Types of pseudomonotonicity in the study of variational inequalities

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Abstract. Pseudomonotonicity is the basic tool in proving the existence of solutions of variational inequalities. There is a variety of algebraic and topological extensions corresponding to the solvability of the forms of generalized variational-like inequalities. An ordering of a part of these types of pseudomonotone set-valued mappings is presented.

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Let X be a real Banach space, X^* its dual space and $\langle\ ,\ \rangle : X^* \times X \to \mathbb{R}$ their usual pairing. For a given $f \in X^*$, a monotone-like map $A : X \to X^*$ and C a nonempty closed and convex subset of X, the problem of finding an element $x \in C$ such that

$$\langle Ax - f, y - x\rangle \geq 0, \forall y \in C, \tag{1.1}$$

is called a *variational inequality (VI)*. In the last 40 years, variational inequalities have turned out to be appropriate in mathematical modeling. To prove the existence of solutions of VIs in the case of singlevalued operators, Brezis ([1]) and Karamardian ([8]) introduced different concepts of pseudomonotonicity. Throughout the paper, we suppose that X is a reflexive Banach space as infinite-dimensional commodity space, and we replace nets by the usual sequences.

Starting from the monotonicity condition of the operator

$$\langle Ax - Ay, x - y\rangle \geq 0, \forall x, y \in D(A),$$

or $\langle Ax, x-y\rangle \geq \langle Ay, x-y\rangle, \forall x, y \in D(A)$, Karamardian defined the following algebraic pseudomonotonicity: $A : D(A) \to X^*$ is said to be *a-pseudomonotone* if

$$\langle Ay, x - y\rangle \geq 0 \qquad \text{implies} \qquad \langle Ax, x - y\rangle \geq 0, \forall x, y \in D(A).$$

Clearly, a-pseudomonotonicity is weaker then the monotonicity condition. On the order hand, Brezis ([1]) introduced a topological pseudomonotonicity, namely, A :

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$D(A) \to X^*$ is *b-pseudomonotone* if for every sequence $\{x_n\} \subset D(A)$ the conditions $x_n \rightharpoonup x$ in X and $lim\ sup\langle Ax_n, x_n - x\rangle \leq 0$ imply that

$$\langle Ax, x - y\rangle \leq lim\ inf\langle Ax_n, x_n - y\rangle\ for\ all\ y \in D(A).$$

Recall that $A : D(A) \to X^*$ is *hemicontinuous* if the function $t \longmapsto \langle T(x + ty), z\rangle$ is continuous on [0,1] for all $x, y, z \in D(A)$. Note that, a monotone hemicontinuous operator A is b-pseudomonotone.

The above pseudomonotonicities are extended by the following definition: an operator $A : D(A) \to X^*$ is called *c-pseudomonotone* if for every sequence $\{x_n\} \subset D(A)$ the conditions $x_n \rightharpoonup x$ in X and $\langle Ax_n, (1-t)x+ty-x_n\rangle \geq 0$ for all $t \in [0, 1], y \in D(A)$ and $n \in \mathbb{N}$, imply that $\langle Ax, y - x\rangle \geq 0$.

It has been proved in [4] that both any b-pseudomonotone and hemicontinuous a-pseudomonotone operators are c-pseudomonotone. Now, we denote by Conv (X^*) the collection of all convex closed subsets of X^* and introduce, for a set-valued mapping $T : D(T) \subseteq$X$\to Conv(X^*)$, the *upper* and *lower support* functions by

$$[T(x), y]_+ = \sup_{x^* \in T(x)} \langle x^*, y\rangle \qquad \text{and} \qquad [T(x), y]_- = \inf_{x^* \in T(x)} \langle x^*, y\rangle,$$

where the *upper norm* on Conv (X^*) defined by $\|T(x)\|_+ = \sup_{x^* \in T(x)} \|x^*\|_{X^*}$. We distinguish $T(x)$ and $\overline{co}T(x)$, the minimal closed convex set which contains $T(x)$, and let the graph $G(\overline{co}T) = \{(x, x^*) \in D(T) \times X^* : x^* \in \overline{co}T(x)\}$.

Using the above notations, we will be able to present, in a simpler way, three extensions of the pseudomonotonicity proposed by Inoan and Kolumban ([7]) in the case of a set-valued mapping $T : D(T) \subseteq X \to 2^{X^*}$.

Definition 1. *The map T is said to be* α-pseudomonotone *if,* $[T(y), x - y]_+ \geq 0$ *implies* $[T(x), x - y]_+ \geq 0$, *for every* $x, y \in D(T)$.

This definition represents an extension of the algebraic pseudomonotonicity introduced by Karamardian [8].

Definition 2. *The map T is* β-pseudomonotone *if, for every sequence* $\{x_n\} \subset D(T)$ *the coditions* $x_n \rightharpoonup x$ *in* X *and*

$$\lim_{n\to\infty} [T(x_n), x_n - x]_+ \leq 0$$

imply that to each $y \in D(T)$ *there corresponds* $y^* \in T(x)$ *such that*

$$\langle y^*, x - y\rangle \leq \lim_{n\to\infty} [T(x_n), x_n - y]_-.$$

This definition extends the topological pseudomonotonicity considered by Brezis [1].

Definition 3. *The map T is* γ-pseudomonotone *if, for every* $x, y \in D(T)$ *and every sequence* $\{x_n\} \subset D(A)$, *the conditions* $x_n \rightharpoonup x$ *in* X *and*

$$[T(x_n), x_n - (1 - t)x + ty]_- \leq 0, \forall t \in [0, 1], \forall n \in \mathbb{N},$$

imply that $[T(x), y - x]_+ \geq 0$.

Likewise, it has proved in [7] that both *α-pseudomonotone* and *β-pseudomonotone* mappings are *γ-pseudomonotone*, for certain classes of multivalued mappings.

We are in position to introduce a general definition of pseudomonotonicity useful to the solution of variational inequalities. Let C be a closed convex set of a real reflexive Banach space X.

Definition 4. *The mapping $T : C \to 2^{X^*}$ is said to be* pseudomonotone *if the following conditions hold:* (i) *For each $x \in C$, the image $T(x)$ is nonempty bounded closed and convex subset in X^**; (ii) *For each finite-dimensional subspace F of X, the map T is upper semicontinuous from $C \cap F$ into 2^{X^*}, with X^* given its weak topology;* (iii) *If $\{x_n\}$ is a sequence in C converging weakly to $x \in C$ and $\lim_{n\to\infty}[T(x_n), x - x_n]_- \geq 0$, to each $y \in C$ there corresponds an element $y^* \in T(x)$, with the property that*

$$\lim_{n\to\infty}[T(x_n), y - x_n]_+ \leq \langle y^*, y - x\rangle.$$

Of course, the condition (iii) can be extended by the *γ-pseudomonotonicity* of the mapping T.

Moreover, Hess and Browder ([3]) introduced a more general concept:

Definition 5. *The mapping $T : C \to 2^{X^*}$ is said to be* generalized pseudomonotone *if it verifies (i) - (ii) while (iii) will be replaced by*
(iii') *for any sequence $\{(x_n, x_n^*)\} \subset G(\overline{co}T)$ such that $(x_n, x_n^*) \rightharpoonup (x, x^*)$ in $X \times X^*$ and*

$$\lim_{n\to\infty}[x_n^*, x - x_n] \geq 0,$$

it follows that $x^* \in \overline{co}T(x)$ and $\langle x_n^*, x\rangle \to \langle x^*, x\rangle$.

It is easy to check that every pseudomonotone mapping is generalized pseudomonotone and a bounded generalized pseudomonotone mapping is pseudomonotone [11]. Along with the initial problem (1.1), we will investigate the multivalued variational inequality (MVI), i. e., finding an element $u \in C$ such that

$$[T(u), y - u]_+ \geq \langle f, y - u\rangle, \forall y \in C,$$

where $C \subset D(T)$ be a nonempty closed convex subset of a reflexive Banach space X, (see, e. g. [14]).

In proving the existence of solutions (VMI)'s we are based on the following:

Theorem 1. *(The Fixed Point Theorem [2]) Let K be a nonempty compact and convex set in a locally convex space X and $S : K \to 2^K$ a multivalued mapping with the properties that $S(x)$ is a nonempty convex subset in X^* for all $x \in K$ and the preimages $S^{-1}(y)$ are relatively open with respect to K for all $y \in K$. Then S has a fixed point.*

Now we consider an auxiliary multivalued variational inequality for a mapping $T : K \to 2^{X^*}$. Find a pair such that

$$\langle g, v - u\rangle \geq 0 \;\; for \;\; all \;\; v \in K. \tag{1.2}$$

As a consequence of Browder's theorem, we extend a result from [6] to multivalued mappings:

Theorem 2. *Assume that the mapping $T : K \to 2^{X^*}$ is locally bounded, upper semi-continuous and generalized pseudomonotone. Then the variational inequality* (1.2) *has a solution $(u, g) \in K \times T(u)$.*

Proof. In the contrary case, to each $h \in T(u)$ would correspond an element $w \in K$ such that

$$\langle h, w - u\rangle < 0. \tag{1.3}$$

Define the multivalued map $S : K \to 2^K$ by $S(u) := \{w \in K : \langle h, w - u\rangle < 0\}$. According to assumption (1.3), the set $S(u)$ is nonempty for all $u \in K$. Moreover, $S(u)$ is convex. We show that its preimage

$$S^{-1}(w) := \{u \in K : \langle h, w - u\rangle < 0\}$$

is relatively open in K. Let $\{u_n\}$ be a sequence in $K - S^{-1}(w)$ with $u_n \to z$ and $h_n \in T(u_n)$, so that

$$\langle h_n, w - u_n\rangle \geq 0 \qquad \text{or} \qquad \langle h_n, u_n - w\rangle \leq 0, \ \ \forall n \in \mathbb{N}.$$

From the local boundedness of T, we can assume that $h_n \rightharpoonup h$ in X^*. The generalized pseudomonotonicity of T implies that $h \in T(z)$. Thus, $K - S^{-1}(w)$ is relatively closed and $S^{-1}(w)$ is relatively open in K.

In virtue of Browder's Theorem, there exists a fixed point $u \in S(u)$. This leads to the contradiction $\langle h, u - u\rangle < 0$. Hence, there exists a $g \in T(u)$ with $u \in K$, satisfying of the variational inequality (1.2). □

Similar to [3, Proposition 16], using a finite-dimensional procedure, we can replace the compactness condition on C by the coerciveness of T. Considering a reflexive Banach space X and a closed convex subset C of X, a multivalued map $T : C \to 2^{X^*}$ is said to be *coercive* if there exists a function $c : R^+ \to R$ with $c(r) \to \infty$ as $r \to \infty$, such that $\langle f, x\rangle \geq c(\|x\|)\,\|x\|$ for all $x \in C$ and $f \in T(x)$. In a manner similar to [3, Proof of Theorem 15], we establish:

Theorem 3. *Let C be a nonempty closed convex subset of a real reflexive Banach space X. Assume that $0 \in C$ and $T : C \to 2^{X^*}$ is a coercive generalized pseudomonotone mapping. Then for each $f \in X^*$, there exists a solution $u \in C$ of the multivariational inequality*

$$[T(u), y - u]_+ \geq \langle f, y - u\rangle, y \in C.$$

Related to the condition (iii) in Definition 4, a subclass of multivalued monotone maps has been introduced by:

Definition 6. *[12]. The mapping $T : C \to 2^{X^*}$ is of type (S) if it satisfies* (i) - (ii) *while* (iii) *will be replaced by*
(iii") *If $\{x_n\}$ is a sequence in C converging weakly to $x \in X$ such that*

$$lim\ sup\langle f_n, x_n - x\rangle \leq 0 \ \ for \ \ some \ \ f_n \in Tx_n,$$

it follows that $x_n \longrightarrow x \in C$ and $\{f_n\}$ has a subsequence which converges weakly to $f \in Tx$ in X^*.

Using the above notations, we have also:

Definition 7. *[7]. The mapping* $T : C \to Conv(X^*)$ *is of type* (α) *if for any sequence* $\{x_n\} \subset C$ *converging weakly to* $x \in X$ *such that*

$$\lim_{n\to\infty} [T(x_n), x_n - x]_+ \leq 0,$$

there exists a subsequence $\{x_m\} \subset \{x_n\}$ *such that* $x_m \to x$ *in* C.

The alikeness between the mappings of type (S) and of type (α) are discussed in Skrypnik's book [13]. The mappings of type (S) are a good substitute of compactness in proving the existence of solutions of the nonlinear problems, in particular, of the general variational inequalities.

The similarity with compact operators directed the first author ([5]) to introduce a topological degree for the multivalued mappings of type (S). On the other hand, the variational inequalities can be converted to inclusions defined by sums between mappings of monotone-like and a subdifferentials. The second author ([10]) surveyed the actual progress in defining an appropriate topological degree for operators (S) with maximal monotone perturbations.

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