

Principal flat connections on second order tangent spaces

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*Dedicated to the 70-th anniversary
of Professor Constantin Udriste*

Abstract. Considering two natural principal bundle structures on T^2M , one prove that a Finsler metric defines flat connections in these bundles.

M.S.C. 2000: 53A40, 53C05, 53C60, 70H50.

Key words: Principal bundle; flat connection; Finsler map; Lagrangian; second order tangent bundle.

1 Introduction

Considering natural Lie group structures on $G^2 = \mathbb{R}^* \times \mathbb{R}$ and $G_+^2 = \mathbb{R}_+^* \times \mathbb{R}$, one defines natural actions on $\mathbb{R}_{**n}^{2n} = \mathbb{R}^{2n} \setminus (\{0_n\} \times \mathbb{R}^n)$. The orbit spaces $\mathbb{R}P^{(2)n-1}$ and $S^{(2)n-1}$ of the actions are vector bundles over the projective space $\mathbb{R}P^{n-1}$ and the sphere S^{n-1} respectively. The space \mathbb{R}_{**n}^{2n} becomes principal bundle over $\mathbb{R}P^{(2)n-1}$ and $S^{(2)n-1}$ (Propositions 2.2 and 2.3). There exist natural right actions of G^2 and G_+^2 on the fibers of the second tangent bundle $T^2M \rightarrow M$ such that the natural affine bundle $T^2M \rightarrow TM$ is a G^2 -equivariant bundle and a G_+^2 -equivariant bundle as well. The orbit spaces P^2M and S^2M are affine bundles over PM and SM respectively. The manifold T^2M has two structures of principal bundles over P^2M and S^2M respectively (Proposition 2.5). We also prove that a certain second order Lagrangian $L : T^2M \rightarrow g^2$ satisfying the conditions (3.1) defines a principal flat connection (Proposition 3.2). Starting from a Finsler metric we find such a connection. An example is considered starting from the canonical Euclidian metric on \mathbb{R}^2 . Further applications can be related to Lie groups and invariant Lagrangians, as the settings in [1] and [6].

2 Basic constructions

The motivation to construct the groups G^2 and G_+^2 , as well as their actions is given, as follows.

Let $(a, b) = I \xrightarrow{\gamma} M$ be a curve having the local form $t \xrightarrow{\gamma} \gamma^i(t)$. One can induce the maps $I \xrightarrow{\gamma^{(1)}} T^1M$ and $I \xrightarrow{\gamma^{(2)}} T^2M$ given locally by the formulas $t \xrightarrow{\gamma^{(1)}} \frac{d\gamma^i}{dt}$ and $t \xrightarrow{\gamma^{(2)}} \frac{d^2\gamma^i}{dt^2}$.

Let us suppose that the parameter t changes according to the formula $t' = t'(t)$ and let $t' \xrightarrow{\tilde{\gamma}} \tilde{\gamma}^i(t')$ be the local form of γ in the new coordinates. Obviously $\gamma = \tilde{\gamma} \circ t'$ and $\frac{dt'}{dt} \neq 0$. Then $\frac{d\gamma^i}{dt} = \frac{d\tilde{\gamma}^i}{dt'} \frac{dt'}{dt}$, $\frac{d^2\gamma^i}{dt^2} = \frac{d^2\tilde{\gamma}^i}{dt'^2} \left(\frac{dt'}{dt}\right)^2 + \frac{d\tilde{\gamma}^i}{dt'} \frac{d^2t'}{dt^2}$.

These facts suggest us to consider the sets $G^2 = \mathbb{R}^* \times \mathbb{R}$, $V^{2m} = \mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m$ and the map $\Phi : V^{2m} \times G^2 \rightarrow V^{2m}$ given by the formula $\Phi((y^i, z^i), (\alpha, \beta)) = (\alpha y^i, \frac{1}{2}\beta y^i + \alpha^2 z^i)$.

Let us consider $m = 1$. Then Φ induces a map $\Phi' : G^2 \times G^2 \rightarrow G^2$, i.e. a binary operation on G^2 , denoted by $*$, i.e. $\Phi'(g_1, g_2) = g_1 * g_2$.

Proposition 2.1. *The binary operation $*$ defines a group structure on G^2 .*

A normal subgroup of G^2 is $G^2_+ = \{(\alpha, \beta) \in G^2 : \alpha > 0\}$. It is easy to see that $G^2/G^2_+ = \mathbb{R}^* = \mathbb{R} \setminus \{0\}$, the group of non-null reals. Other subgroups (which are not normal) are $H^1 = \{(\alpha, 0) \in G^2 : \alpha \in \mathbb{R}^*\}$ (called the *subgroup of homotheties*) and $H^+ = \{(\alpha, 0) \in G^2 : \alpha > 0\}$ (called the *subgroup of positive homotheties*).

The Lie algebra $g^{(2)}$ of G^2 is the only non-commutative Lie algebra of second order, generated by $\{A, B\}$, $[A, B] = A$.

We denote below also by $*$ or Φ the right action of G^2 on $\mathbb{R}^{2m}_* = \mathbb{R}^{2m} \setminus (\{0\} \times \mathbb{R}^m)$ (i.e. given by the same formula). More explicitly, if $g = (\alpha, \beta) \in G^2$, $W^i = (y^i, z^i)$, $i = \overline{1, m}$, and $W = (W^i)_{i=\overline{1, m}} \in V^{2m}$, then $W * g = (W^i * g)_{i=\overline{1, m}}$. We denote also by $\Phi : G^2 \times \mathbb{R}^{2m}_* \rightarrow \mathbb{R}^{2m}_*$, $\Phi(g, W) = W * g$.

Proposition 2.2. *The application Φ defines a right action of G^2 on \mathbb{R}^{2m}_* . The orbit space $\mathbb{R}P^{(2)m-1}$ of the action is a vector bundle over the projective space $\mathbb{R}P^{m-1}$. The space \mathbb{R}^{2m}_* becomes a principal G^2 -bundle over $\mathbb{R}P^{(2)m-1}$.*

It is also easy to see that Φ defines a linear representation $\Phi' : G^2 \rightarrow \text{End}(\mathbb{R}^{2m})$.

The projective space $\mathbb{R}P^{n-1}$ is obtained as the orbit space of the homothety action $\mathbb{R} \times \mathbb{R}^n_* \rightarrow \mathbb{R}^n_*$. Analogously, the sphere S^{n-1} is obtained as the orbit space of the positive homothety action $\mathbb{R}^*_+ \times \mathbb{R}^n_* \rightarrow \mathbb{R}^n_*$. Specifically, if one considers \mathbb{R}^n_* , the following equivalence relations:

$$(O) \quad \bar{x}_1 \sim \bar{x}_2 \text{ iff } (\exists)\lambda \in \mathbb{R}^* \text{ such that } \bar{x}_1 = \lambda \bar{x}_2,$$

$$(O^+) \quad \bar{x}_1 \approx \bar{x}_2 \text{ iff } (\exists)\lambda \in \mathbb{R}^*_+ \text{ such that } \bar{x}_1 = \lambda \bar{x}_2,$$

then the quotient spaces \mathbb{R}^n_*/\sim and \mathbb{R}^n_*/\approx are the projective space $\mathbb{R}P^{n-1}$ and the (hyper)sphere S^{n-1} , respectively. The inclusion of $G^2_+ \subset G^2$ induces the projection $S^{n-1} \rightarrow \mathbb{R}P^{n-1}$, that is a 2-cover.

The projection $\mathbb{R}^n_* \xrightarrow{\pi_1} \mathbb{R}P^{n-1}$ is a principal bundle with structural group \mathbb{R}^* . The projection $\mathbb{R}^n_* \xrightarrow{\pi_2} S^{n-1}$ is also a principal bundle with structural group \mathbb{R}^*_+ . While the bundle π_2 is isomorphic with the trivial bundle $\mathbb{R}^*_+ \times S^{n-1} \rightarrow S^{n-1}$, the bundle π_1 is not trivial.

Let $n \geq 3$. The orbit space of the vectors in \mathbb{R}_{*n}^{2n} of the form $(0, z^1, y^2, z^2, \dots, y^n, z^n)$ is a vector bundle $B^{(n-1)}$ with the fiber \mathbb{R} over $\mathbb{R}P^{(2)n-1}$. Indeed, if $y^2 \neq 0$ and $y^3 \neq 0$, then a representative element has the form $(0, z^1, 1, 0, y^3, z^3, \dots, y^n, z^n)$ and $(0, z^{1'}, y^{2'}, z^{2'}, 1, 0, y^{4'}, z^{4'}, \dots, y^{n'}, z^{n'})$ respectively, where $z^{1'} = \frac{z^1}{y^3}$. The vector bundle $B^{(n-1)}$ plays the role of $\mathbb{R}P^{(1)n-2}$ in the classical case, when $\mathbb{R}P^{(1)n-1} = \mathbb{R}P^{(1)n-2} \sqcup \mathbb{R}^{n-1}$. In the second order case considered in this paper one have $\mathbb{R}P^{(2)n-1} = B^{(n-1)} \sqcup \mathbb{R}^{2n-2}$.

Let us consider the set $M_k = \mathbb{R}^k \times \mathbb{R}_{*n}^{2n}$ and the right action of G^2 on M_k given by the formula $((w^\alpha), (y^i, z^i)) \cdot (\alpha, \beta) = ((\alpha w^\alpha), (\alpha y^i, \frac{1}{2}\beta y^i + \alpha^2 z^i))$. The quotient space M_k/G^2 is the total space of a vector bundle over $\mathbb{R}P^{(2)n-1}$. This vector bundle $M_k/G^2 \rightarrow \mathbb{R}P^{(2)n-1}$ is associated with the principal bundle $\mathbb{R}_{*n}^{2n} \rightarrow \mathbb{R}P^{(2)n-1}$ and the left action of G^2 on \mathbb{R}^k by homotheties. The vector bundle $B^{(n-1)}$ is obtained for $k = 1$.

It is easy to see that the map Φ restricts to a map $\Phi^+ : \mathbb{R}_{*n}^{2n} \times G_+^2 \rightarrow \mathbb{R}_{*n}^{2n}$. In the case of the action Φ^+ , the Proposition 2.2 becomes:

Proposition 2.3. *The application Φ^+ defines a right action of G_+^2 on \mathbb{R}_{*n}^{2n} . The orbit space of the action is a vector bundle over the sphere S^{n-1} . The space \mathbb{R}_{*n}^{2n} becomes a principal G_+^2 -bundle over S^{n-1} .*

Proposition 2.4. *The Lie algebra \mathfrak{g}^2 is generated by the vector fields*

$$\left\{ A = x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y}, \quad B = x \frac{\partial}{\partial y} \right\}.$$

The projective bundle PM and the sphere bundle SM of a manifold M (or, more general, PE and SE of a vector bundle $E \rightarrow M$ instead of $TM \rightarrow M$) are obtained as the orbit spaces of the homothety action and of the positively homothety action respectively on the fibers of the (first) tangent bundle $TM \rightarrow M$.

There are canonical actions of G^2 and G_+^2 on the fibers of the second tangent bundle $T^2M \rightarrow M$.

Notice that G^2 and G_+^2 act on $TM \rightarrow M$ by homotheties on the fibers: if $(\alpha, \beta) \in G^2$ (or G_+^2), then the action is the homothety by α .

A G -equivariant bundle is a bundle $\pi : E \rightarrow B$ such that there exist (right) actions of a Lie group G on E and B such that $\pi(eg) = \pi(e)g$, $(\forall)g \in G, e \in E$.

Proposition 2.5. *There exist natural right actions of G^2 and G_+^2 on the fibers of the second tangent bundle $T^2M \rightarrow M$ such that the natural affine bundle $T^2M \rightarrow TM$ is a G^2 -equivariant bundle and a G_+^2 -equivariant bundle as well. The orbit spaces P^2M and S^2M are affine bundles over PM and SM respectively. The manifold T^2M has two structures of principal bundle over P^2M and S^2M , respectively.*

3 Construction of principal flat connections on second order tangent spaces

Proposition 3.1. *Let $F_0 : TM_* \rightarrow \mathbb{R}$ be a Finsler map and $F_1 : T^2M \rightarrow \mathbb{R}$ be a (-3) -homogeneous map. Then there exists a second order Lagrangian $L : T^2M_* \rightarrow \mathfrak{g}^2$*

constructed using F_0 and F_1 and satisfying the conditions:

$$(3.1) \quad \tilde{L}^1 = \alpha^{-1}L^1, \quad \tilde{L}^2 = -\beta\alpha^{-2}L^1 + L^2.$$

Example 1. Using F_0 , one can consider $F_1 = \frac{1}{F_0^3}$.

Notice that more elaborated examples can be considered.

A connection in the principal bundle $T^2M_* \rightarrow S^2M$ is defined by a differential 1-form $\omega : TT^2M_* \rightarrow g^2$ such that $\omega(A^*) = A$, $\omega(B^*) = A$, $R_g^*\omega = ad(g^{-1})\omega$.

Let us consider $w^{(2)} \in T^2M$ having local coordinates $(x^i, y^{(1)i}, y^{(2)i})$. We denote by $\omega_{(b)i}^a = \omega^a \left(\frac{\partial}{\partial y^{(b)i}} \right)$ considered at the point $w^{(2)}$ and let be $\tilde{\omega}_{(b)i}^a$ the corresponding local functions considered at the point $\tilde{w}^{(2)} = w^{(2)}g$, where $g = (\alpha, \beta)$.

Proposition 3.2. Let $L : T^2M \rightarrow g^2$ be a second order Lagrangian satisfying the conditions (3.1) and let $T^2M_L \subset T^2M_*$ be the set where $L^1 \neq 0$, such that $T^2M_L \rightarrow M$ is an open principal subbundle of $T^2M_* \rightarrow M$. Then there exists a connection in the principal bundle $T^2M_L \rightarrow M$ defined using L , and having a null curvature.

Corollary 3.3. There is a connection in the G^2 -principal bundle $T^2M_* \rightarrow M$ defined using a Finslerian F and having a null curvature.

These connections are constructed using Proposition 3.1 as in Example 1. The integral leaves of the involutive horizontal distribution are given by $L^1 = c_1$, $L^2 = c_2$.

Example 2. Let us consider the Euclidian Finslerian function $F_0 : T\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$, $F_0(x^i, y^i) = \sqrt{(y^1)^2 + (y^2)^2}$, $F_1 = 1/F_0^3$. Then the fundamental vector fields on $T\mathbb{R}_*^2 = \mathbb{R}_*^2 \times \mathbb{R}^2$ are $A^* = y^{(1)1} \frac{\partial}{\partial y^{(1)1}} + y^{(1)2} \frac{\partial}{\partial y^{(1)2}} + 2y^{(2)1} \frac{\partial}{\partial y^{(2)1}} + 2y^{(2)2} \frac{\partial}{\partial y^{(2)2}}$ and $B^* = y^{(1)1} \frac{\partial}{\partial y^{(2)1}} + y^{(1)2} \frac{\partial}{\partial y^{(2)2}}$. Let us consider the 2-Lagrangian given by Proposition 3.1 using F_0 and $F_1 = F_0^{-3}$. The horizontal bundle $H = \ker \omega$ of the connection 1-form ω given by Proposition 3.2 is generated by $X = X^{(0)i} \frac{\partial}{\partial x^i} + X^{(1)i} \frac{\partial}{\partial y^{(1)i}} + X^{(2)i} \frac{\partial}{\partial y^{(2)i}}$; the relation $\omega(X) = 0$ gives $-y^{(1)1}X^{(1)1} + y^{(1)2}X^{(1)2} = 0$ and $-y^{(2)1}X^{(1)1} + y^{(2)2}X^{(1)2} - y^{(1)1}X^{(2)1} + y^{(1)2}X^{(2)2} = 0$. Thus the horizontal distribution is generated by the vector fields

$$C_1 = \frac{\partial}{\partial x^1}, \quad C_2 = \frac{\partial}{\partial x^2}, \quad C_3 = -y^{(1)2} \frac{\partial}{\partial y^{(2)1}} + y^{(1)1} \frac{\partial}{\partial y^{(2)2}},$$

$$C_4 = -y^{(1)2} \frac{\partial}{\partial y^{(1)1}} + y^{(1)1} \frac{\partial}{\partial y^{(1)2}} - y^{(2)2} \frac{\partial}{\partial y^{(2)1}} + y^{(2)1} \frac{\partial}{\partial y^{(2)2}}.$$

This is an integrable distribution, giving rise to a 4-dimensional foliation on $\mathbb{R}^2 \times \mathbb{R}_*^2 \times \mathbb{R}^2$ having as leaves $\mathbb{R}^2 \times TS^1$, where a TS^1 has the form

$$\left(y^{(1)1}\right)^2 + \left(y^{(1)2}\right)^2 = c_1, \quad y^{(1)1}y^{(2)1} + y^{(1)2}y^{(2)2} = c_2$$

(tangent planes to homothetic circles αS^1 , $\alpha > 0$, where S^1 is the unit circle centered at the origin).

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