A finite element technique for solving an integral-differential equation

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Dedicated to the 70-th anniversary of Professor Constantin Udriste

Abstract. In this paper is solved a neutron transport problem by the finite element simulations. We present a method that replaces a boundary value problem for one-dimensional transport equation with a boundary value problem for a diffusion equation. To solve this new problem we will construct and then we minimize a functional, using the specific techniques of the finite element methods. Numerical examples allow us a detailed analysis of the errors according to the mesh step.

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1 Introduction

This paper deals with a typical problem of the mathematical-physics: the solving of neutron transport equation. In a reactor, the neutrons are yield at the fission of the nucleus and they are named the fast neutrons. These neutrons have an average velocity equal to $2 \cdot 10^7 m/s$, and are subjected to a slowing process: their energy decreases until these are in an equilibrium state with the other atoms of the environment. When the reactor is in a stationary state, the particles have the tendency to move from a region with a great density to another with a small density and thus we get a uniform density. This process is named the diffusion. The main problem in the nuclear reactor theory is to find the neutrons distribution in the reactor, hence its density, which is the solution of an integral-differential equation named the neutron transport equation.

An exact solution of integral-differential equation was found only in the particular cases. We remind the papers of the authors: Brezis ([5]), Cardona and Vilhena ([6]), Davidson and Sykes ([10]), Kadem ([10]), Siewert ([23]), Wilson and Sen ([29]). Generally, these are obtained with the help of the methods of mathematical analysis, abstract functional analysis and the spectral methods.

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In this paper we approach the finite element method techniques to solve a boundary value problem for the linearized version of the Boltzmann equation with numerous applications in physics and astrophysics.

2 Problem formulation

Let us consider the integral-differential equation of transport theory for the stationary case:

$$\begin{aligned}&\frac{\partial u}{\partial x} + u(x,\mu) = \int_{-1}^{1} u(x,\mu')d\mu' + g(x,\mu) \\ &(x,\mu) \in D,\ D = [0,1] \times [-1,1] = D_1 \times D_2\end{aligned} \tag{2.1}$$

with

$$\begin{aligned}&u(0,\mu) \quad \text{and} \quad \frac{\partial u}{\partial x}(1,\mu) = 0 \quad \text{if} \quad \mu \in [0,1] \\ &u(1,\mu) \quad \text{and} \quad \frac{\partial u}{\partial x}(0,\mu) = 0 \quad \text{if} \quad \mu \in [-1,0),\end{aligned} \tag{2.2}$$

where g is the source function and u is the density of neutrons. These migrate towards a direction defined by the angle α against Ox axis and we denote $\mu = \cos\alpha$. Suppose that the functions, which appear in (2.1) are continuous together with their first and second derivatives in the domain D.

Let $\Delta_1 = (x_0, x_1, \ldots, x_n)$ be a partition of the interval D_1 into n subintervals of length $h = x_{i+1} - x_i$, $i = 0, 1, \ldots, n-1$. Also, let $\Delta_2 = (\mu_0, \mu_1, \ldots, \mu_p)$ be a partition of the interval D_2 and $\tau = \mu_{l+1} - \mu_l$, $\forall l = 0, 1, \ldots, p-1$. To solve the problem (2.1) - (2.2) by finite element method, we replace the equation (2.1) with the partial derivative of this with respect to variable x:

$$\frac{\partial^2 u(x,\mu)}{\partial x^2} + \frac{\partial u(x,\mu)}{\partial x} = \int_{-1}^{1} \frac{\partial u(x,\mu')}{\partial x} d\mu' + \frac{\partial g(x,\mu)}{\partial x} \tag{2.3}$$

and using again (2.1), we get

$$\frac{\partial^2 u(x,\mu)}{\partial x^2} - u(x,\mu) = \gamma(x,\mu) \tag{2.4}$$

where

$$\gamma(x,\mu) = \int_{-1}^{1} g(x,\mu')d\mu' + \frac{\partial g(x,\mu)}{\partial x} - g(x,\mu) \tag{2.5}$$

Let us consider that for every fixed value $\mu_l \in D_2$, the function $u \in C^2(D_1)$. Also, u together with the first and the second derivatives belong to a Hilbert space with the scalar product defined by the formula:

$$(w,v) = \int_0^1 w(x)v(x)dx \tag{2.6}$$

Now, we rewrite (2.4) as an operatorial equation

$$Lu = f \tag{2.7}$$

where

$$Lu = -\frac{d^2u(x,\mu_l)}{dx^2} + u(x,\mu_l), \qquad f(x,\mu_l) = -\gamma(x,\mu_l)$$

and

$$\begin{cases} u(0,\mu_l) \text{ and } \dfrac{\partial u}{\partial x}(1,\mu_l) = 0, \text{ if } \mu_l > 0 \\ \\ u(1,\mu_l) \text{ and } \dfrac{\partial u}{\partial x}(0,\mu_l) = 0, \text{ if } \mu_l < 0. \end{cases} \tag{2.8}$$

In order to obtain a finite element approximation for the solution of the problem (2.7)-(2.8), we minimize the functional:

$$J(u) = (Lu,u) - 2(u,f) \tag{2.9}$$

Let us now consider that $\mu_l > 0$. Then, our problem of finding a solution of (2.7)-(2.8) can be reduced to the problem of minimizing the functional (2.9). Applying the method of integration by parts and (2.8), we get:

$$J(u) = \int_0^1 \left(u\left(\frac{d^2u}{dx^2} - u\right) - 2fu \right) dx = -\int_0^1 \left(\left(\frac{du}{dx}\right)^2 + u^2 + 2fu \right) dx \tag{2.10}$$

where u is a function of x and $\mu = \mu_l$ fixed. The functional $J(u)$ will be minimized directly, using the finite element method. According to Ritz approximation, the actual solution is represented by a set of functions that are linear in each segment $[x_j, x_{j+1}]$, continuous at the nodes $x_j = jh$ and equal to zero at $x = 0$. Therefore, the solution will be of the form

$$u_n^h(x) = \sum_{j=1}^{n} \alpha_j \varphi_j(x) \tag{2.11}$$

where

$$\varphi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{h}, & x \in [x_{j-1}, x_j] \\ \dfrac{x_{j+1} - x}{h}, & x \in [x_j, x_{j+1}] \\ 0, & \text{in rest} \end{cases} \tag{2.12}$$

with $\varphi_j(x_k) = \delta_{jk}$, where δ_{jk} is the Kroneker's symbol. Two properties of the approximation (2.11) would be mentioned. First, $u^h(x_j) = \alpha_j$, i.e. α_j are the nodal values of u^h. Second, the functions φ_{j-1} and φ_{j+1} are orthogonal, because where one of these is non-vanish, the other is vanish. Thus the basis introduced is "almost orthogonal". This is the reason of appearance of the band matrices in the method of finite element.

Let us return to the problem of minimizing of the functional (2.9) that is now of the form

$$J(u^h) = \int_0^1 \left[\left(\frac{du^h}{dx} \right)^2 + (u^h)^2 - 2fu^h \right] dx \tag{2.13}$$

In view of (2.11), this integral is a function of coordinates $\alpha_1, \ldots, \alpha_{n-1}$ and it may be computed as the sum of integrals over all the segments of $[0, 1]$. Since the function u^h is linear, we get for the j - th segment $[x_{j-1}, x_j]$:

$$u^h[(j-1)h] = \alpha_{j-1}, \quad u^h(jh) = \alpha_l \tag{2.14}$$

and

$$(u^h)' = \frac{\alpha_j - \alpha_{j-1}}{h}$$

These lead to the following equalities

$$\int_{(j-1)h}^{jh} \left(\frac{du^h}{dx} \right)^2 dx = \frac{(\alpha_j - \alpha_{j-1})^2}{h} \tag{2.15}$$

and

$$\int_{(j-1)h}^{jh} (u^h)^2 dx = \frac{h}{3}(\alpha_{j-1}^2 + \alpha_{j-1}\alpha_j + \alpha_j^2) \tag{2.16}$$

where $\alpha_0 = 0$, according to the boundary conditions. We observe that the formula (2.15) can be rewritten in the matrix form

$$\begin{aligned} \int_{(j-1)h}^{jh} [(u^h)']^2 dx &= [\alpha_{j-1} \quad \alpha_j] \frac{1}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_{j-1} \\ \alpha_j \end{bmatrix} = \\ &= [\alpha_{j-1} \quad \alpha_j] k_1^j \begin{bmatrix} \alpha_{j-1} \\ \alpha_j \end{bmatrix} \end{aligned} \tag{2.17}$$

Similarly, the expression (2.16) can be written in the following form

$$\begin{aligned} \int_{(j-1)h}^{jh} (u^h)^2 dx &= [\alpha_{j-1} \quad \alpha_j] \frac{h}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \alpha_{j-1} \\ \alpha_j \end{bmatrix} = \\ &= [\alpha_{j-1} \quad \alpha_j] k_0^j \begin{bmatrix} \alpha_{j-1} \\ \alpha_j \end{bmatrix} \end{aligned} \tag{2.18}$$

where k_0^j is called "the mass" matrix of j element. Then, the computation of the term $\int_0^1 fu^h dx$ leads to

$$\int_0^1 \left(\sum_{j=1}^{n-1} \alpha_j \varphi_j \right) f dx = \sum_{j=1}^{n-1} \alpha_j \int_0^1 f \varphi_j dx = \mathbf{F}^T \alpha$$

where $\alpha = [\alpha_1\ \alpha_2 \ldots \alpha_{n-1}]$ and $\mathbf{F}^T$ is a row matrix with elements

$$F_j = \int_0^1 f\varphi_j dx = \int_{(j-1)h}^{jh} \frac{x - x_j}{h} f dx + \int_{jh}^{(j+1)h} \frac{x_{j+1} - x}{h} f dx. \tag{2.19}$$

In order to construct the global rigidity matrix

$$\mathbf{K} = \mathbf{K}_1 + \mathbf{K}_0 \tag{2.20}$$

we define now a matrix A that relates the coefficients α_j of each element j with the column matrix α. The matrix A is a rectangular matrix in which each line consists of zeros except one or two equal to unity. Using the boundary condition: $\alpha_0 = 0$, we have

$$\overline{\alpha} = \begin{bmatrix} \alpha_1 \\ -- \\ \alpha_1 \\ \alpha_2 \\ -- \\ \vdots \\ -- \\ \alpha_{n-2} \\ \alpha_{n-1} \\ -- \\ \alpha_{n-1} \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-2} \\ \alpha_{n-1} \\ \alpha_n \end{bmatrix} = \mathbf{A}\alpha. \tag{2.21}$$

Applying the techniques of the finite element method, we obtain the matrix $\mathbf{K}_1$ in the general coordinate system of the form

$$\mathbf{K}_1 = \mathbf{A}^T \overline{\mathbf{K}}_1 \mathbf{A} = \mathbf{A}^T \cdot \frac{1}{h} \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & -1 & 1 \end{bmatrix} \cdot \mathbf{A} = \tag{2.22}$$

$$= \frac{1}{h} \cdot \begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & & 0\dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 1 \end{bmatrix}$$

Similarly, the global "mass matrix" is of the form

$$\mathbf{K}_0 = \mathbf{A}^T \overline{\mathbf{K}}_0 \mathbf{A} = \mathbf{A}^T \cdot \frac{h}{6} \begin{bmatrix} 2 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 2 \end{bmatrix} \cdot \mathbf{A} =$$

$$= \frac{h}{6} \cdot \begin{bmatrix} 4 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 4 & 1 & & 0\dots & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 2 \end{bmatrix}$$

Finally, the integrals of the functional on the entire interval [0,1] will be

$$\int_0^1 [(u^h)']^2 dx = [\alpha_1, \dots, \alpha_{n-1}] \cdot \mathbf{K}_1 \cdot \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{bmatrix} = \alpha^T \mathbf{K}_1 \alpha \tag{2.23}$$

$$\int_0^1 (u^h)^2 dx = \alpha^T \mathbf{K}_0 \alpha$$

and using (2.20), the functional $J(u^h)$ becomes

$$J(u^h) = \alpha^T \mathbf{K} \alpha - 2\mathbf{F}^T \alpha \tag{2.24}$$

where the global matrix $\mathbf{K}$ is a symmetrical matrix. Therefore

$$J(u^h) = \sum_{i,j} K_{ij}\alpha_i\alpha_j - 2\sum_i F_i\alpha_i \tag{2.25}$$

Minimization of this functional leads to a system of $(n-1)$ equations

$$\frac{\partial J}{\partial \alpha_m} = 0, \quad m = 1, 2, \dots, n-1 \tag{2.26}$$

where

$$\frac{\partial J}{\partial \alpha_m} = \sum_{j=1}^{n} K_{mj}\alpha_j + \sum_{j=1}^{n} K_{jm}\alpha_j - 2F_m = 2\left(\sum_{j=1}^{n} K_{mj}\alpha_j - F_m\right)$$

Solving the matrix equation that corresponds to (2.26)

$$\mathbf{K}\cdot\alpha = \mathbf{F} \tag{2.27}$$

we can find the approximate nodal values of the function u:

$$u^h(x_j) = \alpha_j, \quad j = 1, 2, \dots, n-1 \tag{2.28}$$

for every fixed $\mu_l > 0$. The algorithm will be repeated for each $l = 1, 2, \dots, p$. Similarly, we can study the case $\mu_l < 0$.

3 Numerical results

Let us consider the neutron transport equation

$$\frac{\partial u(x,\mu)}{\partial x} + u(x,\mu) = \int_{-1}^{1} u(x,\mu')d\mu' + g(x,\mu) \tag{3.1}$$

where

$$g(x,\mu) = \begin{cases} \mu\left(\frac{\pi}{2}\cos\frac{\pi x}{2} + \sin\frac{\pi x}{2}\right), & \mu \geq 0 \\ \mu\left(\frac{-\pi}{2}\sin\frac{\pi x}{2} + \cos\frac{\pi x}{2}\right), & \mu < 0 \end{cases} \tag{3.2}$$

$$(x,\mu) \in [0,1]\times[-1,1] = D_1 \times D_2$$

and

$$u(0,\mu) = 0, \;\; \frac{\partial u}{\partial x}(1,\mu) = 0 \text{ if } \mu > 0 \tag{3.3}$$

$$u(1,\mu) = 0, \;\; \frac{\partial u}{\partial x}(0,\mu) = 0 \text{ if } \mu < 0 \tag{3.4}$$

For the domain D_1 we use the partition Δ_1 with the step $h = \frac{1}{8}$ and for the domain D_2 we use a partition Δ_2 with the step $\tau = \frac{1}{4}$. The accuracy of this method is verified by comparing the numerical results with the corresponding values of the exact solution

$$ex(x,\mu) = \begin{cases} \mu \sin \dfrac{\pi x}{2}, & \mu \geq 0 \\ \mu \cos \dfrac{\pi x}{2}, & \mu < 0. \end{cases} \tag{3.5}$$

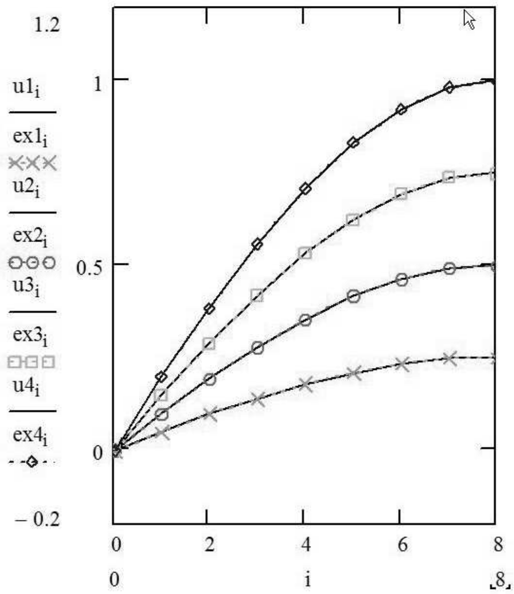

Fig. 1

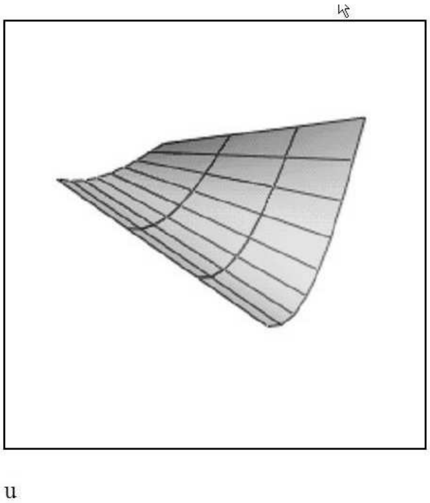

Fig. 2

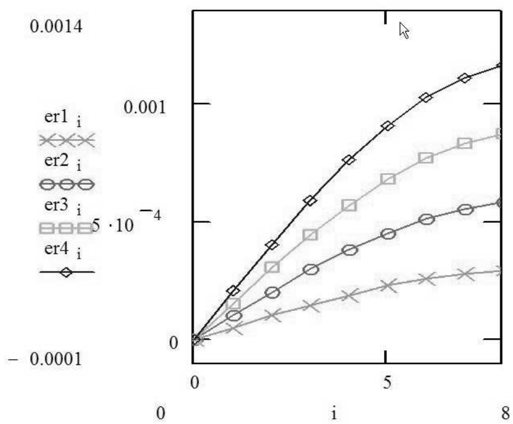

Fig.3

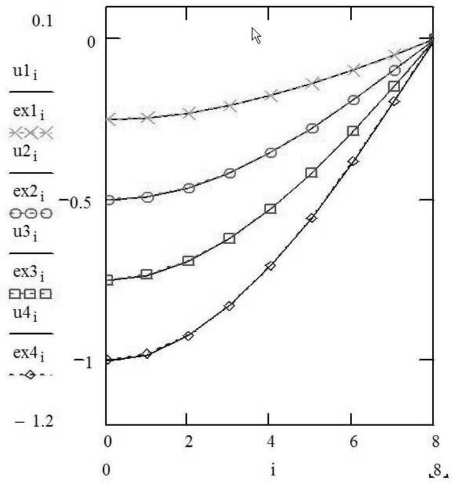

Fig. 4

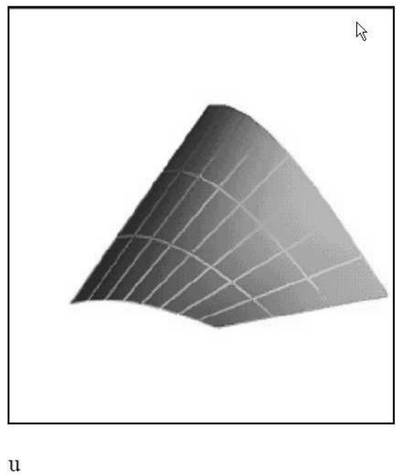

Fig.5

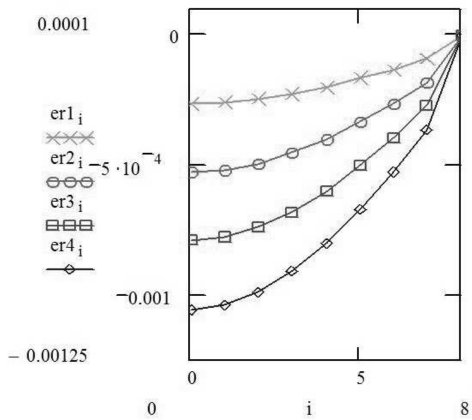

Fig. 6

The dependence of the density u^h on the values of μ_l, where $u1_i = u^h(x_i, 0.25)$; $u2_i = u^h(x_i, 0.5)$; $u3_i = u^h(x_i, 0.75)$; $u4_i = u^h(x_i, 1)$, $i = 1, 2, \ldots, 8$ and the comparison of the numerical values u^h with the exact values ex, where $ex1_i = ex(x_i, 0.25)$; $ex2_i = ex(x_i, 0.5)$; $ex3_i = ex(x_i, 0.75)$; $ex4_i = ex(x_i, 1)$, $i = 1, 2, \ldots, 8$ are presented in Figure 1. Figure 2 and Figure 5 show the surface $u = u(x, \mu)$ for $\mu > 0$ and $\mu < 0$, respectively. The values of the exact (ex) and the numerical solution of the problem (3.1)-(3.4) in the nodes x_i for $\mu < 0$ are presented in the Figure 4: $u1_i = u^h(x_i, -0.25)$; $u2_i = u^h(x_i, -0.5)$; $u3_i = u^h(x_i, -0.75)$; $u4_i = u^h(x_i, -1.)$, $i = 1, 2, \ldots, 8$.

Let us define the errors

$$er_l = \max_{0 \le i \le n} |u^h(x_i, \mu_l) - ex(x_i, \mu_l)|, \quad l = 0, 1, 2, 3, 4 \tag{3.6}$$

with respect to x and $\mu > 0$ are shown in Figure 3 and with respect to x and $\mu < 0$ are shown in Figure 6. We observe that the errors increase with the increasing of μ and don't exceed the value 0.0012. As is shown in the Figures 1 and 4, the graphics of the numerical solution, u^h and of the exact solution, ex, practically coincide.

4 Approximation study

Let us now consider the question of estimating the interpolation error when the solution is obtained by a polynomial of the form (2.11). We assume that $\overline{u}(x_i) = u^h(x_i)$, $\forall i = 1, 2, \ldots, n$, where we denoted the exact solution $u(x, \mu_l) = \overline{u}(x)$ for a fixed μ_l. For any $x \in [x_{i-1}, x_i]$, we estimate now the difference

(4.1)
$$\begin{aligned}
\overline{u}(x) - u^h(x) &= \int_{x_{i-1}}^{x} \frac{d}{dt}(\overline{u} - u^h)(t)dt = \int_{x_{i-1}}^{x} \frac{d\overline{u}}{dt}(t)dt - \int_{x_{i-1}}^{x} \frac{\overline{u}_i - \overline{u}_{i-1}}{h} dt = \\
&= \frac{1}{h}\int_{x_{i-1}}^{x_i} dt' \int_{x_{i-1}}^{x} \left(\frac{d\overline{u}}{dt}(t) - \frac{d\overline{u}}{dt}(t')\right) dt = \\
&= \frac{1}{h}\int_{x_{i-1}}^{x_i} dt' \int_{x_{i-1}}^{x} dt \int_{t'}^{t} \frac{d^2\overline{u}}{d\xi^2}(\xi) d\xi, \quad \overline{u}_i = \overline{u}(x_i)
\end{aligned}$$

Then

$$|\overline{u}(x) - u^h(x)| \leq \frac{1}{h}\int_{x_{i-1}}^{x_i} dt' \int_{x_{i-1}}^{x} dt \int_{t'}^{t} \left|\frac{d^2\overline{u}}{d\xi^2}(\xi)\right| d\xi \leq h \int_{x_{i-1}}^{x_i} \left|\frac{d^2\overline{u}}{d\xi^2}(\xi)\right| d\xi \tag{4.2}$$

Taking into account (2.7) - (2.8), we get the inequality

$$\begin{aligned}
\int_0^1 f^2 dx &= \int_0^1 \left(-\frac{d^2u}{dx^2} + u\right)^2 dx = \\
&= -2u\left.\frac{du}{dx}\right|_0^1 + \int_0^1 \left[\left(\frac{d^2u}{dx^2}\right)^2 + 2\left(\frac{du}{dx}\right)^2 + u^2\right] dx \geq \int_0^1 \left(\frac{d^2u}{dx^2}\right)^2 dx
\end{aligned} \tag{4.3}$$

Now, integrating (4.2) on the interval $[x_{i-1}, x_i]$ and using the Hölder's inequality, we obtain

$$\begin{aligned}
\int_{x_{i-1}}^{x_i} |\overline{u}(x) - u^h(x)|^2 dx &\leq \int_{x_{i-1}}^{x_i} h^2 \left(\int_{x_{i-1}}^{x_i} 1 \cdot \frac{d^2u}{d\xi^2} d\xi\right)^2 dx \leq \\
&\leq h^3 \int_{x_{i-1}}^{x_i} dx \int_{x_{i-1}}^{x_i} \left(\frac{d^2\overline{u}}{d\xi^2}\right)^2 d\xi = h^4 \int_{x_{i-1}}^{x_i} \left(\frac{d^2\overline{u}}{d\xi^2}\right)^2 d\xi.
\end{aligned}$$

From (4.3) and assuming that in any interval (μ_{l-1}, μ_l), the difference (4.2) varies linearly with respect to μ, the following estimation is obtained

$$\begin{aligned}
||u(x, \mu) - \tilde{u}^h(x, \mu)||^2_{L_2(0,1)} &= \max_{1 \leq i \leq n} \int_0^\tau K^2 \mu^2 d\mu \int_{x_{i-1}}^{x_i} |\overline{u}(x) - u^h(x)|^2 dx \leq \\
&\leq h^4 \int_0^\tau K^2 \mu^2 d\mu \int_0^1 \left(\frac{d^2\overline{u}}{d\xi^2}\right)^2 d\xi \leq K^2 h^4 \int_0^\tau \mu^2 d\mu \int_0^1 f^2 dx.
\end{aligned} \tag{4.4}$$

where $\tilde{u}^h(x, \mu_l) = u^h(x)$. Finally, we get

$$||u(x, \mu) - \tilde{u}^h(x, \mu)||_{L_2(D)} \leq M h^2 \tau^{3/2} ||f||^2_{L_2(0,1)}. \tag{4.5}$$

5 Conclusions

The contributions of this study to solving a neutron transport problem are the following:

(a) a method that replaces an integral-differential equation with specific boundary conditions for a transport problem with a diffusion problem;

(b) determining of a functional for which the Euler's equation leads to the given transport equation;

(c) an algorithm that uses the techniques of finite element method is presented to solve the new boundary problem;

(d) finally, if we approximate the exact solution of the problem (2.7) - (2.8) by the finite sum

$$\tilde{u}^h(x,\mu) = \sum_{j=1}^{n} \alpha_j(\mu)\varphi_j(x), \quad \text{where} \quad \varphi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{h}, & x \in [x_{j-1}, x_j] \\ \dfrac{x_{j+1} - x}{h}, & x \in [x_j, x_{j+1}] \\ 0, & \text{in rest} \end{cases}$$

and α_j varies linearly with respect to μ in any interval from Δ_2, then, in view of (4.5), the approximation of the exact solution according to the steps of Δ_1 and Δ_2 is of the order of h^2 and $\tau^{3/2}$, respectively. From the study of the errors obtained in our numerical example, we get for (4.5) that the constant $M = 0.4$.

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