Optimal control problems related to kinematics

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Dedicated to the 70-th anniversary
of Professor Constantin Udriste

Abstract. The paper considers three main control problems related to kinematics in the n-dimensional real space. The aim of the first problem is to maximize the kinetic energy by means of a control field. The initial field of forces and the control field are not assumed to be conservative. We give explicit solutions for the controlled motion and maximal kinetic energy. We also derive relations between the work and the maximal kinetic energy, as well as other consequences. Another problem is concerned with the control of the motion of a material particle, under a Newton's like field of forces in the n-dimensional real space. We prove a relation between the norm of the velocity and the distance to the gravitational attraction center. In particular, this formula allows the control of the velocity at the impact moment. The last main result concerns the optimal control of the motion of a material particle under a field of conservative forces, on an arbitrary smooth hypersurface. Examples and related comments are presented.

M.S.C. 2000: 26A12, 49J15, 49J40, 49S05, 34A40, 53A17, 53A04, 53A05.
Key words: Optimal control problems; extremal problems; kinematics.

1 Introduction

The problem of controlling some physical and economical processes is very important, for several prediction and low-cost reasons ([1], [2], [3], [4], [6], [7], [9], [13], [15], [18], [19], [20], [22], [23], [21], [24]). Recent works on dynamical systems show the connection between this field and nonlinear inequalities, as well as the importance of related iterative methods [10]. Our first goal is to optimize the free motion of a material particle by means of a suitable chosen control vector field. We show the possible connection between this first general result and the maximum principle proven in [3] using time minimization. For generalizations of similar maximum principles with differential constraints, see [23]. For vector optimization with motivated constraints see [2], [4] [13], [18]. Our proof starts from an equivalent principle: the conservation of the whole energy and minimizing the potential energy lead to the maximization of the kinetic energy, hence to the maximization of the norm of the velocity field. We derive the law of optimal motion explicitly, for the general case of nonconservative forces.

BSG Proceedings 17. The International Conference "Differential Geometry-Dynamical Systems 2009" (DGDS-2009), October 8-11, 2009, Bucharest-Romania, pp. 163-171.

Some relations between work and maximal kinetic energy are also derived. Finally, we consider the optimal control problem of the motion on hypersurfaces. Similar results are contained in [3], [9], [15], [19], [21], [24], [22]. Another quite similar problem has been solved in [17], by means of the maximum modulus principle for holomorphic functions. For general background see [1], [3], [5], [7], [8], [9], [19], [24],[22]. Recent related results using other methods can be found in [2], [4], [11], [13], [18], [20], [22], [23], [21]. Other constrained optimization problems in infinite dimensional spaces have been studied in [16]. For "algebraic" equations with operator coefficients and differential equations involving operator coefficients, see [12].

The rest of the paper is organized as follows. The first part of Section 2 is concerned with two "free" control problems. The first problem, solved in Theorem 2.1, leads to some "difference inequalities" concerning the relationship between the work and the kinetic energy (Corollary 2.1). We prove the explicit expression of the maximized kinetic energy, under the hypothesis that the forces are conservative, in Corollary 2.2. Theorem 2.2, the last result of Section 2, is concerned with another control problem, related to a gravitational Newton's like field of forces. Some equalities involving the norms of the basic fields of motion at any moment of time are established. We then derive the possibility of controlling the velocity at the impact moment. In Section 3, we solve an optimal control problem for a constrained motion. The material particle is moving on an arbitrary given hypersurface (Theorem 3.1). In example 3.1, an application to a particular hypersurface is briefly discussed.

2 A general optimal control problem related to kinematics

In the following theorem a material particle of unit mass is moving in $\mathbb{R}^n$ under the action of a field of forces $F(Y) = (f_1, ..., f_n)(Y)$. Our goal is to add a control field $H(t) = (h_j(t))_{j=1}^n, t \in [t_0, t_1]$, so that the kinetic energy has the maximum possible value at any moment t. Here $Y(t) = (y_j(t))_{j=1}^n$ is the state vector field of the moving particle, while t_0 is the starting moment. For simplicity we additionally assume that $Y(t_0) = \dot{Y}(t_0) = (0, ..., 0)$, i.e. the material particle starts from the origin of the space, and has zero velocity at the starting moment.

Theorem 2.1 *Let*

$$\begin{aligned} \ddot{y}_j(t) &= f_j\left(y_1\left(t\right), ..., y_n\left(t\right)\right) + h_j\left(t\right), t \in \left[t_0, t\right], \\ \dot{y}_j\left(t_0\right) &= 0,\ j = 1, ..., n,\ y_j\left(t_0\right) = 0, j = 1, ..., n \end{aligned} \tag{2.1}$$

be the law of the controlled motion described above. In order to maximize the kinetic energy T at any moment t, the control field H and the law of motion are given by:

$$H(t_1) = \lambda \dot{Y}(t_1),\ y_j(t_1, \lambda) = (1/\lambda) \cdot \int_{t_0}^{t_1} \left[\left(\exp\left(\lambda\left(t_1 - u\right)\right)\right) - 1\right] \cdot f_j(Y(u))du, \tag{2.2}$$

$\forall t_1 \geq t_0$, *where $\lambda = const.$ is a free parameter, and the maximum of the kinetic energy is:*

$$T_{\max}(t_1,\lambda) = \exp(2 \cdot |\lambda| \cdot t_1) \int_{t_0}^{t_1} \left[\exp\left(-2\left|\lambda\right| t\right) \frac{d}{dt}\left(W\left(t\right)\right)\right] dt,\ \forall t_1 \geq t_0, \tag{2.3}$$

where $W(Y(t)) = \int_{t_0}^{t} \left[\sum_{j=1}^{n} f_j\left(Y\left(u\right)\right)\dot{y}_j\left(u\right)\right] du$ *stands for the work along the path of the motion, in time interval* $[t_0, t]$.

Proof. We multiply each equation (2.1) by $\dot{y}_j$. This leads to:

$$\begin{aligned}\ddot{y}_j\dot{y}_j &= f_j(Y)\dot{y}_j + h_j\dot{y}_j \Rightarrow \\ T\left(t\right) &= \frac{1}{2}\left[\sum_{j=1}^{n}\left(\dot{y}_j\left(t\right)\right)^2\right] \\ &= \int_{t_0}^{t}\sum_{j=1}^{n} f_j\left(Y\left(u\right)\right)\dot{y}_j\left(u\right)du + \int_{t_0}^{t}\sum_{j=1}^{n}\left(h_j\dot{y}_j\right)\left(u\right)du \\ &\leq W\left(Y\left(t\right)\right) + \left[\int_{t_0}^{t}\sum_{j=1}^{n} h_j^2\right]^{1/2}\left[\int_{t_0}^{t}\sum_{j=1}^{n}\dot{y}_j^2\right]^{1/2},\end{aligned} \tag{2.4}$$

and equality holds if and only if there exists $\lambda(u) \in \mathbb{R}$ such that $H(u) = \lambda(u)\dot{Y}(u)$ $\forall u \geq t_0$. This parallelism condition leads to $\max T(t)$. Since H is a control vector field, we can choose its components such that $\lambda(u) = \lambda = const.$ Hence $H = \lambda\,\dot{Y}$, with $\lambda = const.$, so that the first relation (2.2) is proved.

Introducing relations $h_j = \lambda\dot{y}_j$ into (2.1), one obtains the equations:

$$\ddot{y}_j - \lambda\dot{y}_j = f_j(Y),\ j = 1, ..., n \Rightarrow \dot{y}_j(t) - \lambda y_j(t) = \int_{t_0}^{t} f_j(Y(u))du.$$

Further integration leads to (also using the initial conditions):

$$\begin{aligned} y_j(t_1) &= \exp(\lambda t_1)\int_{t_0}^{t_1}\left[\exp(-\lambda t)\cdot\int_{t_0}^{t} f_j\left(Y\left(u\right)\right)du\right]dt \\ &= \frac{1}{\lambda}\exp\left(\lambda t_1\right)\cdot\int_{t_0}^{t_1} f_j\left(Y\left(u\right)\right)\left[\exp\left(-\lambda u\right) - \exp\left(-\lambda t_1\right)\right]du \\ &= \frac{1}{\lambda}\int_{t_0}^{t_1} f_j\left(Y\left(u\right)\right)\left[\exp\left(\lambda\left(t_1 - u\right)\right) - 1\right]du, \forall t_1 \geq t_0.\end{aligned}$$

In the above computation the change of the order of integration in the double integral was made on the triangle of vertices (t_0, t_0), (t_1, t_0), (t_1, t_1). Thus, the second relation (2.2) is also proved. Finally, in order to prove (2.3), we see that all the preceding relations were obtained from the first relation (2.2), derived from the requirement to maximize T. From (2.4) and from the comment which follows it, we have (via (2.2)):

$$\begin{aligned} T_{\max}\left(t, \lambda\right) &= W\left(t\right) + \left|\lambda\right|\int_{t_0}^{t}\left[\sum_{j=1}^{n}\left(\dot{y}_j\left(u\right)\right)^2\right]du \Rightarrow \\ \frac{d}{dt}\left(T_{\max}\right) &= \frac{d}{dt}\left(W\right) + \left|\lambda\right| 2T_{\max}.\end{aligned}$$

Integrating this linear equation, one obtains the solution given by (2.3). □

Remark 2.1 Although λ seems to be arbitrary, from its definition we infer that some bounds for:

$$\lambda = \frac{h_j(t)}{\dot{y}_j(t)} = \frac{\left[\int_{t_0}^{t_1} \sum_{j=1}^{n} h_j^2\right]^{1/2}}{\left[\int_{t_0}^{t_1} \sum_{j=1}^{n} (\dot{y}_j)^2\right]^{1/2}} = const.$$

can be deduced from estimations of $\dot{y}_j$, h_j. In the above expression of λ, the usual convention on the case $\frac{0}{0}$ is made, both functions being assumed analytic. From the proof of Theorem 2.1 it follows that the goal of maximizing the norm of velocity works for $\lambda > 0$, when the motion, as well as the kinetic energy seem to be not stable. This is not always the case, because in some cases the time can take negative values, or upper bounded positive values. This corresponds to models similar to the case when the particle is attracted by a material body, following Newton's gravitation law (see Theorem 2.2).

Remark 2.2 From Theorem 2.1, relations (2.1), (2.2), we have: $\dot{h}_j = \lambda \ddot{y}_j = \lambda (f_j (Y) + h_j)$. This yields:

$$\ddot{h}_j = \lambda \left[\sum_{k=1}^{n} \frac{\partial f_j}{\partial x_k} \dot{y}_k + \dot{h}_j \right] \Rightarrow \ddot{h}_j - \lambda \dot{h}_j = \sum_{k=1}^{n} \frac{\partial f_j}{\partial x_k} h_k \,.$$

This relation is analogue to the maximum necessary condition (1.25) from [3] (maximum principle), systematically applied in [3] to several linear and nonlinear optimal control problems.

Remark 2.3 The assumptions $Y(t_0) = \dot{Y}(t_0) = (0, ..., 0)$ from Theorem 2.1 are not essential and usually may be not suitable for various classical problems related to kinematics. For example, the importance of studying the motion on an ellipse is well known, the attractor "point" being situated at a foci of the ellipse. Then we usually choose the origin in the symmetry center of the ellipse or at the foci, when clearly we may have non-vanishing Y, $\dot{Y}$ for the moving point on the ellipse. That is why we give below the general formulae for the motion law and for the optimum kinetic energy, the proof being the same.

$$\text{(2.5)} \quad \begin{aligned} & y_j(t_1, \lambda) - y_j(t_0, \lambda) \\ = \; & \frac{1}{\lambda} \left\{ \int_{t_0}^{t_1} f_j (Y(u)) \left[\exp(\lambda (t_1 - u)) - 1 \right] du + \dot{y}_j \ (t_0) \cdot \left[\exp(\lambda (t_1 - t_0)) - 1 \right] \right\} \end{aligned}$$

$$\text{(2.6)} \quad \begin{aligned} & T_{\max}(t_1, \lambda) \\ = \; & \exp(2 |\lambda| t_1) \int_{t_0}^{t_1} \exp(-2 |\lambda| t) \, dW(Y(t)) + T_{\max}(t_0) \exp\left[2 \cdot |\lambda| \cdot (t_1 - t_0)\right]. \end{aligned}$$

Corollary 2.1 *If* $\frac{d}{dt} W(t) > 0 \ \forall t \in [t_0, t_1]$, *then the following inequalities hold:*

$$\begin{aligned} \left[W(Y(t_1)) - W(Y(t_0)) \right] & < T_{\max}(t_1) - T_{\max}(t_0) \cdot \exp\left[2 \cdot |\lambda| \cdot (t_1 - t_0)\right] \\ & < \exp\left[2 \cdot |\lambda| \cdot (t_1 - t_0)\right] \cdot \left[W(Y(t_1)) - W(Y(t_0)) \right]. \end{aligned}$$

Proof. One uses the inequalities:

$$\exp(-2 \cdot |\lambda| \cdot t_1) < \exp(-2 \cdot |\lambda| \cdot t) < \exp(-2 \cdot |\lambda| \cdot t_0) \ \ \forall t \in \,]t_0, t_1[\ ,$$

multiplied by $dW(t) > 0$, under the integral sign from (2.6). □

Corollary 2.2 *Under the same hypothesis, assume additionally that the material particle is moving on a closed path in the time interval* $[t_0, t_1]$, *while the field of forces* F *is conservative* $(F = \nabla U)$. *Then we have:*

$$T_{\max}(t_1, \lambda) = T_{\max}(t_0, \lambda) + 2\,|\lambda| \int_{t_0}^{t_1} T(u)du.$$

Proof. Using the proof of Theorem 2.1 and the special hypothesis, one obtains:

$$\left[\tfrac{1}{2}\sum_{j=1}^{n}(\dot{y}_j)^2\right]\Bigg|_{t_0}^{t_1} = \int_{t_0}^{t_1}\left[\sum_{j=1}^{n}\frac{\partial U}{\partial y_j}(Y(u))\cdot \dot{y}_j(u)\right]du + |\lambda| \int_{t_0}^{t_1}\left[\sum(\dot{y}_j(u))^2\right]du$$
$$= 2\,|\lambda| \int_{t_0}^{t_1} T(u)\,du,$$

because of the assumption that the path is closed $(Y(t_1) = Y(t_0))$, which lead to the conclusion that the first integral in the above formula is zero. □

The next theorem studies the motion behavior when the material particle is under a Newton-like field of forces. This time we add a control field in order to reduce the velocity at the moment of impact. We denote this control field as above and we consider it to be conservative: $H = \nabla V$.

Theorem 2.2 *Let*

$$\ddot{Y} = F + H,\ F = -k\frac{(y_1, ..., y_n)}{\left(\sum_{j=1}^{n} y_j^2\right)^{3/2}} = -k\frac{Y}{||Y||^3} = \nabla U, \tag{2.7}$$

$U = \frac{k}{||Y||}, Y \neq (0, ..., 0), H = \nabla V$ *be the equation of the controlled motion, (for* $n = 3$, F *is exactly the Newtonian gravitational attraction field, centered at the origin* $(0, ..., 0) = Y(t_1)$*). Then for* $V(t) = \lambda(t)\,||\dot{Y}(t)||^2$, *where* $\lambda = \lambda(t) < 1/2$ *is a control function, we have:*

$$||\dot{Y}||^2||Y|| \cdot [(1/2) - \lambda] = k = const, ||\ddot{Y}|| \cdot ||Y||^2 = k, \frac{||\dot{Y}||}{||\dot{Y}||} \cdot \frac{||Y||}{||\dot{Y}||} \cdot \frac{2}{1 - 2\lambda} = 1. \tag{2.8}$$

In particular, it follows that around moment t_1 *at which the material particle meets the attractor point* $O(0, ..., 0) = Y(t_1)$, *since* $||Y|| \to 0$, *we have* $||\dot{Y}|| \to \infty$, *with the control relation:*

$$||\dot{Y}||^2 = \frac{2k}{(1 - 2\lambda) \cdot ||Y||} \leq M for\ 1 - 2\lambda \geq \frac{2k}{M \cdot ||Y||}.$$

Hence, by choosing $\lambda = \lambda(t) \to -\infty$ *such that the last relation is satisfied at any moment, the velocity at the impact moment is bounded by* $\sqrt{M}$, $M > 0$ *being a given suitable threshold.*

Proof. We have:

$$\begin{aligned}\sum_{j=1}^{n} \ddot{y}_j \cdot \dot{y}_j &= \left\langle \nabla U(Y), \dot{Y} \right\rangle + \left\langle \nabla V(Y), \dot{Y} \right\rangle \Rightarrow \\ \frac{1}{2}\left[\sum_{j=1}^{n} (\dot{y}_j(t))^2\right] &= U(Y(t)) - U(-\infty) + V(Y(t)) - V(-\infty) = \\ &= U(Y(t)) + V(Y(t)), \ \ \forall t < t_1\end{aligned}$$

where t_1 is the impact moment, hence $Y(t_1) = O(0, ..., 0)$, and we additionally assume that the kinetic energy at $-\infty$, as well as U and V at $-\infty$ are zero (on the expression of the Newtonian like potential U, this assumption is natural: a long (infinite) period of time for a motion, usually also means a long distance, so that $\lim_{t\to-\infty} U(t) = \frac{1}{\infty} = 0$). The last relation can be rewritten as:

$$\begin{aligned}T(t) &= \frac{1}{2}||\dot{Y}||^2 = \frac{k}{||Y(t)||} + V(t) \Leftrightarrow \\ \left(\frac{1}{2}||\dot{Y}||^2 - V\right) \cdot ||Y|| &= k = const. \Leftrightarrow ||\dot{Y}||^2 ||Y|| \left(\frac{1}{2} - \lambda\right) = k\end{aligned}$$

where we choose the control potential $V = \lambda || \dot{Y} ||^2$, where $\lambda = \lambda(t) < \frac{1}{2}$ is a free control scalar function. Hence we have:

$$||\dot{Y}||^2 \cdot ||Y|| (1/2 - \lambda) = k, \tag{2.9}$$

which proves the first relation (2.8). From (2.7), going to norms, we obtain:

$$||\ddot{Y}|| \cdot ||Y||^2 = k$$

The last two relations lead to the conclusion:

$$\frac{||\ddot{Y}||}{||\dot{Y}||} \cdot \frac{||Y||}{||\dot{Y}||} \cdot \frac{2}{1 - 2\lambda} = 1.$$

The last control relation from the statement and its meaning are consequences of the first equality (2.8). The proof is complete. □

3 Restricted optimal control problems

Let $\sum$ be a hypersurface in $\mathbb{R}^n$, given by the equation $G(x_1, ..., x_n) = 0, G \in C^1(D)$. We consider the problem of optimal control motion of a material particle on $\sum$, given by the vector equation:

$$\ddot{X} = F + H = -\nabla U - \nabla V \tag{3.1}$$

where $F = -\nabla U$ is the field of the forces, $H = -\nabla V$ is the field of the control-"forces", both of them being assumed conservative, while X is the state vector field.

Theorem 3.1 *The optimal motions satisfy the condition:*

$$\ddot{X} = -\alpha \nabla G, \ \ \alpha = const. \tag{3.2}$$

Along a path of such an optimal motion, the norm of the velocity field is constant $||\dot{X}|| = v = const.$ *We have* $ds = vdt$*, and the path is on a geodesic of the hypersurface* $\sum$*. The following condition on* V *is necessary and sufficient for such a law of motion:*

$$\nabla(U + V) = \alpha \nabla G, \tag{3.3}$$

along the path.

Proof. Relations (3.1) yield:

$$\begin{aligned} \sum_{j=1}^{n} \ddot{x}_j \dot{x}_j &= -\frac{dU\left(X\left(t\right)\right)}{dt} - \frac{dV\left(X\left(t\right)\right)}{dt} \Rightarrow \\ T\left(t\right) - T\left(t_0\right) &= (1/2)\left[\sum_{j=1}^{n} \left(\dot{x}_j\left(t\right)\right)^2 - \sum_{j=1}^{n} \left(\dot{x}_j\left(t_0\right)\right)^2\right] = -U\left(X\left(t\right)\right) - V\left(X\left(t\right)\right), \end{aligned}$$

$t > t_0$ where we choose the potentials U, V such that both of them are vanishing at $X\left(t_0\right)$. By the theory of conditional extrema for functionals, there exists $\lambda \in \mathbb{R}$, such that the following functional involving the controlled action has a critical function X:

$$J\left[X\right] = \int_{t_0}^{b} \left[\frac{1}{2}\left(\sum_{j=1}^{n} \dot{x}_j^2\right) + U\left(X\right) + V\left(X\right) + \lambda G\left(X\right)\right](t)\,dt.$$

The functional J must usually have a minimum, due to the minimum action principle. The system of Euler-Lagrange equations leads to:

$$\ddot{X} = \nabla\left(U + V\right) + \lambda \nabla G. \tag{3.4}$$

From relations (3.1) and (3.4), by addition, one obtains: $\ddot{X} = \frac{\lambda}{2}\nabla G = -\alpha \nabla G$, where $\alpha = -\frac{\lambda}{2}$. This proves (3.2), which leads to (3.3) via (3.1). Now the basic conclusion (3.2) yields:

$$\begin{aligned} \left\langle \ddot{X}, \dot{X} \right\rangle &= 0, \text{ which is equivalent to } ||\dot{X}|| = v = const. \Rightarrow \\ s\left(t\right) &= \int_{t_0}^{t} vdt = v\left(t - t_0\right) \Rightarrow ds = vdt \end{aligned} \tag{3.5}$$

along the motion path. Hence it only remains to prove the fact that such motions are along the geodesics of $\sum$. But we have already proved relation (3.2), which together with relation (3.5) yields:

$$\frac{d^2X}{ds^2} = \frac{1}{v^2}\ddot{X} = -\frac{\alpha}{v^2} \cdot \nabla G,$$

which are well known to be the equation of a geodesic, when the parameter is the natural one, s. By t_0 we denote the starting moment, where $s = s\left(t_0\right) = 0$. The proof is complete. □

Example 3.1 Let $\sum$ be the hyperellipsoid defined by the equation:

$$G(X) = \sum_{j=1}^{n} \frac{x_j^2}{a_j^2} - 1 = 0.$$

If the controlled motion is given by (3.1) and the control field satisfies (3.2), then the paths of the optimal motions are given by:

$$\ddot{x}_j(t) = -\alpha \frac{2x_j(t)}{a_j^2}, \ j = 1, ..., n. \tag{3.6}$$

Assuming that we do not have an upper bound for the time of the motion and also using the fact that the ellipsoid is bounded and $d(O, \sum) > 0$, in equations (3.6) we must have $\alpha > 0$. Hence the solutions for (3.6) are given by:

$$x_j(t) = \alpha_j \cos(\omega_j t) + \beta_j \sin(\omega_j t), \ j = 1, ..., n, \tag{3.7}$$

where $\omega_j = \sqrt{2\alpha}/a_j$.

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