Wentzel-Freidlin estimates for jump processes in semi-group theory: lower bound

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Dedicated to the 70-th anniversary of Professor Constantin Udriste

Abstract. In the present paper we translate in semi-group theory the proof of Wentzel-Freidlin of large deviation estimates for jump processes whose jumps are smaller and smaller and whose number is increasing: we study the lower bound.

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1 Introduction

The object of the large deviation theory [32] is to estimate the logarithm of the probability of rare events. Bismut [1] pointed out the relationship between large deviation estimates and the Malliavin Calculus in order to get short time asymptotics of heat kernels associated to diffusion semi-groups. This relationship was fully performed by Léandre in [7, 19]. The reader interested in short time asymptotics of heat-kernels by using the Malliavin Calculus as a tool can look at the review of Léandre [11], Kusuoka [6] and Watanabe [31].

Léandre has translated plenty of tools of stochastic analysis in semi-group theory [22, 27, 23, 25, 18, 20, 28, 26, 29, 17], by using the fact that in the proof of these tools, there are suitable stochastic differential equations and therefore suitable parabolic equations which appear. Stochastic analysis tools were introduced by Léandre [14, 15, 16, 13] in the fields of partial differential equations different of the classical parabolic equations whose generators satisfied the maximum principle.

Léandre [8, 9] pointed out that the small time behaviour of the transition density of a jump process has a polynomial decay related to the number of jump necessary to join two points. This relationship was fully performed by Ishikawa in [2]. Small time asymptotics of jump processes is not simply related to large deviation theory.

Large deviation for jump processes is related to the fact that the process has more and more jumps which are smaller and smaller. The mixture of large deviation estimates for jump processes and the Malliavin Calculus was performed by Ishikawa

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[3] and Ishikawa-Léandre [4]. Our goal is to translate in semi-group theory Wentzel-Freidlin estimates for jump processes, in the thematic we have chosen to translate Malliavin Calculus of Bismut type for jump process in semi-group theory [12, 20, 24]. The estimate is true only for the semi-group and not for the path space of the process.

We consider the symbol $(x, \xi) \to H(x, \xi)$ of the operator. We consider the operator L^h associated to the symbol $H^h(x, \xi) = H(x, h\xi)$. We consider the heat-equation associated to $1/hL^h$ and his behaviour when $h \to 0$. Since in this case, we have a probability semi-group, it is convenient to study lower-bound of the behaviour when $h \to 0$ of its solution. In semi-classical analysis [30], people consider the Schroedinger equation associated to $1/hL^h$. In such a case there is no representation of the solution of it by using probability measures, and people look asymptotics expansion of the solution of the Schroedinger equation whose main term is oscillatory.

The reader interested by this topic can see the reviews of Léandre [10, 21].

2 Statement of the main theorem

Let $\mu(x, dz) = g(x, z)dz$ be a positive measure on $\mathbb{R}^d$ such that $\sup_x \int_{\mathbb{R}^d} |z|^2 \mu(x, dz) < \infty$. It is the Levy measure of the associated jump process. We suppose that $(x, z) \to g(x, z)$ is continuous if $z \neq 0$. We introduce the Hamiltonian defined for $(x, \xi) \in \mathbb{R} \times \mathbb{R}^d$

$$H(x, \xi) = \int_{\mathbb{R}^d} (\exp[\langle z, \xi\rangle] - 1 - \langle z, \xi\rangle)\mu_x(dz) \tag{2.1}$$

Hypothesis H.1H is continuous convex, bounded uniformy in x by $H_1(\xi)$, a convex function on $\mathbb{R}^d$.

We consider the Legendre transform of H

$$L(x, \alpha) = \sup_{\xi} \{\langle \alpha, \xi\rangle - H(x, \xi)\} \tag{2.2}$$

Hypothesis H.2: There exists a unique continuous in (x, α) $\xi(x, \alpha)$ such that

$$L(x, \alpha) = \langle \alpha, \xi(x, \alpha)\rangle - H(x, \xi(x, \alpha)) \tag{2.3}$$

We suppose that $\sup_{x, |\alpha| \leq R} |\xi(x, \alpha)| \leq \xi(R) < \infty$ for all $R > 0$

We consider a piecewise C^1 curve $\phi(t)$ and we consider the action:

$$S(\phi) = \int_0^1 L(\phi(t), d/dt\phi(t))dt \tag{2.4}$$

We put

$$l(x, y) = \inf_{\phi(0)=x, \phi(1)=y} S(\phi) \tag{2.5}$$

Under the previous assumption, $(x, y) \to l(x, y)$ is continuous. We define the generator L^h defined on smooth functions with bounded derivatives at each order

$$L^h f(x) = \int_{\mathbb{R}^d} (f(x + hz) - f(x) - h\langle z, f'(x)\rangle)\mu_x(dz) \tag{2.6}$$

Under these previous assumptions, we will get in the sequel Markovian semi-groups. In particular, $1/hL^h$ generates a semi-group P_t^h. In such a case, P_t^h is a semi-group in probability measures ([5, 32])

Theorem*(Wentzel-Freidlin [32])Let O be an open ball of $\mathbb{R}^d$. When $h \to 0$,*

$$\underline{\lim} h \, Log P_1^h[1_O](x) \geq -\inf_{y \in O} l(x,y) \tag{2.7}$$

Remark: We can adapt the proof of this theorem to get Wentzel-Freidlin estimates, lower bound, for diffusion ([28, 17]).

Remark:$H(x, i\xi)$ is the symbol associated to L^1 ([5]).

3 Proof of the theorem

The proof is the translation in semi-group theory of the proof of the same theorem for the whole path space in [32]. We give only the details of the algebra of the proof: for the estimates we refer to [12] or to [20], Appendix. In the sequel, ϕ_t is a piecewise C^1 curve starting from x.

We introduce the generator acting on bounded function $F(x,y)$ on $\mathbb{R}^d \times \mathbb{R}$

$$Q_t^h F(x,y) = \frac{1}{h} \int_{\mathbb{R}^d} (F(x+hz, y \exp[\langle \xi(x, d/dt\phi_t), z\rangle]) - F(x,y) - \langle \exp[\langle \xi(x, d/dt\phi_t), z\rangle] - 1, y D_y F(x,y)\rangle - \langle hz, D_x F(x,y)\rangle)\mu_x(dz) \tag{3.1}$$

Let $\overline{f}(x,y)$ be the function $(x,y) \to f(x)y$. A simple computation shows that

$$Q_t^h \overline{f}(x,1) = \frac{1}{h} \tilde{L}_t^h f(x) + \langle D_\xi H(x, \xi(x, d/dt\phi(t))), D_x f(x)\rangle \tag{3.2}$$

where

$$\tilde{L}_t^h = \int_{\mathbb{R}^d} (f(x+hz) - f(x) - h\langle z, f'(x)\rangle)\tilde{\mu}_{x,t}(dz) \tag{3.3}$$

and where $\tilde{\mu}_{x,t}(dz) = \exp[\langle \xi(x, d/dt\phi_t), z\rangle]\mu_x(dz)$. Moreover in such a case, we have the equality $D_\xi H(x, \xi(x, d/dt\phi(t))) = d/dt\phi_t$. Q_t^h generates a time inhomogeneous semi-group $R_{s,t}^h$ such that $R_{s,t}^h \overline{f}(x,y) = R_{s,t}^h \overline{f}(x,1)y$. Therefore, if we consider the semi-group $\tilde{P}_{s,t}^h$ generated by the generator $f \to \frac{1}{h}\tilde{L}_t^h f(x) + \langle d/dt\phi_t, D_x f(x)\rangle$, we have the relation

$$R_{0,t}^h \overline{f}(x,1) = \tilde{P}_{0,t}^h f(x) \tag{3.4}$$

We consider the generator

$$\tilde{Q}_t^h F(x,y) = \frac{1}{h} \int_{\mathbb{R}^d} (F(x+hz, y \exp[-\langle \xi(x, d/dt\phi_t), z\rangle] - F(x,y) - \langle \exp[-\langle \xi(x, d/dt\phi_t), z\rangle] - 1, y D_y F(x,y)\rangle - \langle hz, D_x F(x,y)\rangle)\tilde{\mu}_{t,x}(dz) + \langle d/dt\phi_t, D_x F(x,y)\rangle \tag{3.5}$$

It generates a semi-group $\tilde{R}^h_{s,t}$ and analog considerations show that

$$\tilde{R}^h_{0,t}\overline{f}(x,1) = P^h_t f(x) \tag{3.6}$$

Let us consider the generator

$$\begin{aligned}\overline{Q}^h_t F(x,y) = 1/h\int_{\mathbb{R}^d}(F(x+zh, y - \langle\xi(x, d/dt\phi_t), z\rangle) - F(x,y)\\ - \langle hz, D_x F(x,y)\rangle + \langle\xi(x,d/dt\phi_t), z\rangle D_y F(x,y))\tilde{\mu}_x(dz) + \langle d/dt\phi_t, D_x F(x,y)\rangle\\ - 1/h\langle d/dt\phi_t, \xi(x, d/dt\phi_t)\rangle D_y F(x,y) + 1/hH(x, d/dt\phi_t) D_y F(x,y)\end{aligned} \tag{3.7}$$

Q^h_t is transformed into $\tilde{Q}^h_t$ by the transformation $(x,y) \to (x, \exp[y])$. Let $\tilde{f}(x,y) = f(x)\exp[y]$ and let $\overline{R}^h_{s,t}$ be the time inhomogeneous semi-group generated by $\overline{Q}^h_t$. We get

$$P^h_t f(x) = \tilde{R}^h_{0,t}\overline{f}(x,1) = \overline{R}^h_{0,t}\tilde{f}(x,0) \tag{3.8}$$

Let us remark that

$$\langle d/dt\phi_t, \xi(x, d/dt\phi_t)\rangle - H(x, d/dt\phi_t) = L(x, d/dt\phi_t) \tag{3.9}$$

We consider the generator $\overline{Q}^{h,1}_t$ acting on functions on $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$:

$$\begin{aligned}\overline{Q}^{h,1}_t F(x, y_1, y_2, y_3) = 1/h\int_{\mathbb{R}^d}(F(x+zh, y_1 - \langle\xi(x, d/dt\phi_t), z\rangle, y_2, y_3)\\ - F(x, y_1, y_2, y_3) - \langle hz, D_x F(x, y_1, y_2, y_3)\rangle\\ + \langle\xi(x, d/dt\phi_t), z\rangle D_{y_1} F(x, y_1, y_2, y_3))\tilde{\mu}_x(dz) + \langle d/dt\phi_t, D_x F(x, y_1, y_2, y_3)\rangle\\ + 1/h(L(\phi_t, d/dt\phi_t) - -L(x, d/dt\phi_t)) D_{y_2} F(x, y_1, y_2, y_3)\\ - 1/hL(\phi_t, d/dt\phi_t) D_{y_3} F(x, y_1, y_2, y_3)\end{aligned} \tag{3.10}$$

We put $\overline{F}(x, y_1, y_2, y_3) = f(x)g(y_1 + y_2 + y_3)$ and let $F(x,y) = f(x)g(y)$. We introduce the Markovian semi-group $\overline{R}^{h,1}_{s,t}$ associated to $\overline{Q}^{h,1}_t$. We get:

$$\overline{R}^{h,1}_{0,t}\overline{F}^1(x,0,0,0) = \overline{R}^h_{0,t} F(x,0) \tag{3.11}$$

For that we remark, that $D_{y_1}, D_{y_2}, D_{y_3}$ commute with $\overline{Q}^{t,1}_t$ and therefore with the involved semi-group. This show that $\overline{R}^{h,1}_{0,t}\overline{F}^1(x, y_1, y_2, y_3)$ is a function of x and $y_1 + y_2 + y_3$ only, because the functions on $\mathbb{R}^3$ of $y_1 + y_2 + y_3$ are characterized by the fact $D_{y_1} - D_{y_2}$ and $D_{y_1} - D_{y_3}$ vanish on them. D_y commute with $\overline{Q}^h_t$ and therefore with the involved semi-group.

We remark that $\overline{R}^{h,1}_{s,t}$ is a semi-group in probability measures.

In the sequel, we can suppose without restriction that $x = 0$. We consider the function y_1^2. We get that

$$\left|\frac{\partial}{\partial t}\overline{R}^{h,1}_{0,t}[y_1^2](0,0,0,0)\right| \leq C/h \tag{3.12}$$

Therefore $|\overline{R}^{h,1}_{0,t}[y_1^2](0,0,0,0)| \leq C/h$ and by Tchebitchev inequality, for all η we can find a C such that

$$|\overline{R}^{h,1}_{0,1}[|y_1| \leq Ch^{-1/2}](0,0,0,0) \geq 1-\eta \tag{3.13}$$

Let us estimate the second quantity y_2. By considering a function with bounded first order derivative which tends to the function $y_2 \to |y_2|$, we have that

$$|\frac{\partial}{\partial t}\overline{R}^{h,1}_{0,t}[|y_2|](0,0,0,0)| \leq 1/h|\overline{R}^{h,1}_{0,t}[|L(\phi_t, d/dt\phi_t) - L(x, d/dt\phi_t)|](0,0,0,0)| \tag{3.14}$$

But

$$\begin{aligned} &|\overline{R}^{h,1}_{0,t}[|L(\phi_t, d/dt\phi_t) - L(x, d/dt\phi_t)|](0,0,0,0)| \\ &\quad \leq \{|\overline{R}^{h,1}_{0,t}[|L(\phi_t, d/dt\phi_t) - L(x, d/dt\phi_t)|^2](0,0,0,0)|\}^{1/2} \\ &\leq \delta + C\overline{R}^{h,1}_{0,t}[|x-\phi_t| > \delta_1](0,0,0,0) \leq \delta + C(\delta)\{\overline{R}^{h,1}_{0,t}[|x-\phi_t|^2](0,0,0,0)\}^{1/2} \end{aligned} \tag{3.15}$$

by Hypothesis H(1) and Hypothesis H(2) for any small δ and some $C(\delta)$.

$$\begin{aligned} \frac{\partial}{\partial t}\overline{R}^{h,1}_{0,t}[|x-\phi_t|^2](0,0,0,0) &= \frac{\partial}{\partial t}\overline{R}^{h,1}_{0,t}[|x|^2](0,0,0,0) - \\ - 2\overline{R}^{h,1}_{0,t}[\langle x, d/dt\phi_t\rangle](0,0,0,0) &- 2\frac{\partial}{\partial t}\overline{R}^{h,1}_{0,t}[\langle x, .](0,0,0,0), \phi_t\rangle \\ &+ 2\langle \phi_t, d/dt\phi_t\rangle \end{aligned} \tag{3.16}$$

The fist term is bounded by h, the second term vanishes because his derivative vanishes and the third term is equal to $-2 < \phi_t, d/dt\phi_t >$. Therefore we get the bound

$$\overline{R}^{h,1}_{0,1}[|y_2|](0,0,0,0) \leq \delta/h + C\sqrt{h}^{-1} \tag{3.17}$$

By Tchebitchev inequality, for all η, we can find a $C(\eta)$ such that for $0 < h \leq h(\eta)$:

$$\overline{R}^{h,1}_{0,1}[|y_2| \leq C(\eta)h^{-1}](0,0,0,0) \geq 1-\eta \tag{3.18}$$

Moreover, when $\eta \to 0$, $C(\eta) \to 0$. By doing by the same procedure, we find that for h enough small that

$$\overline{R}^{h,1}_{0,1}[|x-\phi_1| < \delta](0,0,0,0) \geq 1-\eta \tag{3.19}$$

Therefore for $0 < h < h(\eta)$, $C(\eta) \to 0$ when $\eta \to 0$

$$\overline{R}^{h,1}_{0,1}[|x-\phi_1| < \delta; |y_1+y_2| \leq C(\eta)h^{-1}](0,0,0,0) \geq 1-3\eta \tag{3.20}$$

It remains to remark that under the probability measure$\overline{R}^{h,1}_{0,1}[](0,0,0,0)$ y_3 is deterministic and equal to $-S(\phi)/h$. The result follows from (3.8), (3.11) and from the fact we consider semi-groups in **probability** measures.

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