Kaehler submanifolds with the second fundamental form satisfying certain conditions

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Dedicated to the 70-th anniversary
of Professor Constantin Udriste

Abstract. Let M^n be a Kaehler submanifold in a complex space form and x any point of M^n. Then there exists a neighborhood U of x which a local field ξ of any normal vector and the second fundamental form A_ξ in the direction of ξ are defined on U. In the present paper we will give a characterization of a Kaehler submanifold in a real space form which satisfies $(R(X,Y)A_\xi)Z = 0$ for all vectors X, Y and Z tangent to M^n and A_ξ in the direction of any normal ξ, where R is the curvature tensor of M^n which is defined on U.

M.S.C. 2000: 53C40, 53B25.
Key words: Complex space form; Kaehler submanifold; totally geodesic.

1 Introduction

Let $\tilde{M}^{n+p}(c)$ be a complex $(n+p)$-dimensional complex space form of constant holomorphic sectional curvature c (i.e. complete, simply connected Kaehler manifold with constant holomorphic sectional curvature, say, c). For each real number c, there is (up to holomorphic isometry) exactly one complex space form in every dimension with holomorphic sectional curvature c. The complex space forms of holomorphic sectional curvature c are denoted by $P^{n+p}(C)$, C^{n+p} and D^{n+p} depending on whether c is positive, zero or negative, respectively. $P^{n+p}(C)$ is the complex projective space with Fubini-Study metric of constant holomorphic sectional curvature c, C^{n+p} is the complex Euclidean space. D^{n+p} is the open unit ball in C^{n+p} endowed with Bergman metric of constant holomorphic sectional curvature c.

Let M^n be a connected manifold of complex dimension $n(\geq 2)$ isometrically and holomorphically immersed in $\tilde{M}^{n+p}(c)$. Then we call M^n a Kaehler submanifold of $\tilde{M}^{n+p}(c)$. We denote the complex structure by $\tilde{J}$ (resp. J) and the Kaehler metric by $\tilde{g}$ (resp. g) of $\tilde{M}^{n+p}(c)$ (resp. M^n). Let $\tilde{\nabla}$ (resp. ∇) denote the covariant differentiation in $\tilde{M}^{n+p}(c)$ (resp. M^n). Now, we denote by A_ξ the second fundamental form in the direction of ξ which is defined on a neighborhood U of any x of M^n. Let

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BSG Proceedings 17. The International Conference "Differential Geometry-Dynamical Systems 2009" (DGDS-2009), October 8-11, 2009, Bucharest-Romania, pp. 132-139.

R be the curvature tensor of M^n. Mashiko, Kurosu and Matsuyama ([3]) showed: If a Kaehler hypersurface M^n of $\tilde{M}^{n+1}(c)$ satisfies

$$(R(X,Y)A)Z = 0,$$

then M^n is totally geodesic, where A denotes the second fundamental form. In the case of a real submanifold of a real space form Matsuyama ([4]) proved: Let M^n be a real n-dimensional manifold which is minimally immersed in a real $(n+p)$-dimensional space form $\tilde{M}^{n+p}(c)$ of constant curvature c. We assume that

$$(R(X,Y)A_\xi)Z = 0$$

for all vectors X, Y and Z tangent to M^n and any normal ξ. (1) If M^n is compact (or the scalar curvature of M^n is constant), $c > 0$ and the length $||\sigma||^2$ of the second fundamental forms satisfies $||\sigma||^2 \geq pnc$, then M^n is parallel, $||\sigma||^2 = pnc$ and M^n is classified. (2) If the scalar curvature of M^n is constant and $c \leq 0$, then M^n is totally geodesic.

In the case of a minimal Lagrangian submanifold of a complex projective space Matsuyama ([5]) also proved: Let M^n be a minimal Lagrangian submanifold in $CP^n(c)$ which satisfies

$$(R(X,Y)A_\xi)Z = 0$$

for the shape operator A_ξ in the direction of any normal ξ. Then M^n is totally geodesic or locally congruent to a flat torus minimally embedded in $P^2(C)$ with parallel second fundamental form.

The purpose of this paper is to consider a characterization of a Kaehler submanifold in a complex space form which satisfies $(R(X,Y)A_\xi)Z = 0$ for all vectors X, Y and Z tangent to M^n and A_ξ in the direction of any normal ξ. Then we obtain the following:

Theorem. *Let M^n be a complex n-dimensional Kaehler manifold which is holomorphically and isometrically immersed in a complex space form $\tilde{M}^{n+p}(c)$ of constant holomorphic sectional curvature c and R the curvature tensor of M^n. Let x be any point of M^n. Then there exists a neighborhood U of x which local field ξ of normal vectors and the second fundamental form A_ξ in the direction of ξ are defined on U. We assume that*

$$(R(X,Y)A_\xi)Z = 0$$

for all vectors X, Y and Z tangent to M^n and any normal vector ξ. Then M^n totally geodesic.

2 Preliminaries

Let M^n be a connected manifold of complex dimension n (≥ 2) isometrically and holomorphically immersed in a complex space form $\tilde{M}^{n+p}(c)$ of complex dimension $n+p$. Then we call M^n a Kaehler submanifold of $\tilde{M}^{n+p}(c)$. The complex structure $\tilde{J}$ and the Kaehler metric $\tilde{g}$ of $\tilde{M}^{n+p}(c)$ induce the complex structure J and the Kaehler metric g on M^n, respectively. Let ∇ (resp. $\tilde{\nabla}$) denote the covariant differentiation in M^n (resp. $\tilde{M}^{n+p}(c)$). Extend ξ to a normal vector field defined in a neighborhood U

of $x \in M^n$ and define $-A_\xi X$ to be the tangential component of $\tilde{\nabla}_X \xi$ for $X \in T_x M$. $A_\xi X$ depends only on ξ at x and X, and we call A_ξ the second fundamental form. Let R be the curvature tensor of M^n and X, Y and Z the tangent vectors on M^n. Then we have the following relationships ([1], [2]):

$$\begin{aligned}
&(2.1) & \tilde{\nabla}_X Y &= \nabla_X Y + \sum_\alpha g(A_\alpha X, Y)\xi_\alpha + \sum_\alpha g(JA_\alpha X, Y)J\xi_\alpha, \\
&(2.2) & g(A_\alpha X, Y) &= g(X, A_\alpha Y), \\
&(2.3) & \tilde{\nabla}_X \xi_\alpha &= -A_\alpha X + \sum_\beta s_{\alpha\beta}(X)J\xi_\beta, \\
&(2.4) & s_{\alpha\beta} &+ s_{\beta\alpha} = 0, \\
&(2.5) & A_\alpha J &= -JA_\alpha, \\
&(2.6) & R(X,Y)Z &= \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\
& & &\quad -g(JX,Z)JY + 2g(X,JY)JZ\} \\
& & &\quad +\sum_\alpha g(A_\alpha Y, Z)A_\alpha X - \sum_\alpha g(A_\alpha X, Z)A_\alpha Y \\
& & &\quad +\sum_\alpha g(JA_\alpha Y, Z)JA_\alpha X - \sum_\alpha g(JA_\alpha X, Z)JA_\alpha Y,
\end{aligned}$$

— Gauss equation

$$(2.7) \qquad (\nabla_X A_\alpha)Y - \sum_\beta s_{\alpha\beta}(X)JA_\beta Y = (\nabla_Y A_\alpha)X - \sum_\beta s_{\beta\alpha}(Y)JA_\beta X,$$

— Codazzi equation

where we write $A_\alpha = A_{\xi_\alpha}$.

Now, we prepare the following results without proof.

Theorem A.(See [6], [8], [7], [10]). *If M^n is a Kaehler hypersurface in a complex space form $\tilde{M}^{n+p}(c)$, then the following conditions are equivalent on M^n*

(1) *M^n is locally symmetric.*

(2) *M^n is of Einstein.*

Theorem B.(See [6]). *Let M^n be a Kaehler hypersurface of complex dimension $n \geq 1$ in a complex space form $\tilde{M}^{n+1}(c)$ of constant holomorphic sectional curvature c. If M^n is of Einstein, then M^n is locally symmetric and either M^n is of constant holomorphic sectional curvature c and totally geodesic in $\tilde{M}^{n+1}(c)$ or M^n is locally holomorphically isometric to the complex quadric Q^n in the complex projective space* $P^{n+1}(c)$, the latter case arising only when $c > 0$.

We next must introduce the concept of the first normal space of M^n at $x \in M^n$.

Definition. For $x \in M^n$, the *first normal space*, $N_1(x)$, is the orthogonal complement in $T_x(M^n)$ of the set

$$N_0(x) = \{\xi \in T_x | A_\xi = 0\}$$

We define a new inner product, $<,>$, on $N_1(x)$ by

$$< J\xi, J\eta >= \mathrm{trace} A_\xi A_\eta \text{ for } \xi, \eta \in N_1(x).$$

One easily checks that $<,>$ is a positive definite inner product on $N_1(x)$, and that for $\xi, \eta \in N_1(x)$,

$$< J\xi, J\eta >=< \xi, \eta > \text{ and } < \xi, J\xi >= 0.$$

For $\xi \in N_1(x)$ we assume that $A_\xi^2 = \lambda^2 I$ for $\lambda > 0$. Then it is well known the following:
Theorem C.(See [1]). *Let $x \in M^n$ and let k be the complex dimension of $N_1(x)$. Then $k \leq 1$.*

3 Proof of Theorem

Now, we prove the following theorem:
Theorem. *Let M^n be an n-dimensional manifold which is isometrically and holomorphically immersed in a complex space form $\tilde{M}^{n+p}(c)$ of constant holomorphic sectional curvature c and R the curvature tensor of M^n. Let x be any point of M^n. Then there exists a neighborhood U of x which local field ξ of normal vectors and the second fundamental form A_ξ in the direction of ξ are defined on U. We assume that*

$$(R(X,Y)A_\xi)Z = 0 \tag{3.1}$$

for all vectors X, Y and Z tangent to M^n and any normal vector ξ. Then M^n is totally geodesic.
Proof. Let M^n be a connected manifold of complex dimension n (≥ 2) holomorphically and isometrically immersed in a complex space form $\tilde{M}^{n+p}(c)$. Setting $A_\xi X$ in the place of X in (3.1) , i.e.,

$$(R(A_\xi X,Y)A_\xi)Z = 0 \tag{3.2}$$

for X, Y and $Z \in T_x(M)$ and using (2.6) , we have

$$\begin{aligned}
&\frac{c}{4}\{g(Y,A_\xi Z)A_\xi X - g(A_\xi X, A_\xi Z)Y + g(JY, A_\xi Z)JA_\xi X \\
&-g(JA_\xi X, A_\xi Z)JY + 2g(A_\xi X, JY)JA_\xi Z\} \\
&+\sum_\alpha g(A_\alpha Y, A_\xi Z)A_\alpha A_\xi X - \sum_\alpha g(A_\alpha A_\xi X, A_\xi Z)A_\alpha Y \\
&+\sum_\alpha g(JA_\alpha Y, A_\xi Z)JA_\alpha A_\xi X - \sum_\alpha g(JA_\alpha A_\xi X, A_\xi Z)JA_\alpha Y \\
&-\frac{c}{4}\{g(Y,Z)A_\xi^2 X - g(A_\xi X, Z)A_\xi Y + g(JY,Z)A_\xi JA_\xi X \\
&-g(JA_\xi X, Z)A_\xi JY + 2g(A_\xi X, JY)A_\xi JZ\} \\
&-\sum_\alpha g(A_\alpha Y, Z)A_\xi A_\alpha A_\xi X + \sum_\alpha g(A_\alpha A_\xi X, Z)A_\xi A_\alpha Y \\
&-\sum_\alpha g(JA_\alpha Y, Z)A_\xi JA_\alpha A_\xi X - \sum_\alpha g(JA_\alpha A_\xi X, Z)A_\xi JA_\alpha Y = 0.
\end{aligned} \tag{3.3}$$

We choose a local field of orthonormal frames $e_1, \ldots, e_{2n}$. Putting $Y = e_j$ and $Z = e_j$ in (3.3) , we have

$$
\begin{aligned}
&\frac{c}{4}\{g(e_j, A_\xi e_j)A_\xi X - g(A_\xi X, A_\xi e_j)e_j + g(Je_j, A_\xi e_j)JA_\xi X \\
&-g(JA_\xi X, A_\xi e_j)Je_j + 2g(A_\xi X, Je_j)JA_\xi e_j\} \\
&+\sum_\alpha g(A_\alpha e_j, A_\xi e_j)A_\alpha A_\xi X - \sum_\alpha g(A_\alpha A_\xi X, A_\xi e_j)A_\alpha e_j \\
&+\sum_\alpha g(JA_\alpha e_j, A_\xi e_j)JA_\alpha A_\xi X - \sum_\alpha g(JA_\alpha A_\xi X, A_\xi e_j)JA_\alpha e_j \\
&-\frac{c}{4}\{g(e_j, e_j)A_\xi^2 X - g(A_\xi X, e_j)A_\xi e_j + g(Je_j, e_j)A_\xi JA_\xi X \\
&-g(JA_\xi X, e_j)A_\xi Je_j + 2g(A_\xi X, Je_j)A_\xi Je_j\} \\
&-\sum_\alpha g(A_\alpha e_j, e_j)A_\xi A_\alpha A_\xi X + \sum_\alpha g(A_\alpha A_\xi X, e_j)A_\xi A_\alpha e_j \\
&-\sum_\alpha g(JA_\alpha e_j, e_j)A_\xi JA_\alpha A_\xi X - \sum_\alpha g(JA_\alpha A_\xi X, e_j)A_\xi JA_\alpha e_j = 0.
\end{aligned}
$$

Taking the summation on j, we obtain

$$
\begin{aligned}
&-\frac{n+3}{2}cA_\xi^2 X + \sum_\alpha (\mathrm{trace} A_\xi A_\alpha)A_\alpha A_\xi X \\
&+\sum_\alpha (\mathrm{trace} A_\xi JA_\alpha)JA_\alpha A_\xi X + 2\sum_\alpha A_\xi A_\alpha^2 A_\xi X = 0.
\end{aligned}
$$

Setting $\lambda = 1, \ldots, 2p$, we obtain

$$
-\frac{n+3}{2}cA_\xi^2 X + \sum_\lambda (\mathrm{trace} A_\xi A_\lambda)A_\lambda A_\xi X + \sum_\lambda A_\xi A_\lambda^2 A_\xi X = 0.
$$

Taking the summation on β, we get

$$
(3.4) \qquad -\frac{n+3}{2}c\sum_\beta A_\beta^2 X + \sum_{\beta,\lambda} (\mathrm{trace} A_\beta A_\lambda)A_\lambda A_\beta X + \sum_{\beta,\lambda} A_\beta A_\lambda^2 A_\beta X = 0.
$$

Putting $X = e_j$ and $Z = e_j$ in (3.3) , we have

$$\begin{aligned}
&\frac{c}{4}\{g(Y, A_\xi e_j)A_\xi e_j - g(A_\xi e_j, A_\xi e_j)Y + g(JY, A_\xi e_j)JA_\xi e_j\\
&-g(JA_\xi e_j, A_\xi e_j)JY + 2g(A_\xi e_j, JY)JA_\xi e_j\}\\
&+\sum_\alpha g(A_\alpha Y, A_\xi e_j)A_\alpha A_\xi e_j - \sum_\alpha g(A_\alpha A_\xi e_j, A_\xi e_j)A_\alpha Y\\
&+\sum_\alpha g(JA_\alpha Y, A_\xi e_j)JA_\alpha A_\xi e_j - \sum_\alpha g(JA_\alpha A_\xi e_j, A_\xi e_j)JA_\alpha Y\\
&-\frac{c}{4}\{g(Y, e_j)A_\xi^2 e_j - g(A_\xi e_j, e_j)A_\xi Y + g(JY, e_j)A_\xi JA_\xi e_j\\
&-g(JA_\xi e_j, e_j)A_\xi JY + 2g(A_\xi e_j, JY)A_\xi Je_j\}\\
&-\sum_\alpha g(A_\alpha Y, e_j)A_\xi A_\alpha A_\xi e_j + \sum_\alpha g(A_\alpha A_\xi e_j, e_j)A_\xi A_\alpha Y\\
&-\sum_\alpha g(JA_\alpha Y, e_j)A_\xi JA_\alpha A_\xi e_j + \sum_\alpha g(JA_\alpha A_\xi e_j, e_j)A_\xi JA_\alpha Y = 0.
\end{aligned}$$

Taking the summation on j, we obtain

$$\begin{aligned}
&\frac{c}{4}\{A_\xi^2 Y - (\mathrm{trace}A_\xi^2)Y - 2A_\xi^2 Y\}\\
&+\sum_\alpha A_\alpha A_\xi^2 A_\alpha Y - \sum_\alpha(\mathrm{trace}A_\xi A_\alpha A_\xi)A_\alpha Y\\
&+\sum_\alpha A_\alpha A_\xi^2 A_\alpha Y - \sum_\alpha(\mathrm{trace}A_\xi JA_\alpha A_\xi)JA_\alpha Y\\
&-\frac{c}{4}\{A_\xi^2 Y + A_\xi^2 Y + 2A_\xi^2 Y\}\\
&-\sum_\alpha A_\xi A_\alpha A_\xi A_\alpha Y + \sum_\alpha(\mathrm{trace}A_\alpha A_\xi)A_\xi A_\alpha Y\\
&+\sum_\alpha A_\xi A_\alpha A_\xi A_\alpha Y + \sum_\alpha(\mathrm{trace}JA_\alpha A_\xi)A_\xi JA_\alpha Y = 0
\end{aligned}$$

Hence we get

$$\begin{aligned}
&-\frac{3}{2}cA_\xi^2 Y - \frac{c}{4}(\mathrm{trace}A_\xi^2)Y + \sum_\lambda A_\lambda A_\xi^2 A_\lambda Y\\
&-\sum_\lambda(\mathrm{trace}A_\xi A_\lambda A_\xi)A_\lambda Y + \sum_\lambda(\mathrm{trace}A_\lambda A_\xi)A_\xi A_\lambda Y = 0.
\end{aligned} \tag{3.5}$$

Taking the summation on β, we obtain

$$\begin{aligned}
&-\frac{3}{2}c\sum_\beta A_\beta^2 Y - \frac{c}{4}\sum_\beta(\mathrm{trace}A_\beta^2)Y + \sum_{\beta,\lambda} A_\lambda A_\beta^2 A_\lambda Y\\
&+\sum_{\beta,\lambda}(\mathrm{trace}A_\lambda A_\beta)A_\beta A_\lambda Y = 0,
\end{aligned} \tag{3.6}$$

since trace$A_\xi A_\lambda A_\xi = 0$ for each λ.
Comparing (3.4) with (3.6) , we obtain

$$-\frac{n+3}{2}c\sum_\mu A_\mu^2 = -\frac{3}{2}c\sum_\mu A_\mu^2 - \frac{c}{4}\sum_\mu \mathrm{trace}A_\mu^2,$$

where $\mu = 1, \dots, 2p$. Then we define $||\sigma||^2$ by

$$||\sigma||^2 = \sum_\lambda \mathrm{trace}A_\lambda^2.$$

If we assume that $c \neq 0$, then we get

$$\sum_\lambda A_\lambda^2 = \frac{||\sigma||^2}{2n}I,$$

where I is the identity transformation. From the equation we see that M is Einstein.
If $c = 0$, then, from (3.6) we have

$$2\sum_\alpha(\mathrm{trace}A_\alpha^2)^2 + 4\mathrm{trace}(\sum_\alpha A_\alpha^2)^2 = 0.$$

Hece we get

$$A_\alpha = 0.$$

We consider the case of $c < 0$. In the case it is well known that M which is of Eintein is totally geodesic (See [9]).
We consider the case of $c > 0$. We choose an orthonormal frames $\xi_i, \dots, \xi_p, J\xi_i, \dots, J\xi_p$ such that trace$A_\xi A_\eta = 0$ for normal vectors ξ, η, $\xi \neq \eta$ Then from (3.4) we have

$$-\frac{n+3}{2}cA_\xi^2X + (\mathrm{trace}A_\xi^2)A_\xi^2X + \frac{||\sigma||^2}{2n}A_\xi^2X = 0. \tag{3.7}$$

Setting JA_ξ int the place of A_ξ in (3.5) , we have

$$-\frac{3}{2}cA_\xi^2Y - \frac{c}{4}(\mathrm{trace}A_\xi^2)Y + \sum_\lambda A_\lambda A_\xi^2 A_\lambda Y + (\mathrm{trace}A_\xi^2)A_\xi^2Y = 0, \tag{3.8}$$

since trace$JA_\xi A_\lambda JA_\xi = -$trace$A_\xi A_\lambda A_\xi = 0$. Noting that A_λ is symmetric, we obtain

$$-\frac{3}{2}cA_\xi^2Y - \frac{c}{4}(\mathrm{trace}A_\xi^2)Y + \frac{||\sigma||^2}{2n}A_\xi^2Y + (\mathrm{trace}A_\xi^2)A_\xi^2Y = 0, \tag{3.9}$$

Comparing (3.7) with (3.9) we get

$$-\frac{n+3}{2}cA_\xi^2 = -\frac{3}{2}cA_\xi^2 - \frac{c}{4}(\mathrm{trace}A_\xi^2)I,$$

i.e.,

$$A_\xi^2 = \frac{\mathrm{trace}A_\xi^2}{2n}I.$$

Thus we know that $p = 1$ (See [1]). Hence we obtain the conclusion (See [3]).

References

- [1] T. E. Cecil, *Geometric applications of critical point theory to submanifolds of complex projective space,* Nagoya Math. J. 55 (1974), 5-31.
- [2] J. Erbacher, *Reduction of the codimension of an siometric immersion,* J. Differential Geometry 5 (1971), 333-340.
- [3] Y Mashiko, S. Kurosu and Y. Matsuyama, *On a Kaehler hypersurfaces of a complex space form with recurrent second fundamental form,* Differ. Geom. Dyn. Syst. (DGDS) 10 (2008), 221-225.
- [4] Y. Matsuyama, *A characterization of minimal submanifolds in a real space form,* preprint.
- [5] Y. Matsuyama, *A characterization of minimal Lagrangian submanifolds in a complex projective space,* Int. J. Pure Appl. Math. 44 (2008), 363-372.
- [6] K. Nomizu and B. Smyth, *Differential geometry of complex hypersurfaces II,* J. Math. Soc. Japan 20 (1968), 498-521.
- [7] P. J. Ryan, *A class of complex hypersurfaces,* Colloquium Mathematicum 26 (1972), 177-182.
- [8] B. Smyth, *Differential geometry of complex hpersurfaces,* Ann. of Math. 85 (1967), 246-266.
- [9] M. Umehara, *Einstein Kaehler subamnifolds of a complex linear or hyperbolic space,* Tohoku Math. J. 39 (1987), 385-389.
- [10] P. J. Ryan, *Hypersurfaces with parallel Ricci tensor,* Osaka. J. Math. 8(1971), 251-259.

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