Geometrical structures associated to implicit first order dynamical systems with some holonomic constraints

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Dedicated to the 70-th anniversary
of Professor Constantin Udriste

Abstract. In [5], [6], [7], [8], it was shown that to any implicit first (or higher) order dynamical system without constraints, it corresponds, on the configuration space, a geometrical structure given by a (nondegenerate and in general not symmetric) generalized d-metric and a nonlinear d-connection. Thus, there exist a covariant differential operator, a parallel transport of the vector fields and autoparallel curves. The following property holds: *if the equations which describe the dynamics of the system do not depend explicitly on time, the set of autoparallel curves associated to the structure build in this way, coincide with the set of solutions of the given dynamical system.* In this paper we generalize these results to the case of implicit first order dynamical systems with holonomic constraints.

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1 Geometrical structures defined on a differentiable manifold

Let $M = M_m$ be a differentiable manifold of dimension m. On it we define the following geometrical structures ([6]).

1.1 Natural frames ([1], [2])

On a change of local chart, the coordinates on TM change by the rules:

$$\bar{x}^h = \bar{x}^h(x^i), \;\; \dot{\bar{x}}^h = \frac{\partial \bar{x}^h}{\partial x^i}\dot{x}^i, \;\; \frac{\partial \dot{\bar{x}}^h}{\partial \dot{x}^i} = \frac{\partial \bar{x}^h}{\partial x^i}. \tag{1.0}$$

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The corresponding natural frames change respectively by the rules:

$$\begin{aligned}\frac{\partial}{\partial \bar{x}^h} &= \frac{\partial x^i}{\partial \bar{x}^h}\frac{\partial}{\partial x^i} + \frac{\partial \dot{x}^i}{\partial \bar{x}^h}\frac{\partial}{\partial \dot{x}^i}, \quad \frac{\partial}{\partial \dot{\bar{x}}^h} = \frac{\partial x^i}{\partial \bar{x}^h}\frac{\partial}{\partial \dot{x}^i},\\ d\bar{x}^h &= \frac{\partial \bar{x}^h}{\partial x^i}dx^i, \quad d\dot{\bar{x}}^h = \frac{\partial \dot{\bar{x}}^h}{\partial x^i}dx^i + \frac{\partial \bar{x}^h}{\partial x^i}d\dot{x}^i.\end{aligned} \tag{1.0'}$$

It follows, by the formulas (1.0′), that in each point x of the domain of the considered chart, the vectors $\dfrac{\partial}{\partial \dot{x}^i}$ generate a subspace of the space TM_x, which does not depend on the chosen chart, called *vertical subspace.*

1.2 Generalized Lagrange d-metric

It is called generalized Lagrange d-metric defined on M, a d-tensor of the second rank, two times covariant and nondegenerated (in general not symmetric). In a local chart (U, χ), with the coordinated (x^i) of a current point x, the components of the d-tensor are given by the functions $a^1_{ij} = a^1_{ij}\left(t, x^h, \dot{x}^h\right)$, $\left(t, x^h, \dot{x}^h\right) \in J^1M$, with the property that, on a change of local chart, they change by the rules:

$$\bar{a}^1_{hk} = \frac{\partial x^i}{\partial \bar{x}^h}\frac{\partial x^j}{\partial \bar{x}^k}a^1_{ij}, \quad det\left(a^1_{ij}\right) \neq 0, \ \forall x \in U. \tag{1.1}$$

We make no additional hypothesis related to the symmetry or signature.

By the definition, it follows the existence of the functions $a_1^{ij} = a_1^{ij}\left(t, x^h, \dot{x}^h\right)$, components of the reciprocal matrix of (a^1_{ij}). On a change of local chart, we have:

$$\bar{a}_1^{hk} = \frac{\partial \bar{x}^h}{\partial x^i}\frac{\partial \bar{x}^k}{\partial x^j}a_1^{ij}. \tag{1.1'}$$

1.3 Nonlinear d-connection

We say that on the manifold M it is given a nonlinear d-connection, if in each local chart we have a set of functions $a^0_{ij} = a^0_{ij}\left(t, x^h, \dot{x}^h\right)$, which change, on a change of chart, by the rules:

$$\bar{a}^0_{hk} = \frac{\partial x^i}{\partial \bar{x}^h}\left[\frac{\partial x^j}{\partial \bar{x}^k}a^0_{ij} + \frac{\partial \dot{x}^j}{\partial \bar{x}^k}a^1_{ij}\right]. \tag{1.2}$$

The functions a^0_{ij} are called *Christoffel symbols of first type* of the d-connection.

1.4 Mixed coefficients of connection

By contracting the above coefficients with the contravariant components of the d-metric, we obtain the functions:

$$M^i_j = a_1^{ih}a^0_{hj}, \tag{1.2'}$$

called *the Christoffel symbols of second type* or coefficients of the nonlinear connection. They change, on a change of local chart, by the rules ([3], [4]):

$$\overline{M}^i_j = \frac{\partial \bar{x}^i}{\partial x^h}\left[\frac{\partial x^k}{\partial \bar{x}^j}M^h_k + \frac{\partial \dot{x}^k}{\partial \bar{x}^j}\delta^h_k\right]. \tag{1.3}$$

1.5 Adapted frames

Given a d-connection, we can build an adapted dual frame and, thus, an adapted frame by ([4]):

$$\text{(1.4)}\qquad \begin{cases} \delta x^i = dx^i (= \delta^i_j dx^j), \\ \delta \dot{x}^i = d\dot{x}^i + M^i_j dx^j = (M^i_j dx^j + \delta^i_j d\dot{x}^j) \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{\delta}{\delta x^i} = \dfrac{\partial}{\partial x^i} - M^j_i \dfrac{\partial}{\partial \dot{x}^j}, \\ \dfrac{\delta}{\delta \dot{x}^i} = \dfrac{\partial}{\partial \dot{x}^i}, \end{cases}$$

respectively.

On a change of local chart, the rules of change:

$$\text{(1.5)}\qquad \delta \bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^h} \delta x^h,\ \delta \dot{\bar{x}}^i = \frac{\partial \bar{x}^i}{\partial x^h} \delta \dot{x}^h,\ \text{and}\ \frac{\delta}{\delta \bar{x}^i} = \frac{\partial x^h}{\partial \bar{x}^i} \frac{\delta}{\delta x^h},\ \frac{\delta}{\delta \dot{\bar{x}}^i} = \frac{\partial x^h}{\partial \bar{x}^i} \frac{\delta}{\delta \dot{x}^h}$$

hold respectively.

By the formulas (1.5) it follows that the vectors $\dfrac{\delta}{\delta x^i}$ generate a second subspace, called *horizontal subspace*, and the tangent space to TM in a point $(x, \dot{x})$ is the direct sum of the vertical and horizontal spaces in that point.

The Lie brackets of the vector fields $\dfrac{\delta}{\delta x^i}$ and $\dfrac{\delta}{\delta x^j}$ are expressed by the formulas:

$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = R^h_{ij} \frac{\delta}{\delta \dot{x}^h}, \quad \text{where:} \quad R^h_{ij} = \frac{\delta M^h_i}{\delta x^j} - \frac{\delta M^h_j}{\delta x^i}.$$

This tensor is called *tensor of curvature*. On a change of local chart, its components change by the rules: $\overline{R}^q_{hk} = \dfrac{\partial \bar{x}^q}{\partial x^p} \dfrac{\partial x^i}{\partial \bar{x}^h} \dfrac{\partial x^j}{\partial \bar{x}^k} R^p_{ij}$.

1.6 Differential operator, covariant derivative on M associated to a nonlinear d-connection

We call *d-vector field* on M a special vector field on $J^1 M$, locally written as $X = X^i(t, x, \dot{x}) \dfrac{\partial}{\partial x^i}$, which change, on a change of local chart, by the rules:

$$\text{(1.6)}\qquad \overline{X}^i = \frac{\partial \bar{x}^i}{\partial x^h} X^h$$

and we denote there set by $\mathfrak{X}^d(M)$. It is obviously that $\mathfrak{X}(M) \subset \mathfrak{X}^d(M)$.

Let given the spaces (rings) $\Lambda^o(J^1 M)$, $\Lambda^o(J^2 M)$, the modulus $\mathfrak{X}^d_1(M)$, where $X_1 \in \mathfrak{X}^d_1(M)$ if $X_1 = X^i_1(t, x, \dot{x}) \dfrac{\partial}{\partial x^i}$, and a nonlinear connection N of local components M^j_i. A differential operator $D : \Lambda^o(J^1 M) \to \Lambda^o(J^2 M)$, is defined, along a curve $x^i = c^i(t)$, by $D : f \to Df = \dfrac{df}{dt} = \dfrac{\partial f}{\partial t} + \dot{x}^i \dfrac{\partial f}{\partial x^i} + \ddot{x}^i \dfrac{\partial f}{\partial \dot{x}^i}$. To the connection N it is associated a linear differential operator D_N, defined on the set $\mathfrak{X}^d_1(M)$ of the d-vector fields and with values in $\mathfrak{X}^d_2(M)$, given by its values: $D_N \left(\dfrac{\partial}{\partial x^i}\right) = M^j_i \dfrac{\partial}{\partial x^j}$ on the bases vectors and satisfying the relation $D_N(fX) = fD_N(X) + D(f)X$, for any $f \in \Lambda^o(J^1M)$, $X \in \mathfrak{X}_1^d(M)$. On an arbitrary *d*-vector field X, the local expression of the operator D_N is given by $D_N(X) = \left[\dfrac{dX^i}{dt} + M_j^i X^j\right] \dfrac{\partial}{\partial x^i}$ and it is called the *N-covariant derivative* of X. The expressions $\dfrac{\delta X^i}{\delta t} = \dfrac{dX^i}{dt} + M_j^i X^j$ are called *covariant derivatives* of the components of the *d*-vector field X with respect to the nonlinear connection N. On a change of local chart, the covariant derivatives $\dfrac{\delta X^i}{\delta t}$ change by the rules: $\dfrac{\delta \overline{X}^i}{\delta t} = \dfrac{\partial \bar{x}^i}{\partial x^h} \dfrac{\delta X^h}{\delta t}$.

We say that a *d*-vector field $X = X^i \dfrac{\partial}{\partial x^i}$ is *transported by parallelism*, along a curve $x^i = c^i(t)$, if, along this curve, its covariant derivative vanish: $\dfrac{dX^i}{dt} + M_j^i X^j = 0$.

A curve $x^i = c^i(t)$, is called *autoparallel* if $\dfrac{d^2c^i}{dt^2} + M_j^i \dfrac{dc^j}{dt} = 0$, that means its tangent vector $\dfrac{dc^i}{dt}$ is transported by parallelism.

2 Submanifolds V_p of a structured differentiable manifold $(M_m, a_{ij}^1, a_{ij}^0)$

Let $V = V_p$ be a submanifold of the differentiable manifold M, locally defined by the functions

$$x^i = x^i(u^a), \ i = \overline{1,m}, \ a = \overline{1,p}, \ 1 \le p < m, \tag{2.0}$$

which give us a parametrical representation of V [9]. By the definition of the submanifold, it is assumed that the condition:

$$rank \left\| \frac{\partial x^i}{\partial u^a} \right\| = p \tag{2.0'}$$

is fulfilled. The atlas of the submanifold V is not necessarily the restriction to V of the atlas of M.

2.1 Induced frames on V

Given a canonical frame by the relations (1.0′), it induces on V the frame $\left(\dfrac{\partial}{\partial u^a}\right)$ and on TV the frame $\left(\dfrac{\partial}{\partial u^a}, \dfrac{\partial}{\partial \dot{u}^a}\right)$ by the formulas:

$$\frac{\partial}{\partial \dot{u}^a} = \frac{\partial x^i}{\partial u^a} \frac{\partial}{\partial \dot{x}^i}, \ \frac{\partial}{\partial u^a} = \frac{\partial x^i}{\partial u^a} \frac{\partial}{\partial x^i} + \frac{\partial \dot{x}^i}{\partial u^a} \frac{\partial}{\partial \dot{x}^i}. \tag{2.1}$$

It is necessary that, on a change of local chart on M and a change of parameters on V, respectively, the elements of the frame $\left(\dfrac{\partial}{\partial u^a}, \dfrac{\partial}{\partial \dot{u}^a}\right)$ to change by the rules:

$$\frac{\partial}{\partial \bar{u}^b} = \frac{\partial u^a}{\partial \bar{u}^b} \frac{\partial}{\partial u^a} + \frac{\partial \dot{u}^a}{\partial \bar{u}^b} \frac{\partial}{\partial \dot{u}^a}, \ \frac{\partial}{\partial \dot{\bar{u}}^b} = \frac{\partial u^a}{\partial \bar{u}^b} \frac{\partial}{\partial \dot{u}^a}. \tag{2.2}$$

We have:

Lemma 1. *The following diagram is commutative:*

$$\begin{tikzcd}
\left\{\dfrac{\partial}{\partial \dot{x}^i}\right\} \arrow[r, "h_1"] \arrow[d, "v_1"'] \arrow[dr, shift left] \arrow[dr, shift right] & \left\{\dfrac{\partial}{\partial \dot{\bar{x}}^h}\right\} \arrow[d, "v_2"] \\
\left\{\dfrac{\partial}{\partial \dot{u}^a}\right\} \arrow[r, "h_2"] & \left\{\dfrac{\partial}{\partial \dot{\bar{u}}^c}\right\}
\end{tikzcd}$$

Indeed, $v_2 h_1 = h_2 v_1 : \dfrac{\partial}{\partial \dot{x}^i} \to \dfrac{\partial}{\partial \dot{\bar{u}}^c} = \dfrac{\partial x^i}{\partial \bar{u}^c}\dfrac{\partial}{\partial \dot{x}^i}$.

Lemma 2. *The following diagram is commutative:*

$$\begin{tikzcd}
\left\{\dfrac{\partial}{\partial x^i}\right\} \arrow[r, "h_1"] \arrow[d, "v_1"'] \arrow[dr, shift left] \arrow[dr, shift right] & \left\{\dfrac{\partial}{\partial \bar{x}^h}\right\} \arrow[d, "v_2"] \\
\left\{\dfrac{\partial}{\partial u^a}\right\} \arrow[r, "h_2"] & \left\{\dfrac{\partial}{\partial \bar{u}^c}\right\}
\end{tikzcd}$$

Indeed, by (1.0′) and (2.2) it follows $v_2 h_1 = h_2 v_1 : \dfrac{\partial}{\partial x^i} \to \dfrac{\partial}{\partial \bar{u}^c} = \dfrac{\partial x^i}{\partial \bar{u}^c}\dfrac{\partial}{\partial x^i} + \dfrac{\partial \dot{x}^i}{\partial \bar{u}^c}\dfrac{\partial}{\partial \dot{x}^i}$. Thus, we have:

Theorem 1. *The frame* (1.0′) *given on TM induces a frame on TV, defined by* (2.1)*, which change, on a change of local chart on M and of parameters on V, by the rules* (2.2)*.*

The frame on TM induces a dual frame on TV, but it is difficult to define it directly. Let us consider the induced frame $\left(\dfrac{\partial}{\partial u^a}, \dfrac{\partial}{\partial \dot{u}^a}\right)$ on TV, we associate to it its dual frame $(du^a, d\dot{u}^a)$. We define the last one by the implicit formulas:

$$dx^i = \frac{\partial x^i}{\partial u^a} du^a, \qquad d\dot{x}^i = \frac{\partial \dot{x}^i}{\partial u^a} du^a + \frac{\partial x^i}{\partial u^a} d\dot{u}^a.$$

It is necessary that, on a change of local chart and a change of parameters, the forms of the frame to change (by hypothesis) by the rules:

$$d\bar{u}^a = \frac{\partial \bar{u}^a}{\partial u^b} du^b, \qquad d\dot{\bar{u}}^a = \frac{\partial \dot{\bar{u}}^a}{\partial u^b} du^b + \frac{\partial \bar{u}^a}{\partial u^b} d\dot{u}^b.$$

This is proved by:

Lemma 3. *The following diagram is commutative:*

$$\begin{tikzcd}
dx^i & d\bar{x}^h \arrow[l, "h_1"'] \\
du^a \arrow[u, "v_1"] & d\bar{u}^c \arrow[l, "h_2"] \arrow[u, "v_2"'] \arrow[ul, shift left] \arrow[ul, shift right]
\end{tikzcd}$$

Indeed, $h_1 v_2 = v_1 h_2 : d\bar{u}^c \to dx^i = \dfrac{\partial x^i}{\partial \bar{u}^c} d\bar{u}^c$.

Lemma 4. *The following diagram is commutative:*

$$\begin{tikzcd}
d\dot{x}^i & d\dot{\bar{x}}^h \arrow[l, "h_1"'] \arrow[dl, phantom, ""] \\
d\dot{u}^a \arrow[u, "v_1"] & d\dot{\bar{u}}^c \arrow[l, "h_2"'] \arrow[u, "v_2"'] \arrow[ul, shift left] \arrow[ul, shift right]
\end{tikzcd}$$

Indeed, $h_1 v_2 = v_1 h_2 : d\dot{\bar{u}}^c \to d\dot{x}^i = \dfrac{\partial \dot{x}^i}{\partial \bar{u}^c} d\bar{u}^c + \dfrac{\partial x^i}{\partial \bar{u}^c} d\dot{\bar{u}}^c$.

Let us consider the equations (2.0), which define the submanifold V. The formula:

$$\frac{\partial x^h}{\partial u^b} \delta^i_h = \frac{\partial x^i}{\partial u^a} \delta^a_b, \tag{2.3}$$

where δ^i_h and δ^a_b are the Kronecker's symbols on M and V respectively, holds.

2.2 Induced d-metric on V

The d-metric, locally given by a^1_{ij}, of the manifold M, induces a d-metric α^1_{ab} on the submanifold V, by the formula:

$$\alpha^1_{ab} = \frac{\partial x^i}{\partial u^a} \frac{\partial x^j}{\partial u^b} a^1_{ij}. \tag{2.4}$$

By the definition, it follows $det\left(\alpha^1_{ab}\right) \neq 0$, $\forall x \in U$.

On a change of local chart on M and on V, respectively, we obtain:

$$\bar{\alpha}^1_{cd} = \frac{\partial u^a}{\partial \bar{u}^c} \frac{\partial u^b}{\partial \bar{u}^d} \alpha^1_{ab}. \tag{2.5}$$

This property is proved by:

Lemma 5. *The following diagram is commutative:*

$$\begin{tikzcd}
a^1_{ij} \arrow[r, "h_1"] \arrow[d, "v_1"'] \arrow[dr, shift left] \arrow[dr, shift right] & \bar{a}^1_{hk} \arrow[d, "v_2"] \\
\alpha^1_{ab} \arrow[r, "h_2"] & \bar{\alpha}^1_{cd}
\end{tikzcd}$$

Indeed, we have $v_2 h_1 = h_2 v_1 : a^1_{ij} \to \bar{\alpha}^1_{cd} = \dfrac{\partial x^i}{\partial \bar{u}^c} \dfrac{\partial x^j}{\partial \bar{u}^d} a^1_{ij}$.

The contravariant components α^{ab}_1 of the induced d-metrical tensor can be implicitly defined by the formulas:

$$a^{ij}_1 = \frac{\partial x^i}{\partial u^a} \frac{\partial x^j}{\partial u^b} \alpha^{ab}_1. \tag{2.6}$$

Lemma 6. *The following diagram is commutative:*

$$\begin{tikzcd}
a_1^{ij} & \bar{a}_1^{hk} \arrow[l, "h_1"'] \\
\alpha_1^{ab} \arrow[u, "v_1"] & \bar{\alpha}_1^{cd} \arrow[l, "h_2"'] \arrow[u, "v_2"'] \arrow[ul, shift left] \arrow[ul, shift right]
\end{tikzcd}$$

Indeed, we have $h_1 v_2 = v_1 h_2 : \bar{\alpha}_1^{cd} \to a_1^{ij} = \dfrac{\partial x^i}{\partial \bar{u}^c}\dfrac{\partial x^j}{\partial \bar{u}^d}\bar{\alpha}_1^{cd}$.

The components α_1^{ab} satisfy the relations: $\alpha_1^{ac}\alpha^1_{cb} = \delta^a_b$.

Indeed, let a_1^{ij} be the contravariant components corresponding to a^1_{hk}, we have $a_1^{ih}a^1_{hj} = \delta^i_j$. By contracting each term with the functions $\dfrac{\partial x^j}{\partial u^b}$, it follows:

$$a_1^{ih}a^1_{hj}\frac{\partial x^j}{\partial u^b} = \delta^i_j\frac{\partial x^j}{\partial u^b} = \frac{\partial x^i}{\partial u^a}\delta^a_b = \frac{\partial x^i}{\partial u^a}\frac{\partial x^h}{\partial u^c}\alpha_1^{ac}a^1_{hj}\frac{\partial x^j}{\partial u^b} = \frac{\partial x^i}{\partial u^a}\alpha_1^{ac}\alpha^1_{cb} \;\Rightarrow\; \alpha_1^{ac}\alpha^1_{cb} = \delta^a_b.$$

2.3 Induced d-connection

Let us consider on the manifold M the d-metric a^1_{ij} and the connection a^0_{ij} and on its submanifold V the induced metric α^1_{ab}. The connection a^0_{ij} induces on V a connection α^0_{ab} by the formulas:

$$\alpha^0_{ab} = \frac{\partial x^i}{\partial u^a}\left[\frac{\partial x^j}{\partial u^b}a^0_{ij} + \frac{\partial \dot{x}^j}{\partial u^b}a^1_{ij}\right], \tag{2.7}$$

such that, on a change of local chart on M and on V respectively, the rules of change:

$$\bar{\alpha}^0_{cd} = \frac{\partial u^a}{\partial \bar{u}^c}\left[\frac{\partial u^b}{\partial \bar{u}^d}\alpha^0_{ab} + \frac{\partial \dot{u}^b}{\partial \bar{u}^d}\alpha^1_{ab}\right] \tag{2.8}$$

hold. We have:

Lemma 7. *The following diagram is commutative:*

$$\begin{tikzcd}
a^0_{ij} \arrow[r, "h_1"] \arrow[d, "v_1"'] \arrow[dr, shift left] \arrow[dr, shift right] & \bar{a}^0_{hk} \arrow[d, "v_2"] \\
\alpha^0_{ab} \arrow[r, "h_2"] & \bar{\alpha}^0_{cd}
\end{tikzcd} \tag{2.9}$$

Indeed, $d_1 = v_2 h_1$ and $d_2 = h_2 v_1$, lead us to the relations: $d_1 = d_2 : a^0_{ij} \to \bar{\alpha}^0_{cd} = \dfrac{\partial x^i}{\partial \bar{u}^c}\left[\dfrac{\partial x^j}{\partial \bar{u}^d}a^0_{ij} + \dfrac{\partial \dot{x}^j}{\partial \bar{u}^d}a^1_{ij}\right]$.

By the Lemmas 5 and 7, it follows:

Theorem 2. *The following diagram is commutative:*

$$\begin{tikzcd}
\left(a^1_{ij}, a^0_{ij}\right) \arrow[r, "h_1"] \arrow[d, "v_1"'] \arrow[dr, shift left] \arrow[dr, shift right] & \left(\bar{a}^1_{hk}, \bar{a}^0_{hk}\right) \arrow[d, "v_2"] \\
\left(\alpha^1_{ab}, \alpha^0_{ab}\right) \arrow[r, "h_2"] & \left(\bar{\alpha}^1_{cd}, \bar{\alpha}^0_{cd}\right)
\end{tikzcd} \tag{2.10}$$

Thus, we can associate to the submanifold V an induced geometry by the geometry of M.

Given the functions α^1_{ab} and α^0_{ab}, they define the coefficients (Christoffel symbols of second type) on V:

$$\mu^a_b = \alpha_1^{ac}\alpha^0_{cb}, \tag{2.11}$$

which change, on a change of local chart, by the rules:

$$\bar\mu^c_d = \bar\alpha_1^{cs}\bar\alpha^0_{sd} = \frac{\partial \bar u^c}{\partial u^a}\left[\frac{\partial u^b}{\partial \bar u^d}\mu^a_b + \frac{\partial \dot u^b}{\partial \bar u^d}\delta^a_b\right]. \tag{2.12}$$

By the relations (2.6) and (2.7), between the metric and the connection on M and on V, respectively, we will express the relations between the coefficients M^i_j and μ^a_b. These relations will determine μ^a_b implicitly, as function of M^i_j.

Theorem 3. *Given the coefficients of connection M^i_j on M, the induced coefficients μ^a_b on V are obtained, implicitly, by the formulas:* $\dfrac{\partial x^i}{\partial u^a}\mu^a_b = \dfrac{\partial x^j}{\partial u^b}M^i_j + \dfrac{\partial \dot x^j}{\partial u^b}\delta^i_j$.

Indeed, we have:

$$\begin{aligned}\frac{\partial x^i}{\partial u^a}\mu^a_b &= \frac{\partial x^i}{\partial u^a}\alpha_1^{ac}\alpha^0_{cb} = \frac{\partial x^i}{\partial u^a}\alpha_1^{ac}\frac{\partial x^h}{\partial u^c}\left[\frac{\partial x^j}{\partial u^b}a^0_{hj} + \frac{\partial \dot x^j}{\partial u^b}a^1_{hj}\right] = \\ &= a_1^{ih}\left[\frac{\partial x^j}{\partial u^b}a^0_{hj} + \frac{\partial \dot x^j}{\partial u^b}a^1_{hj}\right] = \frac{\partial x^j}{\partial u^b}M^i_j + \frac{\partial \dot x^j}{\partial u^b}\delta^i_j.\end{aligned}$$

We have:

Lemma 8. *The following diagram is commutative:*

$$\begin{tikzcd}
M^i_i \arrow[r, "h_1"] \arrow[d, "v_1"'] \arrow[rd, shift left] \arrow[rd, shift right] & \overline{M}^h_k \arrow[d, "v_2"] \\
\mu^a_b \arrow[r, "h_2"] & \bar\mu^c_d
\end{tikzcd}$$

Indeed, by the implicit formula: $\dfrac{\partial x^i}{\partial \bar u^c}\bar\mu^c_d = \dfrac{\partial x^j}{\partial \bar u^d}M^i_j + \dfrac{\partial \dot x^j}{\partial \bar u^d}\delta^i_j$, it follows $v_2h_1 = h_2v_1 : M^i_j \to \bar\mu^c_d$, which proves the above statement (formula (2.12)).

With help of the metric α^1_{ab} and the connection α^0_{ab}, we can build a geometrical structure on V, called the *induced structure* of the structure on M.

2.4 Induced adapted frames

On the submanifold V we define an *induced adapted dual frame* by the formulas:

$$\delta u^a = du^a, \ \ \delta\dot u^a = d\dot u^a + \mu^a_b du^b.$$

On a change of parameters, they change by the rules:

$$\delta\bar u^b = \frac{\partial \bar u^b}{\partial u^a}\delta u^a, \ \ \delta\dot{\bar u}^b = \frac{\partial \bar u^b}{\partial u^a}\delta\dot u^a.$$

The relation between the adapted dual frame on M and the induced adapted frame on V is given by the formulas:

$$\delta x^i = \frac{\partial x^i}{\partial u^a}\delta u^a,\ \delta \dot{x}^i = \frac{\partial x^i}{\partial u^a}\delta \dot{u}^a.$$

The *induced adapted frame* is defined by:

$$\frac{\delta}{\delta u^a} = \frac{\partial}{\partial u^a} - \mu_a^b \frac{\partial}{\partial \dot{u}^b},\ \frac{\delta}{\delta \dot{u}^a} = \frac{\partial}{\partial \dot{u}^a}.$$

It is obtained by the adapted frame on M by the formulas:

$$\frac{\delta}{\delta u^a} = \frac{\partial x^i}{\partial u^a}\frac{\delta}{\delta x^i},\ \frac{\delta}{\delta \dot{u}^a} = \frac{\partial x^i}{\partial u^a}\frac{\delta}{\delta \dot{x}^i}.$$

On a change of parameters, the vectors of this frame change by the rules:

$$\frac{\delta}{\delta \bar{u}^b} = \frac{\partial u^a}{\partial \bar{u}^b}\frac{\delta}{\delta u^a},\ \frac{\delta}{\delta \dot{\bar{u}}^b} = \frac{\partial u^a}{\partial \bar{u}^b}\frac{\delta}{\delta \dot{u}^a}.$$

The tensor of curvature is defined by the brackets:

$$\left[\frac{\delta}{\delta u^a}, \frac{\delta}{\delta u^b}\right] = \rho_{ab}^c \frac{\delta}{\delta \dot{u}^c},\ \text{where: } \rho_{ab}^c = \frac{\delta \mu_a^c}{\delta u^b} - \frac{\delta \mu_b^c}{\delta u^a}.$$

The relations between R_{ij}^h and ρ_{ab}^c are given by the relations: $\dfrac{\partial x^h}{\partial u^c}\rho_{ab}^c = \dfrac{\partial x^i}{\partial u^a}\dfrac{\partial x^j}{\partial u^b}R_{ij}^h$.

On a change of parameters, the components ρ_{ab}^c of the tensor of curvature change by the rules: $\bar{\rho}_{cd}^s = \dfrac{\partial \bar{u}^s}{\partial u^r}\dfrac{\partial u^a}{\partial \bar{u}^c}\dfrac{\partial u^b}{\partial \bar{u}^d}\rho_{ab}^r$.

2.5 The covariant differential operator on V

Given a function f on M, we associate to it its restriction to V, and by the invariance of the total derivative operator it follows that $D : f|_V \to Df = \dfrac{df}{dt} = \dfrac{\partial f}{\partial t} + \dot{u}^a \dfrac{\partial f}{\partial u^a} + \ddot{u}^a \dfrac{\partial f}{\partial \dot{u}^a}$. Any vector field $X = X^i \dfrac{\partial}{\partial x^i}$ induces on V a vector field $X_1 = X_1^a \dfrac{\partial}{\partial u^a}$, where the functions X_1^a, $X_1^a = X_1^a(t, x(u), \dot{x}(u, \dot{u}))$, are defined on V implicitly, by the relations $X^i = \dfrac{\partial x^i}{\partial u^a}X_1^a$. We define: $D_N\left(\dfrac{\partial}{\partial u^a}\right) = \mu_a^b \dfrac{\partial}{\partial u^b}$, which has to satisfy the property: $D_N(fX_1) = fD_N(X_1) + D(f)X_1$. Thus, $D_N(X_1) = \left(\dfrac{dX_1^a}{dt} + \mu_b^a X_1^b\right)\dfrac{\partial}{\partial u^a}$. The functions $\dfrac{\delta X_1^a}{\delta t} = \dfrac{dX_1^a}{dt} + \mu_b^a X_1^b$ are the *covariant derivatives* of the components X_1^a. A vector is *transported by parallelism* if the covariant derivatives of its components vanish: $\dfrac{dX_1^a}{dt} + \mu_b^a X_1^b = 0$. A curve $u^a = c^a(t)$ is *autoparallel* if its tangent vector is transported by parallelism.

3 Implicit first order differential dynamical systems with holonomic, scleronomic constraints given by parametric equations ([9])

An implicit first order differential dynamical system is defined by a function $F : (t, x, \dot{x}) \in \mathbf{R} \times_M TM \to T^*M$, which is written, in a local chart, as:

$$F = F_i \left(t, x, \dot{x}\right) dx^i \in T^*_x M.$$

Its kernel is locally expressed by

$$KerF = \left\{(t, x, \dot{x}) \in \mathbf{R} \times TM \left| F_i \left(t, x, \dot{x}\right) dx^i = 0 \right.\right\}.$$

Thus, the system of equations:

$$F_i \left(t, x, \dot{x}\right) = 0 \tag{3.1}$$

represents the dynamical system.

By the definition, it follows that these functions change, on a change of local chart, by the rules:

$$\overline{F}_h = \frac{\partial x^i}{\partial \bar{x}^h} F_i. \tag{3.2}$$

To the dynamical system given by the equations (3.1) we associate the functions:

$$a^1_{ij} = \frac{\partial F_i}{\partial \dot{x}^j}, \quad a^0_{ij} = \frac{\partial F_i}{\partial x^j}, \tag{3.3}$$

which change, on a change of local chart of the form (1.0), by the rules (1.1) and (1.2) respectively.

It follows that the dynamical system (3.1) defines on the space M a geometrical structure (a d-metric and a d-connection) and thus, we have a geometry.

A solution of the dynamical system (3.1) is built by a set of functions $x^i = c^i(t)$ such that they, together with there derivatives $\dot{c}^i = \dfrac{dc^i}{dt}$, transform the equations of the system in identities: $F_i \left(t, c(t), \dot{c}(t)\right) \equiv 0$.

Let us now consider a dynamical system (3.1), whose equations do not depend explicitly on time and let $x^i = c^i(t)$ be a solution of it such that: $F_i \left(c^h(t), \dot{c}^h(t)\right) \equiv 0$. By derivation, these relations lead us to: $\dfrac{\partial F_i}{\partial \dot{x}^j} \dfrac{d\dot{c}^j}{dt} + \dfrac{\partial F_i}{\partial x^j} \dfrac{dc^j}{dt} = 0$. By $\dfrac{d\dot{c}^j}{dt} = \dfrac{d^2 c^j}{dt^2}$, (3.3) and (1.2′), we obtain: $\dfrac{d^2 c^i}{dt^2} + M^i_j \dfrac{dc^j}{dt} = 0$. Thus, we have:

Theorem 4. *Any solution of the given system is an autoparallel curve with respect to the geometrical structure imposed by the system.*

Conversely, let $x^i = c^i(t)$ be an autoparallel curve. The equation $\dfrac{d^2 c^i}{dt^2} + M^i_j \dfrac{dc^j}{dt} = 0$ can be written in the form: $\dfrac{d\dot{c}^i}{dt} + a_1^{ih} a^0_{hj} \dfrac{dc^j}{dt} = 0$ and, by contraction with a^1_{ki}, it follows that: $\dfrac{\partial F_i}{\partial \dot{c}^j} \dfrac{d\dot{c}^j}{dt} + \dfrac{\partial F_i}{\partial c^j} \dfrac{dc^j}{dt} = \dfrac{dF_i}{dt} = 0$. If we integrate, we obtain: $F_i = C_i$ (constant, holonomic manifold). For the constants $C_i = 0$ it follows that the autoparallel curves are solutions of the given system.

4 Implicit first order differential dynamical systems with holonomic constraints given by parametric equations

Let us consider a dynamical system (3.1) on a differentiable manifold M and the submanifold V of the holonomic constraints imposed to the system, assumed as given by the parametric equations (2.0) with the condition (2.0′).

Thus, the equations of the system with these constraints, lead us to the restriction of the system to the submanifold V.

The restrictions to V of the equations of the system given by the kernel of the function $F = F_i dx^i$, defined on $\mathbf{R} \times TM$ with values in T^*M, are:

$$F\,|_V = F_i\left(t, x^h\left(u^b\right), \dot{x}^h\left(u^b, \dot{u}^b\right)\right)\frac{\partial x^i}{\partial u^a}du^a = \Phi_a du^a = 0.$$

We have: $\Phi_a = \dfrac{\partial x^i}{\partial u^a}F_i\,|_V = 0$. To the system $\Phi_a = 0$ we associate the functions $\alpha^1_{ab} = \dfrac{\partial \Phi_a}{\partial \dot{u}^b}$ and $\alpha^0_{ab} = \dfrac{\partial \Phi_a}{\partial u^b}$. On a change of local chart on M and a change of parameters on V, these functions change by the rules: $\bar{\alpha}^1_{cd} = \dfrac{\partial u^a}{\partial \bar{u}^c}\dfrac{\partial u^b}{\partial \bar{u}^d}\alpha^1_{ab}$, $det\left(\alpha^1_{ab}\right) \neq 0$, and $\bar{\alpha}^0_{cd} = \dfrac{\partial u^a}{\partial \bar{u}^c}\left[\dfrac{\partial u^b}{\partial \bar{u}^d}\alpha^0_{ab} + \dfrac{\partial \dot{u}^b}{\partial \bar{u}^d}\alpha^1_{ab}\right]$ respectively. These functions represent nothing else but the restrictions of the functions $a^1_{ij} = \dfrac{\partial F_i}{\partial \dot{x}^j}$, $a^0_{ij} = \dfrac{\partial F_i}{\partial x^j}$ of the geometrical structure, to the submanifold V.

Lemma 9. *The following diagram is commutative:*

$$\begin{tikzcd}
F_i \arrow[r, "h_1"] \arrow[d, "v_1"'] \arrow[dr, phantom, "\searrow\searrow"] & \overline{F}_h = \dfrac{\partial x^i}{\partial \bar{x}^h}F_i \arrow[d, "v_2"] \\
\Phi_a \arrow[r, "h_2"] & \overline{\Phi}_b = \dfrac{\partial u^a}{\partial \bar{u}^b}\Phi_a
\end{tikzcd}$$

Theorem 5. *The constraints manifold (on which we find the admissible solutions of the given system) has as structure the structure of the system reduced to it.*

This follows from Lemma 9 and Theorem 4.

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