Nullity conditions on curvature of 3-$f-$manifolds

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Dedicated to the 70-th anniversary
of Professor Constantin Udriste

Abstract. We study a manifold endowed with a 3-$f-$structure such that there exist the endomorphisms f_l, $l = 1, 2, 3$, the 1-forms η_j and the vector fields ξ_j s.t. $f_l^2 = -Id + \sum_j \eta_j \otimes \xi_j$, $\eta_i(\xi_j) = \delta_{ij}$ and, for any even permutation (k, l, m) of the set $(1, 2, 3)$, one has $f_k f_l = -f_l f_k = f_m$. If a generalized (k, μ)-nullity condition holds on the manifold, then, in the generic case where $\mu^k_{ij} + \mu^k_{rj} \neq 0$, we have $Span(\xi_j)$ involutive, and the associated leaves are totally geodesic and flat.

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1 Preliminaries

Let M be a manifold endowed with a 3-$f-$structure. D will denote the distribution orthogonal to the vector fields ξ_i. Suppose a (k, μ)-nullity condition holds on M, namely ([2, 7]):

$$R(A, B)\xi_j = \sum_i k_{ij}(\eta_i(A)f^2(B) - \eta_i(B)f^2(A)) +$$
$$\sum_{l=1}^3 \sum_i \mu^l_{ij}(\eta_i(B)h_{ik}(A) - \eta_i(A)h_{ik}(B)),$$

where k_{ij}, μ^r_{ij} are some functions, $h_{il} = -\frac{1}{2}\mathcal{L}_{\xi_i} f_l$, and f^2 stands for f_l^2, $l = 1, 2, 3$.

The Riemann curvature tensor is taken with respect to a metric associated to the 3-$f-$structure, i.e., one associated to each $f-$structure, determined by the relation

$$g(fA, fB) = g(A, B) - \sum \eta_j(A)\eta_j(B),$$

where f stands for f_j, $j = 1, 2, 3$. Such conditions were studied by many authors for almost contact, almost $S-$, almost $C-$manifolds (see e.g. [2], [3], [7], [8]). For general reference on these structures, we refer to [1], [5], [10], [6], or more recently, to [4].

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Proposition 1.1 *In the above setting, one gets the following identities*:

$$\begin{cases} R(\xi_r,\xi_i)\xi_j = -\frac{1}{2}\sum_{k=1}^{3}(\mu_{ij}^k+\mu_{rj}^k)f_k([\xi_r,\xi_i]) \\ R(X,Y)\xi_j = 0, \quad g(R(\xi_j,\xi_r)X,Y)=0, \quad g(R(\xi_r,\xi_i)\xi_j,\xi_\theta)=0 \\ R(X,\xi_r)\xi_j = k_{rj}X+\sum_{k=1}^{3}\mu_{rj}^k h_{rk}(X) \\ g(R(X,\xi_j)\xi_p,\xi_i) = \sum_{k=1}^{3}\mu_{jp}^k g(h_{jk}(X),\xi_i) \end{cases}$$

where $X,Y\in\Gamma D$. In particular, it holds

Lemma 1.1 The following relations hold true:

$$\sum_{k=1}^{3}\mu_{ji}^k g(h_{jk}(X),\xi_i)=0.$$

The proofs use the fact that the Riemannian curvature tensor is pairwise symmetric.

Lemma 1.2 *If* (k,l,m) *is an even permutation of* $(1,2,3)$*, then, for any* j*:*

$$h_{jk}=h_{jl}f_m+f_lh_{jm}=-h_{jm}f_l-f_mh_{jl}.$$

For the proof, we differentiate the quaternionic identities satisfied by f_k.

2 The geometric consequence

Theorem 2.1 *Let* M *be a* 3-$f-$*manifold which satisfies the* (k,μ)*-nullity condition. If* $\mu_{ij}^k+\mu_{rj}^k\neq 0$*, then* $Span(\xi_j)$ *is involutive, with totally geodesic flat leaves.*

Proof. From the Lemmas 1.1-1.2, one gets

$$g(h_{j1}(\mu_{ji}^1X-\mu_{ji}^2f_3(X)+\mu_{ji}^3f_2(X)),\xi_i)=0.$$

The vectors obtained by substituting X with $f_1(X)$,$f_2(X)$,$f_3(X)$ in $(\mu_{ji}^1X-\mu_{ji}^2f_3(X)+\mu_{ji}^3f_2(X))$ are independent for μ and not all zero. In particular in this case, expressing X, or $f_1(X)$, or $f_2(X)$, or $f_3(X)$, one gets $g(h_{j1}X,\xi_i)=0$, for $X\in\Gamma D$. Analogously, $g(h_{jk}X,\xi_i)=0$, for $X\in\Gamma D$, and hence $g([\xi_j,X],\xi_i)=0$. Getting back to $R(\xi_r,\xi_i)\xi_j=-\frac{1}{2}\sum_{k=1}^{3}(\mu_{ij}^k+\mu_{rj}^k)f_k([\xi_r,\xi_i])$, it follows from the above and from the properties of the Riemann curvature tensor that $g(\sum_{k=1}^{3}(\mu_{ij}^k+\mu_{rj}^k)f_k([\xi_r,\xi_i]),X)=0$, or, equivalently, $g([\xi_r,\xi_i],\sum_{k=1}^{3}(\mu_{ij}^k+\mu_{rj}^k)f_k(X))=0$, where $X\in\Gamma D$.

Similar calculations show that substituting $f_1(X)$,$f_2(X)$,$f_3(X)$ for X, the rank of the system of vectors provided by $\sum_{k=1}^{3}(\mu_{ij}^k+\mu_{rj}^k)f_k(X)$ is maximal, in case that not all $\mu_{ij}^k+\mu_{rj}^k$ vanish; finally in this case $g([\xi_r,\xi_i],X)=0$ for $X\in\Gamma D$, so $Span(\xi_j)$ is involutive.

The associated leaves are totally geodesic and flat, since $R(\xi_r,\xi_i)\xi_j=0$, and

$$\begin{aligned} g(\nabla_{\xi_j}\xi_r,X) &= g(\nabla_{\xi_r}\xi_j,X)=\tfrac{1}{2}g(\nabla_{\xi_j}\xi_r+\nabla_{\xi_j}\xi_j,X) \\ &= -\tfrac{1}{2}g(\nabla_{\xi_j}X,\xi_r)-\tfrac{1}{2}g(\nabla_{\xi_r}X,\xi_j) \\ &= -\tfrac{1}{2}g(\nabla_X\xi_j,\xi_r)-\tfrac{1}{2}g(\nabla_X\xi_r,\xi_j)=0. \end{aligned}$$

3 Remarks

We believe that it is of interest to study this type of conditions on manifolds with almost contact 3-structures in the sense of Kuo ([9]) and Udrişte ([11]). Similar calculations may be formally done at least for the distribution generated by the vector fields ξ_j.

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