Weak solutions for the pseudo-Laplacian Δ_p^s using a perturbed variational principle

Irina Meghea

Dedicated to the 70-th anniversary
of Professor Constantin Udriste

Abstract. This paper considers two approach methods to obtain and characterize weak solutions for some problems of mathematical physics equations. In the first method, a perturbed variational principle is involved in order to obtain some results characterizing weak solutions. In the second approach, some surjectivity theorems and other theorems of the Fredholm alternative type for operators of the form $\lambda J_\varphi - S$, where J_φ is the duality map and S is the Nemytskii operator, are established to state two original results, which describe new properties of the pseudo-Laplacian.

M.S.C. 2000: 35J05, 35J35, 35S05, 47G30.
Key words: Perturbed variational principle; weak solution; Nemytskii operator; Carathéodory function; Sobolev space; pseudo-Laplacian; duality map.

1 Introduction

This paper is based on the organization of the concepts from [11] corroborated with the results obtained by the author in [12]. The general theory is developed towards two distinct trends. In the first, starting with a linear variational principle ([4], [5], [11]), and following a work of Ghoussoub [7] and the article [12], a variational method to discuss some problems of partial differential equations has been presented in the manner of [11] and [12]. In the second direction, a series of propositions from [6], [3], [8] and others are generalized or used in order to obtain a sequence of original results.

The aim of this work is to compare these two approaches, and particulary to highlight new results in weak solutions for some types of partial differential equations.

2 Critical points and weak solutions for elliptic type equation

In the first approach which characterizes weak solutions for equations involving the pseudo-Laplacian Δ_p^s, we give a *linear* perturbed variational principle. To introduce these results, we first provide the following

BSG Proceedings 17. The International Conference "Differential Geometry-Dynamical Systems 2009" (DGDS-2009), October 8-11, 2009, Bucharest-Romania, pp. 140-150.

Definition. Let X be a real normed space, $f : X \to (-\infty, +\infty]$, C a nonempty subset of X and $x_0 \in C$. *f strongly exposes C from below in x_0*, when

1° $f(x_0) = \inf f(C) < +\infty$, and

2° $x_n \in C \ \forall\, n \geq 1, f(x_n) \to f(x_0) \Rightarrow x_n \to x_0$.

When $C = X$, we get the definition of the strongly minimum point ([11, I, above 1.2]). In the same manner we define: *f strongly exposes C from above at x_0.*

Theorem 1 (The Ghoussoub - Maurey linear principle). *Let X be a reflexive separable space and let $\varphi : X \to (-\infty, +\infty]$ be lower semicontinuous and proper.*

(I) If φ is bounded from below on the closed bounded nonempty subset C, then the set

$$\{\xi \in X^* : \varphi + \xi \text{ strongly exposes } C \text{ from below}\}$$

is an everywhere dense G_δ -set.

(II) If, for any ξ from X^*, $\varphi + \xi$ is bounded from below, then the set

$$\{\xi \in X^* : \varphi + \xi \text{ strongly exposes } X \text{ from below}\}$$

is an everywhere dense G_δ -set.

Remark. We consider the fact that a set is of G_δ *type* when it is a countable intersection of open sets.

We use Theorem 1 for a generalization of the minimization problem form [2] of the form:

$$C_f := \min\left\{\int_\Omega \left(\frac{1}{p}\sum_{i-1}^{N}\left|\frac{\partial u}{\partial x_i}\right| - f(u)\right)\mathrm{d}x : u \in W_0^{1,p}(\Omega), ||u||_{2^*} = 1\right\}, \tag{2.1}$$

where Ω is an open set of C^1 class in $\mathbb{R}^N$, $N \geq 3$, $f \in W^{-1,p'}(\Omega)$ $(= (W_0^{1,p}(\Omega))^*)$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1, 2^* = \dfrac{2N}{N-2}$ – the critical exponent for the Sobolev embedding (for explanations, here and in the following, see [11, 1, §4]).

Let Ω be an open bounded set of class C^1 in $\mathbb{R}^N$, $N \geq 3$. Consider the problem

$$\begin{cases} -\Delta_p^s u = f(x,u) & in \quad \Omega \\ u = 0 & on \quad \partial\Omega, \end{cases} \tag{2.2}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with the growth condition

$$|f(x,s)| \leq c|s|^{p-1} + b(x), \tag{2.3}$$

$$c > 0, 2 \leq p \leq \frac{2N}{N-2}, b \in \mathbb{L}^{p'}(\Omega), \frac{1}{p} + \frac{1}{p'} = 1.$$

The functional $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$,

$$\varphi(u) = \int_\Omega \left(\frac{1}{p}\sum_{i-1}^{N}\left|\frac{\partial u}{\partial x_i}\right|^p - F(x,u(x))\right)\mathrm{d}x, \quad F(x,s) := \int_0^s f(x,t)\mathrm{d}t, \tag{2.4}$$

is of C^1 - Fréchet class and its critical points are weak solutions of (2.2). Let λ_1 be the first eigenvalue for the problem in $W_0^{1,p}(\Omega)$ involving $-\Delta_p^s$ with homogeneous boundary conditions. We have ([12, (7.2)])

$$\lambda_1 = \inf\left\{\frac{|u|_{1,p}^p}{||i(u)||_{0,p}^p} : u \in W_0^{1,p}(\Omega)\setminus\{0\}\right\}. \tag{2.5}$$

We further provide an answer for (2.1). We specify that $|u|_{1,p} = \left(\sum_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\right\|_{\mathbb{L}^p(\Omega)}^p\right)^{1/p}$, and the dual of $(W_0^{1,p}(\Omega), |\cdot|_{1,p})$ is $W^{-1,p'}(\Omega)$, where p' is the conjugate exponent of p (i.e., $\frac{1}{p}+\frac{1}{p'}=1$) and $i : W_0^{1,p}(\Omega) \to \mathbb{L}^p(\Omega)$ is a linear compact embedding.

Proposition 1. *In the above condition, assume satisfied the growth relation:*

$$F(x,s) \le c_1\frac{s^2}{2} + \alpha(x)s, \tag{2.6}$$

where $0 < c_1 < \lambda_1$, $\alpha \in \mathbb{L}^p(\Omega)$ *for some* $2 \le q \le \dfrac{2N}{N-2}$ *and* $f(x-s) = -f(x,s), \forall x \in \Omega, \forall s \in \mathbb{R}$. *Then*

1° *The set of functions* h *from* $W^{-1,p'}(\Omega)$, *having the property that the functional* $\varphi_h : W_0^{1,p}(\Omega) \to \mathbb{R}$,

$$\varphi_h(u) = \int_\Omega \left(\frac{1}{p}\sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^p - F(x,u(x)) + h(u(x))\right)\mathrm{d}x$$

has the minimum attained in only a point, includes an everywhere dense G_δ*-set ;*

2° *The set of functions* h *from* $W^{-1,p'}(\Omega)$, *having the property that the problem*

$$\begin{cases} -\Delta_p^s u = f(x,u) + h(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.7}$$

has solutions in $W_0^{1,p}(\Omega)$ *in the sense of* $W^{-1,p'}(\Omega)$ *includes an everywhere dense* G_δ*-set;*

3° *Moreover, if* $s \to f(x,s)$ *is increasing, then the set of functions* h *from* $W^{-1,p'}(\Omega)$, *having the property that the problem* (2.7) *has a unique solution in* $W_0^{1,p}(\Omega)$ *in the sense of* $W^{-1,p'}(\Omega)$, *includes an everywhere dense* G_δ*-set.*

Proof. It suffices to prove 1°. Consider, for each h from $W^{-1,p'}(\Omega)$, the functional ξ_h from $W^{-1,p'}(\Omega)$, $\xi_h(u) = \int_\Omega h(u(x))\mathrm{d}x$. It is obvious that $\varphi_h = \varphi + \xi_h$ (see (2.4)). Consequently, according to (II) from Theorem 1, if we show that φ_h is bounded from below for any h from $W^{-1,p'}(\Omega)$ (this is sufficient due to the representation Riesz theorem), then 1° is proved. However, taking into account the Sobolev embedding and (2.6), we have $\forall u \in W_0^{1,p}(\Omega)$

$$(\varphi+\xi_h)(u) \ge \int_\Omega \left(\frac{1}{p}\sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^p - c_1|u(x)|^2\right)\mathrm{d}x - ||\alpha||_{q'}||u||_q - ||h||_{W^{-1,p'}}||u||_{W_0^{1,p}} \ge$$

$$\geq \frac{1}{2}\left(1-\frac{c_1}{\lambda_1}\right)\int\limits_{\Omega}\frac{1}{p}\sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|^p \mathrm{d}x - r||u||_{W_0^{1,p}}, r\in\mathbb{R},$$

and hence the conclusion follows, because $1-c_1/\lambda_1>0$. □

We note that similar applications of the variational principle in the Finslerian framework can be found as well in [1].

3 Surjectivity for the operators $\lambda J_\varphi - S$. Applications

3.1 The surjectivity of operators of the form $\lambda T - S$

An extension of the Theorem 1.1 from [6] is proved with weakened assumptions, namely: normed space instead of Banach space and bijection with continuous inverse instead of homeomorphism. Two corollaries are presented as well. Firstly, we have the following

Definition. Let X and Y be normed spaces. Then the mapping $T: X\to Y$, is (K,L,a), where $K>0$, $L>0$, $a>0$, if

$$K||x||^a \leq ||Tx|| \leq L||x||^a,\ \forall\, x\in X.$$

Proposition 2. *Let X,Y be real normed spaces, let $T: X\to Y$ (K,L,a) be an odd bijection with continuous inverse and let $S: X\to Y$ be an odd compact operator. For any $\lambda\neq 0$, if*

$$\lim_{||x||\to+\infty}||\lambda Tx - Sx|| = +\infty,$$

then $\lambda T - S$ is surjective.

Proof. Let z_0 be from Y. We state that

$$\exists\, x_0 \text{ in} X \text{ s.t. } \lambda Tx_0 - Sx_0 = z_0. \tag{3.1.1}$$

We take $R>0$ with the property (see the hypothesis)

$$||x||\geq R \Rightarrow ||\lambda Tx - Sx|| > ||z_0|| \tag{3.1.2}$$

and the open ball from Y, $\sigma: S(0,r), r:=|\lambda|LR^a$. If $y\in\partial\Omega$ and $y=\lambda Tx$, then $||x||\geq R$ and hence (3.1.3) $||\lambda Tx - Sx|| \overset{(3.1.2)}{>} ||z_0||$.

Let be the operator $A: Y\to Y$, $Ay = ST^{-1}(y/\lambda)$. A is compact, odd and $Ay\neq y$ when $y\in\partial\sigma$. *Ad absurdum*, put y in the form λTx and take into account (3.1.3), i.e., $0\notin (I-A)(\partial\sigma)$. Applying the Borsuk theorem, it follows that the Leray - Schauder degree, $\mathrm{d}(I-A,\sigma,0)$, is odd. But

$$H:[0,1]\times\overline{\sigma}\to Y,\ H(t,y) = Ay + tz_0$$

is a homotopy of compact transforms on $\overline{\sigma}$, then we have

$$d(I-H(0,\cdot),\sigma,0) = d(I-H(1,\cdot),\sigma,0) \Longrightarrow d(I-A,\sigma,0) = d(I-A-z_0,0).$$

Consequently, $\mathrm{d}(I-A,\sigma,0)$ is an odd number, particularly different from zero, therefore $\exists\, y_0$ in σ so that $(I-A-z_0)(y_0)=0$ and we have only to take x_0 in X with $y_0=\lambda Tx_0$, to obtain (3.1.1). □

Corollary 1. *Let X, Y be real normed spaces, let $T : X \to Y$ be an odd (K, L, a) bijection with continuous inverse, let $S : X \to Y$ be an odd compact operator and $\alpha := \overline{\lim\limits_{||x||\to+\infty}} \frac{||Sx||}{||x||^a} < +\infty$. If $|\lambda| > \alpha/K$, $\lambda \in \mathbb{R}$, then $\lambda T - S$ is surjective.*

Remarks. According to ([10, vol. II, V, §5, II.9$_1$]), we have $f : X \to Y$, X and Y normed spaces,

$$\overline{\lim_{||x||\to+\infty}} ||f(x)|| \overset{\text{def}}{=} \inf_{\rho>0} \sup_{\substack{x \in X \\ ||x||\geq\rho}} ||f(x)|| = \lim_{\rho\to+\infty} \sup_{\substack{x \in X \\ ||x||\geq\rho}} ||f(x)||.$$

If $\alpha = \overline{\lim\limits_{||x||\to+\infty}} ||f(x)||$, then $x_n \in X$ $\forall n \in \mathbb{N}$ and $||x_n|| \to +\infty$ implies $\lim\limits_{n\to+\infty} ||f(x_n)|| \leq \alpha$. If $\alpha = \overline{\lim\limits_{n\to+\infty}} ||f(x_n)||$ for any (x_n) with $x_n \in X$ and $||x_n|| \to +\infty$, then $\alpha = \overline{\lim\limits_{||x||\to+\infty}} ||f(x)||$.

Proof. It suffices to prove

$$\overline{\lim_{||x||\to+\infty}} ||\lambda Tx - Sx|| = +\infty. \tag{3.1.4}$$

Assuming, *ad absurdum*, the contrary, we obtain $\rho > 0$ and a sequence $(x_n)_{n\geq 0}$, $x_n \in X$, $||x_n|| \to +\infty$ s.t.

$$||\lambda Tx - Sx|| \leq \rho \; \forall \; n \geq 1. \tag{3.1.5}$$

From (3.1.5), we have

$$\lim_{n\to+\infty} \left\| \frac{\lambda Tx_n}{||x_n||^a} - \frac{Sx_n}{||x_n||^a} \right\| = 0 \quad \Rightarrow \quad \lim_{n\to+\infty} \left[\frac{|\lambda|||Tx_n||}{||x_n||^a} - \frac{||S(x_n)||}{||x_n||^a} \right] = 0,$$

and since $\overline{\lim\limits_{n\to+\infty}} \frac{||S(x_n)||}{||x_n||^a} \leq \alpha$, it results

$$\overline{\lim_{n\to+\infty}} \frac{|\lambda|||Tx_n||}{||x_n||^a} \leq \alpha \tag{3.1.6}$$

But the condition (K, L, a) imposes

$$K \leq \overline{\lim_{n\to+\infty}} \frac{||Tx_n||}{||x_n||^a}. \tag{3.1.7}$$

From (3.1.6) and (3.1.7) it results $K \leq \frac{\alpha}{|\lambda|}$. If $\alpha \neq 0$, then $|\lambda| \leq \frac{\alpha}{K}$, which contradicts the hypothesis, and if $\alpha = 0$, then $K = 0$, also in contradiction with the hypothesis, and consequently (3.1.4). □

Corollary 2. *In the conditions of the Corollary 1, if $\alpha = 0$, then $\lambda T - S$ is surjective for any λ from $\mathbb{R} \setminus \{0\}$.*

3.2 Surjectivity of operators of the form $\lambda J_\varphi - S$

Proposition 3. *Let X be a real Banach space, reflexive and with the property (H), let J_φ be a duality map on X with φ (K, L, a) function, let $S : X \to X^*$ be an odd compact operator and*

$$\alpha := \overline{\lim_{x\to+\infty}} \frac{||Sx||}{||x||^a} < +\infty.$$

Then
1° $\alpha > 0 \Rightarrow \lambda J_\varphi - S$ *surjective* $\forall\, \lambda$ *with* $|\lambda| > \dfrac{\alpha}{K}$;
2° $\alpha = 0 \Rightarrow \lambda J_\varphi - S$ *surjective* $\forall\, \lambda \neq 0$.

Remark. We say that a Banach space has the (H) property if it is strictly convex and satisfies:

$$x_n \xrightarrow{w} x \text{ and } ||x_n|| \to ||x|| \Rightarrow x_n \to x.$$

Proof. J_φ is odd and bijective with continuous inverse (X being reflexive, smooth and with the (H) property, and the duality map J_φ on X is bijective with its inverse continuous with respect to the strong topologies on X and X^*. Moreover, since

$$Kt^a \leq \varphi(t) \leq Lt^a \ \forall t \geq 0,$$

we have

$$K||x||^a \leq \varphi(||x||) = ||J_\varphi x|| \leq L||x||^a \ \forall x \in X,$$

i.e., J_φ is a (K, L, a) function. We apply Corollaries 1 and 2 to finish the proof. □

Proposition 4. *Let X be a real reflexive Banach space, smooth with the property (H), and J_φ be the duality map on X with $\varphi(t) = t^{p-1}$, $p \in (1, +\infty)$. Suppose that X is compact embedded by the linear injection i in a Banach space Z,*

$$||i(u)|| \leq c_0||u|| \ \forall u \in X \tag{3.2.1}$$

and $N : Z \to Z^*$ is an odd hemi-continuous operator with the property

$$||Nx|| \leq c_1||x||^{q-1} + c_2 \ \forall x \in Z,\ c_1, c_2 \geq 0,\ q \in (1, p). \tag{3.2.2}$$

Then $\lambda J_\varphi - N$ is surjective for any $\lambda \neq 0$.

Remark. N is the short notation for the operator, which acts from X to X^*, $i' \circ N \circ i$, i' the adjoint of i.

Proof. It suffices to prove that

$$\forall h \in X^* \ \exists\, u \in X \text{ s.t. } \lambda J_\varphi u - (i' \circ N \circ i)u = h. \tag{3.2.3}$$

We apply Proposition 6 with $T = J_\varphi$ (correctly as φ is (K, L, a) with $K = L = 1$, $a = p - 1$), $S = i' \circ N \circ i$. Obviously, S is odd, and also compact: let $(x_n)_{n\in\mathbb{N}}$, $x_n \in X$ be a bounded sequence, $(i(x_n))_{n\geq 1}$ has a convergent subset, let be $i(x_{k_n}) \to \gamma$, $\gamma \in Z$, then $N(i(x_{k_n})) \xrightarrow{w} N(\gamma)$, and consequently $i'(N(i(x_{k_n}))) \to i'(N(\gamma))$ because i' is also compact (Schauder theorem). So, to obtain the conclusion it remains to prove

$$\overline{\lim_{||u||\to+\infty}} \frac{||(i' \circ N \circ i)u||}{||u||^{p-1}} = 0. \tag{3.2.4}$$

Then

$$\begin{aligned} ||i'(N(i(u)))|| &\leq ||i'||||N(i(u))|| \overset{(3.2.1),(3.2.2)}{\leq} c_0(c_1||i(u)||^{q-1}+c_2) \\ &\leq c_0(c_0^{q-1}c_1||u||^{q-1}+c_2), \end{aligned}$$

from which it results (3.2.4) (where we used $||i'|| \leq ||i|| \leq c_0$). □

In the following we search the surjectivity of the operator $\lambda J_\varphi - N$, when N verifies the growth condition (3.2.2), where $q = p$, i.e.,

$$||Nx|| \leq c_1||x||^{p-1} + c_2 \ \forall\, x \text{ from } Z,\ c_1, c_2 \geq 0. \tag{3.2.5}$$

For this reason, we present the following statement

Proposition 5. *Let X be a real reflexive Banach space compactly embedded by the linear injection i in the Banach space Z,*

$$||i(u)|| \leq c_0||u|| \ \forall\, u \ from\ X. \tag{3.2.6}$$

If

$$\lambda_1 := \inf\left\{\frac{||u||^p}{||i(u)||^p} : u \in X \setminus \{0\}\right\},\ p \in (1, +\infty),$$

then

1° *λ_1 is attained and nonzero;*
2° *$\lambda_1^{-\frac{1}{p}}$ is optimal for* (3.2.6) *(i.e. $\lambda_1^{-\frac{1}{p}} \leq c_0$, for any c_0);*
3° *If X and Z are smooth and $J_{XX^*} : X \to X^*$, $J_{YY^*} : Y \to Y^*$ duality maps relative to the same weight $\varphi : \varphi(t) = t^{p-1}$, then λ_1 is the smallest eigenvalue of the couple (J_{XX^*}, J_{YY^*})* ([3]).

Proof. The set from the statement is correctly defined: $u \neq 0 \Rightarrow i(u) \neq 0$. 1°. We have

$$\lambda_1 = \inf\{||\nu||^p : \nu \in X, ||i(\nu)|| = 1\}$$

(the two sets coincide, since $\|i(u/i(u))\| = 1$), let $(\nu_n)_{n\geq 1},\ \nu_n \in X$ be with $||i(\nu_n)|| = 1$ and $||\nu_n|| \to \lambda_1^{\frac{1}{p}}$. Since X is reflexive, then $(\nu_n)_{n\geq 1}$ admits a subsequence, similarly denoted, which is weakly convergent in X, $\nu_n \xrightarrow{w} \nu$ (Kakutani theorem). Then $||\nu|| \leq \lim\limits_{n\to\infty} ||\nu_n||$, whence

$$||\nu||^p \leq \lambda_1. \tag{3.2.7}$$

On the other hand, because i is compact, we have $i(\nu_n) \to i(\nu)$, this implies $||i(\nu_n)|| \to ||i(\nu)||$, $||i(\nu)|| = 1$ and hence $||\nu||^p \geq \lambda_1$, $||\nu||^p \overset{(3.2.7)}{=} \lambda_1$, λ_1 is attained and *a fortiori* nonzero. 2°. We take into account the definition of λ_1 and 1°. 3°. We show that λ_1 is eigenvalue for the couple (J_{ZZ^*}, J_{XX^*}), i.e., $\exists\ u_0 \neq 0$ in X s.t.

$$\lambda_1(i' \circ J_{ZZ^*} \circ i)u_0 = J_{XX^*}u_0. \tag{3.2.8}$$

We consider the functional $\Phi : X \to \mathbb{R}$, $\Phi(u) = \frac{1}{p}||u||^p - \frac{\lambda_1}{p}||i(u)||^p$. Then $\Phi(u) \geq 0\ \forall u \in X$ (see the definition of λ_1) and, for $u_0 \neq 0$ s.t. $\lambda_1 \overset{1^\circ}{=} \left(\frac{||u_0||}{||i(u_0)||}\right)^p$, $\Phi(u_0) = 0$, which (taking into account that X is a smooth space iff its norm is Gâteaux differentiable on $X \setminus \{0\}$) leads to the cancelling of the Gâteaux derivative

$$\Phi'(u_0) = 0. \tag{3.2.9}$$

Then we use the facts: "X smooth $\Rightarrow J_\varphi x = \varphi(||x||)||x||'$, $x \neq 0$" and "if X, Y real normed spaces and $f : X \to Y$ Gâteaux differentiable, $F : Y \to \mathbb{R}$ of Gâteaux C^1 class, then $g := F \circ f$ is Gâteaux differentiable ([10]) and $g'(x) = F'(f(x)) \circ f'(x)$" to obtain: $\forall u \in X, 0 \overset{(3.2.9)}{=} \Phi'(u_0)(u) = \left\langle ||u_0||^{p-1}||\ ||'(u_0), u\right\rangle - \lambda_1 \left\langle ||i(u_0)||^{p-1}||\ ||'(i(u_0)), i(u)\right\rangle = (J_{XX^*}u_0)(u) - \lambda_1(J_{ZZ^*}(i(u_0))(i(u))) = \langle J_{XX^*}u_0 - \lambda_1(i' \circ J_{ZZ^*} \circ i)(u_0), u\rangle$, i.e. (3.2.8). Let now λ be an eigenvalue for the couple (J_{ZZ^*}, J_{XX^*}) and let u be a corresponding eigenvector. Then

$$||u||^p = (J_{XX^*}u)(u) = \lambda \langle J_{ZZ^*}(i(u)), i(u)\rangle = \lambda||i(u)||^p,$$

and hence $\lambda = ||u||^p/||i(u)||^p \geq \lambda_1$. □

We can now state the following

Proposition 6. *Let X be real reflexive Banach space, smooth with the (H) property, let J_φ be a duality map on X with $\varphi(t) = t^{p-1}$, $p \in (1, +\infty)$. Suppose that X is compactly embedded with the linear injection i in the Banach space Z and let $N : Z \to Z^*$ be an odd hemi-continuous operator with:*

$$||Nx|| \leq c_1||x||^{p-1} + c_2\ \forall\ x\ from\ Z,\ c_1, c_2 \geq 0.$$

Then, for any λ, if

$$|\lambda| > \overline{\lim_{||u||\to+\infty}} \frac{||(i' \circ N \circ i)u||}{||u||^{p-1}},$$

for $|\lambda| > c_1\lambda_1^{-1}$, where $\lambda_1 := \inf\{||u||^p/||i(u)||^p \mid u \in X \setminus \{0\}\}$, then $\lambda J_\varphi - N$ is surjective (where by N we denoted $i' \circ N \circ i$).

Proof. Due to Proposition 5, 1°, it follows that $\lambda_1 \neq 0$, hence λ_1^{-1} exists. We apply, as for Proposition 4, Proposition 3 with $T = J_\varphi$, $S = i' \circ N \circ i$. We prove

$$\overline{\lim_{||u||\to+\infty}} \frac{||(i' \circ N \circ i)u||}{||u||^{p-1}} \leq c_1\lambda_1^{-1}, \tag{3.2.10}$$

by using Proposition 5, which is sufficient to establish the conclusion. $||(i' \circ N \circ i)u|| \leq ||i'||||N(i(u))|| \leq \lambda_1^{-\frac{1}{p}}(c_1\lambda_1^{\frac{1-p}{p}}||u||^{p-1} + c_2)$ $(||i'|| \leq ||i|| \leq \lambda_1^{-\frac{1}{p}})$ and (3.2.10) becomes obviously. □

3.3 Existence of the solution

We further investigate the existence of the solution for the problem

$$\begin{cases} -\lambda\Delta_p^s u = f(\cdot, u(\cdot)) + h, & x\ in\ \Omega, \lambda \in \mathbb{R}, p \in (1, +\infty) \\ u \mid \partial\Omega = 0 \end{cases}$$

The pseudo-Laplacian $-\Delta_p^s$, $p \in (1, +\infty)$.

Ω is an open set, with finite Lebesgue measure, in $\mathbb{R}^N$, $N \geq 2$, and for $p \in (1, +\infty)$ we consider the norm on $W_0^{1,p}(\Omega)$:

$$|u|_{1,p} := \left(\sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}},$$

where the derivative involved is weak. The dual of $(W_0^{1,p}(\Omega), |\cdot|_{1,p})$ is denoted by $W^{-1,p'}(\Omega)$, p' being the conjugate of the exponent p. Consider now the operator $-\Delta_p^s : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$,

$$-\Delta_p^s u = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right). \tag{3.3.1}$$

This acts according to ([8]), as follows:

$$\langle -\Delta_p u, h \rangle = \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial h}{\partial x_i} \, \mathrm{d}x, \ \forall\, u, h \in W_0^{1,p}(\Omega). \tag{3.3.2}$$

The problem. We further consider the problem:

$$(*) \qquad \begin{cases} -\lambda \Delta_p^s u = f(\cdot, u(\cdot)) + h, & x \in \Omega, \lambda \in \mathbb{R}. \\ u \mid \partial\Omega = 0. \end{cases}$$

Proposition 7. *Let Ω be an open bounded set of class C^1 in $\mathbb{R}^N$, $N \geq 2$; let $p \in [2, +\infty)$, let h be from $W^{-1,p'}(\Omega)$ and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function with the properties*

1° $f(x, -s) = -f(x, s)$, $\forall\, s \in \mathbb{R}$, $\forall\, x \in \Omega$,

2° $|f(x, s)| \leq c_1 |s|^{q-1} + \beta(x)$ $\forall\, s \in \mathbb{R}$, $\forall\, x \in \Omega \setminus A$, $\mu(A) = 0$,

where $c_1 \geq 0, q \in (1, p), \beta \in L^{q'}(\Omega), \dfrac{1}{q} + \dfrac{1}{q'} = 1$. Then, for any $\lambda \neq 0$, the problem $(*)$ has solutions in $W_0^{1,p}(\Omega)$ in the sense [1] of $W^{-1,p'}(\Omega)$.

Remarks. The relationship $u \mid \partial\Omega = 0$ from $(*)$ is in the sense of the trace [2]. $f(\cdot, u) = N_f u$, N_f Nemytskii operator [3], and so the equation from $(*)$ can be written as

$$-\lambda \Delta_p^s u = N_f u + h. \tag{3.3.3}$$

[1] All the terms from the first relation from $(*)$ are considered as elements of $W^{-1,p}(\Omega)$.

[2] The trace is the unique linear continuous operator $\gamma : W^{1,p}(\Omega) \to W^{1-\frac{1}{p},p}(\partial\Omega)$ such that it is surjective and $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \Rightarrow \gamma(u) = u \mid \partial\Omega$. So we use here the notation $u \mid \partial\Omega$ for $\gamma(u)$.

[3] *Nemytskii operator.* Let be $\mathbb{R}^{\mathbb{N}}, N \geq 1$, let μ be the Lebesgue measure in $\mathbb{R}$, let Ω be an open nonempty Lebesgue measurable and $\mathcal{M}(\Omega) := \{u : \Omega \to \mathbb{R} \mid u \text{ Lebesgue measurable}\}$. By definition $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a *Carathéodory function* if

1° $f(\cdot, s)$ is L.m. $\forall\, s \in \mathbb{R}$,

2° $f(x, \cdot)$ is continuous $\forall\, x \in \Omega \setminus A, \mu(A) = 0$.

In this case, for every $u \in \mathcal{M}(\Omega)$ one can consider the function (*the Nemytskii operator*) $N_f : \mathcal{M}(\Omega) \to \mathcal{M}(\Omega)$, $N_f u : N_f u(x) = f(x, u(x))$.

Let $i' : L^{q'} \to W^{-1,p'}$ be the transposed of i (as $(L^q)^* = L^{q'}$). Then u_0 from $W_0^{1,p}$ is *solution* for $(*)$ in the sense of $W^{-1,p'}$, if

$$-\lambda \Delta_p^s u_0 = (i' \circ N_f \circ i) u_0 + h. \tag{3.3.4}$$

But $(W_0^{1,p}(\Omega), |\cdot|_{1,p})$ is a Banach space which is compact embedded in $\mathbb{L}^q$ because $|\cdot|_{1,p}$ and $||\cdot||_{1,p}$, hence also $||\cdot||_{W^{1,p}}$, are equivalent.

Proof. We have $-\Delta_p^s = J_\varphi$ ([3]). J_φ is the duality map with $\varphi(t) = t^{p-1}$. The Banach space $(W_0^{1,p}, |\cdot|_{1,p})$ is uniform convex ([3]), and consequently has the (H) property and it is reflexive (uniform convex $\Rightarrow$ reflexive). It is also smooth (its norm being Gâteaux differentiable on $W_0^{1,p} \setminus \{0\}$). So, we can apply Proposition 7 (with $X = W_0^{1,p}, Z = L^q, N = N_f$ – odd continuous operator, $Z^* = L^{q'}$, take into account $||N_f u||_{0,q'} \le c_1 ||u||_{0,q}^{q-1} + c_2, c_2 := ||\beta||_{0,q'}, \forall u \in L^q$), the operator $\lambda(-\Delta_p^s) - S : W_0^{1,p} \to W^{-1,p'}$, where $S = i' \circ N_f \circ i$ is surjective, *a fortiori* the operator $-\lambda\Delta_p^s - S - h$ is surjective (commuting group) and hence $\exists\, u_0 \in W_0^{1,p}$ which satisfies (3.3.4). □

Replacing q with p in 2° from Proposition 7 and applying Proposition 6, one obtains

Proposition 8. *Let Ω be an open bounded of C^1 class set in $\mathbb{R}^N$, $N \ge 2, p \in [2, +\infty)$, let $h \in W^{-1,p'}$ and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function with the properties*

1° $f(x, -s) = -f(x, s)\ \forall\, x \in \Omega,\ \forall\, s \in \mathbb{R}$,

2° $|f(x, s)| \le c_1 |s|^{p-1} + \beta(x)\ \forall\, s \in \mathbb{R},\ \forall\, x \in \Omega \setminus A, \mu(A) = 0$,

where $c_1 \ge 0, \beta \in L^{p'}(\Omega), \dfrac{1}{p} + \dfrac{1}{p'} = 1$. Moreover, for any linear compact embedding $i : W_0^{1,p} \to L^p(\Omega)$, and $\lambda \in \mathbb{R}$ s.t.

$$|\lambda| > c_1 \lambda_1^{-1},\ \lambda_1 := \inf \left\{ \frac{|u|_{1,p}^p}{||i(u)||_{0,p}^p} \;\middle|\; u \in W_0^{1,p} \setminus \{0\} \right\},$$

the problem $(*)$ has a solution in $W_0^{1,p}(\Omega)$ in the sense of $W^{-1,p'}(\Omega)$.

Proof. The claim is well stated, since $(W_0^{1,p}, |\cdot|_{1,p})$ is compact embedded in L^p (Rellich-Kondrashev theorem). Then one applies Proposition 6. □

Remark. The condition from Proposition 8 can be replaced (see Proposition 6) by:

$$|\lambda| > \overline{\lim_{||u|| \to +\infty}} \frac{||(i' \circ N_f \circ i)u||}{||u||^{p-1}}.$$

We note that $\lambda_1^{-\frac{1}{p}}$ is optimal for the inequality from the statement, it is attained, nonzero, and is the least eigenvalue for the couple $(J_{L^q L^{q'}}, J_{W_0^{1,p} W^{-1,p'}})$ (see Proposition 5). Concerning the spectrum of the Laplace operator, we can compare the obtained results with the ones from [13].

4 Conclusion

In the first part of the paper, we present an original minimization result (minimization with regular constraints giving global critical points), which involves a linear variational principle, in the framework of characterizations of weak solutions for PDEs. The

second section contains novel results: an extension of a theorem from [6] is proved with weakened assumptions, and two corollaries are derived, while Propositions 3, 4 and 6 are original as well. The Propositions 7 and 8 concern the pseudo-Laplacian Δ_p^s.

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Authors' addresses:

Irina Meghea
University Politehnica of Bucharest, Faculty of Applied Sciences,
Departement of Mathematics II, Splaiul Independenţei 313,
RO-060042, Bucharest, Romania.
E-mail irina.meghea@upb.ro, i_meghea@yahoo.com