T_xM$, $x \in M$ and choose an orthonormal basis $e_1, e_2, ...e_n$ of T_xM such that $e_1 = X$, then we may define the *Ricci curvature* of T_xM at x by

$$Ric(X) = \sum_{j=2}^{n} K_{1j} \tag{1.6}$$

where K_{ij} denotes the sectional curvature of the 2-plane section spanned by e_i,e_j. The *scalar curvature* τ of the hypersurface M is defined by

$$\tau(M) = \sum_{1 \leq i \leq j \leq n} K_{ij}. \tag{1.7}$$

2 Ricci curvature of the H-convex hypersurface in hyperbolic space

Now, following [2] by B.Y.Chen, we may introduce a Riemannian invariant on the hypersurface M of a (n+1)-dimensional hyperbolic space N of constant sectional curvature -1 defined by

$$\theta_n(x) = \frac{1}{n-1} inf Ric(X), (X \in T_pM, \|X\| = 1, x \in M). \tag{2.1}$$

We will prove the following theorem for the hypersurfaces in a hyperbolic space

Theorem 2.1. *Let M be a hypersurface of a (n+1)- dimensional hyperbolic space N , then for any point $x \in N$, we have*

1. *If $\theta_n(x) \neq -1$, then the shape operator at the mean curvature vector satisfies*

$$A_H > \frac{n-1}{n}(\theta_n(x) + 1)I, x \in M. \tag{2.2}$$

2. *If $\theta_n(x) = -1$, then $A_H \geq 0$ at x.*

Proof. Let$\{e_1, e_2, ..., e_n\}$ be an orthonormal basis of T_xM . Considering (1.6), (1.7) and (2.1) we have

$$\tau(x) \geq \frac{n(n-1)}{2}\theta_n(x). \tag{2.3}$$

Then by using the Lemma 1 in [1] we get

$$H^2(x) \geq \frac{2}{n(n-1)}\tau(x) + 1. \tag{2.4}$$

Now, from (2.3) and (2.4) we obtain $H^2(x) \geq \theta_n(x) + 1$. This shows that only when $\theta_n(x) \leq -1$,$H(x) = 0$ and in this case 1) and 2) is clearly satisfied, so we may assume that $H(x) \neq 0$. Choose an orthonormal basis $\{e_1, e_2, ..., e_n, e_{n+1}\}$ at x such that e_{n+1} is in the direction of the mean curvature vector $H(x)$ and $\{e_1, e_2, ..., e_n\}$ diagonalize the shape operator A_H. Then we have

$$A_H = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix}. \tag{2.5}$$

From the equation of Gauss we get

$$a_ia_j = K_{ij} + 1 \tag{2.6}$$

and from (2.6)we obtain

$$a_1(a_2 + ... + a_n) = Ric(e_1) + (n-1). \tag{2.7}$$

if we take $e_1 = X$ in (2.6), taking into account (2.1) we may write

$$a_1(a_2 + ... + a_n) \geq (n-1)(\theta_n(x) + 1) + a_1^2. \tag{2.8}$$

In a similar way we get the following equalities for any $j = 1, 2, ..., n$,

$$a_j(a_1 + ... + a_{j-1} + a_{j+1} + ... + a_n) \geq (n-1)(\theta_n(x) + 1) + a_j^2, \tag{2.9}$$

which yields

$$A_H \geq \frac{n-1}{n}(\theta_n(x) + 1)I. \tag{2.10}$$

Here the equality case only occurs when one of the vectors $e_1, e_2, .., e_n$ is in the null space, but for the hypersurfaces this is impossible so the inequality (2.10) must be sharp, that is

$$A_H > \frac{n-1}{n}(\theta_n(x) + 1)I. \tag{2.11}$$

From Theorem 2.1. we get the following

Corollary 2.1. *Let M be a hypersurface of an $(n+1)$- dimensional hyperbolic space N of constant sectional curvature -1, if the Ricci curvature of M is positive then M is a H-strictly convex hypersurface immersed in N.*

In the previous paper, [3], the first author proved the following theorem.

Theorem 2.2. *Let M be a complete hypersurface in the hyperbolic space N^{n+1}, such that all sectional curvatures of M are bounded away from $-\infty$. If the largest geodesic ball which is contained in M has the radius ρ, then we have*

$$\lim inf_{X \in T_xM, \|X\|=1, x \in M} Ric(X, X) \leq (1-n)(1 - \frac{1}{2sinh(\frac{\rho}{2})} - sinh(\frac{\rho}{2})), \tag{2.12}$$

where $\rho = 2arcsin\frac{ht}{2}$.

Now, we would like to calculate a sharp result for the best possible approximation of the minimum Ricci curvature of a H-strictly convex hypersurface immersed in N^{n+1}. According to the Corollary 2.1 for such an hypersurface the left side of the inequality (2.12) must be positive and so by setting $sinh(\frac{\rho}{2}) = \frac{t}{2} = y$, we get $(n-1)(y + \frac{1}{2y} - 1) \geq 0$ and for $n \geq 2$ we must solve the inequality $y + \frac{1}{2y} - 1 \geq 0$. Let us write $y + \frac{1}{2y} = a$, for being satisfied of the inequality, we have $a \geq 1$ and to get the least value of a, if we solve the equation $y + \frac{1}{2y} = a$ or $y^2 - ay + \frac{1}{2} = 0$, $y \neq 0$, we find $a \geq \sqrt{2}$ and therefore $a = y + \frac{1}{2y} = \sqrt{2}$ is the minimum value. This proves the following

Theorem 2.3. *Let M be a H-strictly convex hypersurface with constant sectional curvatures of the hyperbolic space N^{n+1}. Then for any point $x \in M$ the best possible approximation for the minimum Ricci curvature of M is $(\sqrt{2} - 1)(n-1)$.*

References

[1] B. Y. Chen, *Mean curvature and shape operator of isometric immersions in real valued-space forms*, Glasgow Math. J. 38 (1996), 87-97.

[2] B. Y. Chen, *Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimensions*, Glasgow Math. J. 41 (1999), 33-41.

[3] M. Erdoğan, *A lower bound estimation for the Ricci curvature of a hypersurface in a space form*, Stud. Sci. Math. Hungarica 46 (4) (2009), 539-546.

[4] M. Erdoğan, *On the Ricci curvature of a hypersurface in a space form*, Geometriae Dedicata 61 (1996), 221-225.

[5] M. Erdoğan, *On the Ricci curvature of a hypersurface in a sphere*, Math. Scand. 82 (1998), 186-190.

[6] S. Shichang, A.Y. Han, *Hypersurfaces with constant scalar or mean curvature in a unit sphere*, Balkan J. Geom. Appl. 14, 2 (2009), 90-100.

[7] C. Udrişte, *Convex hypersurfaces*, Analele Şt. Univ. A1. I. Cuza, Iaşi, 32 (1986), 85-87.

[8] C. Udrişte, T. Oprea, *H-convex Riemannian submanifolds*, Balkan J. Geom. Appl., 13 (2) (2008), 112-119.

Authors' address:

Mehmet Erdoğan and Jeta Alo
Department of Mathematics and Computing,
Faculty of Science and Letters, Beykent University,
Ayazağa, Istanbul, Turkey.
E-mail: erdoganm@beykent.edu.tr, jeta@beykent.edu.tr