Semiseparable kernels of 2D generalized hybrid systems

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Dedicated to the 70-th anniversary of Professor Constantin Udriste

Abstract. A model of 2D hybrid system with boundary conditions is provided in the generalized framework of the (continuous) drift matrix in the class of matrix functions of bounded variation and the other coefficient matrices and the control vectors with elements in the space of regulated functions. The properties of the Perron-Stieltjes integral and the theory of generalized differential equations are used in the context of regulated functions. It is shown that the input-output map of this model is characterized by a generalized semiseparable kernel. The existence of realizations of such semiseparable kernels is proved.

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1 Introduction

Systems with boundary conditions appeared in a series of papers on linear estimation theory written by T. Kailath and his co-authors [12], where the systems are described in the form of integral operators with semiseparable kernels. Motivated by the problem of boundary value regulation and the smoothing of non-Markovian linear processes, A.J. Krener introduced the state space representation for time varying linear systems with boundary conditions [13], [14]. M.B. Adams, A.S. Wilsky and B.C. Levy used these systems as models in the linear estimation theory [1], [2].

The study of linear systems with boundary conditions was developed by I. Gohberg and M.A. Kaashoek in several papers [8], [9], [10].

Another important direction in Systems Theory is represented by the 2D hybrid (continuous-discrete) systems [17], which have many applications in domains such as seismology, image processing, linear repetitive processes [6], [19], iterative learning control synthesis [15] etc.

One aim of this paper is to connect the two approaches and to realize a description of a class of 2D hybrid systems with boundary conditions. In order to obtain the most

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general framework, we consider systems with the (continuous) drift matrix in the class of matrix functions of bounded variation and the other coefficient matrices and the control vectors with elements in the space of regulated functions. We use the theory of generalized differential equations, developed by S. Schwabik, M. Tvrdý and O. Vejvoda [20], the properties of the Perron-Stieltjes integral and the study of these devices in the case of regulated functions [21], [22]. The topic of regulated functions was developed in a series of papers ([7], [11]).

Generalized 2D kernels are defined in this framework and some semiseparable such kernels are emphasized. We look for realizations of generalized 2D semiseparable kernels, i.e. generalized 2D hybrid systems whose input-output maps are represented by means of these kernels. To this aim we consider a class of generalized 2D hybrid time-variable linear systems which represents the extension to this framework of Attasi's 2D discrete-time time-invariant systems [3].

The formulæof the state and of the output are obtained firstly for causal systems. Then the input-output map of 2D systems with boundary conditions (i.e. acausal) is derived.

It is shown that these input-output maps include a generalized 2D semiseparable kernel and conversely, each such kernel has a realization within the considered class of systems.

This study can be continued in many directions, for instance stabilization with feedback control [4], multi-time Hamilton systems [5] or adjoint multidimensional systems [18].

We shall use the following definitions and notations. A function $f : [a, b] \to \mathbf{R}$ which posseses finite one sided limits $f(t-)$ and $f(t+)$ for any $t \in [a, b]$ (where by definition $f(a-) = f(a)$ and $f(b+) = f(b)$) is said to be *regulated* on $[a, b]$. The set of all regulated functions denoted by $G(a, b)$, endowed with the supremal norm, is a Banach space; the set $BV(a, b)$ of functions of bounded variation on $[a, b]$ with the norm $||f|| = |f(a)| + \mathrm{var}_a^b f$ is also a Banach space; the Banach space of n-vector valued functions belonging to $G(a, b)$ and $BV(a, b)$ respectively are denoted by $G^n(a, b)$ and $BV^n(a, b)$ (or simply G^n and BV^n); $BV^{n\times m}$ denotes the space of $n\times m$ matrices with entries in $BV(a, b)$. The set of functions $f : [a, b] \times \mathbf{Z} \to \mathbf{R}$ such that $\forall k \in \mathbf{Z}, f(\cdot, k) \in G(a, b)$ $(BV(a, b))$ will be denoted $G_1(a, b)$ $(BV_1(a, b))$ and similar significances will have the above mentioned spaces with subscript 1 (G_1^n, BV_1^n, $BV_1^{n\times m}$).

2 Perron-Stieltjes integral and generalized differential equations

We recall the basic properties of the Perron-Stieltjes integral and of the generalized differential equations, by following [20], [21] and [22].

A pair $D = (d, s)$ where $d = \{t_0, t_1, \ldots, t_m\}$ is a division of $[a, b]$ (i.e. $a = t_0 < t_1 < \ldots < t_m = b$) and $s = \{s_1, \ldots, s_m\}$ verifies $t_{j-1} \le s_j \le t_j$, $j = 1, \ldots, m$ is called a *partition* of $[a, b]$.

A function $\delta : [a, b] \to (0, +\infty)$ is called a *gauge* on $[a, b]$.

Given a gauge δ, the partition (d, s) is said to be *δ-fine* if

$$[t_{j-1}, t_j] \subset (s_j - \delta(s_j), s_j + \delta(s_j)), \quad j = 1, \ldots, m.$$

Given the function $f, g : [a,b] \to \mathbf{R}$ and a partition $D = (d, s)$ of $[a,b]$ let us associate the integral sum

$$S_D(f\Delta g) = \sum_{j=1}^{m} f(s_j)(g(t_j) - g(t_{j-1})).$$

Definition 2.1. The number $I \in \mathbf{R}$ is said to be the *Perron-Stieltjes* (*Kurzweil*) *integral of f with respect to g from a to b* and it is denoted $\int_a^b f \mathrm{d}g$ or $\int_a^b f(t)\mathrm{d}g(t)$ if for any $\varepsilon > 0$ there exists a gauge δ on $[a,b]$ such that $|I - S_D(f\Delta g)| < \varepsilon$ for all δ-fine partitions D of $[a,b]$.

Given $f \in G(a,b)$ and $g \in G([a,b]\times[a,b])$ we define the differences $\Delta^+, \Delta^-, \Delta$ and $\Delta_s^+, \Delta_s^-, \Delta_s$ by $\Delta^+ f(t) = f(t+) - f(t)$, $\Delta^- f(t) = f(t) - f(t-)$, $\Delta f(t) = f(t+) - f(t-)$, $\Delta_s^+ g(t,s) = g(t,s+) - g(t,s)$, $\Delta_s^- g(t,s) = g(t,s) - g(t,s-)$; $\mathbf{D}^-(f)$, $\mathbf{D}^+(f)$ denote respectively the set of the left and right discontinuities of f in $[a,b]$ and similarly we can define for g $\mathbf{D}_t^-(g)$, $\mathbf{D}_t^+(g)$ with respect to the argument t. We denote by $\sum_t$ the sum $\sum_{t\in\mathbf{D}}$ where $\mathbf{D} = \mathbf{D}^-(f) \cup \mathbf{D}^+(f) \cup \mathbf{D}^-(g) \cup \mathbf{D}^+(g)$.

Theorem 2.2. ([21, Theorems 2.8 and 2.15]). *If $f \in G(a,b)$ and $g \in BV(a,b)$ then the Perron-Stieltjes integrals $\int_a^b f dg$ and $\int_a^b g df$ exist and*

$$\int_a^b f \mathrm{d}g + \int_a^b g \mathrm{d}f = f(b)g(b) - f(a)g(a) + \sum_t [\Delta^- f(t)\Delta^- g(t) - \Delta^+ f(t)\Delta^+ g(t)].$$

Theorem 2.3. ([21, Proposition 2.16]). *If $\int_a^b f dg$ exists, then the function $h(t) = \int_a^t f dg$ is defined on $[a,b]$ and*

i) if $g \in G(a,b)$ then $h \in G(a,b)$ and, for any $t \in [a,b]$

$$\Delta^+ h(t) = f(t)\Delta^+ g(t), \quad \Delta^- h(t) = f(t)\Delta^- g(t)$$

ii) if $g \in BV(a,b)$ and f is bounded on $[a,b]$, then $h \in BV(a,b)$.

Theorem 2.4. (substitution, [21, Theorem 2.19]). *Let f, g, h be such that h is bounded on $[a,b]$ and the integral $\int_a^b f \mathrm{d}g$ exists. Then the integral $\int_a^b h(t)f(t)\mathrm{d}g(t)$ exists if and only if the integral $\int_a^b h(t)\mathrm{d}\left[\int_a^t f(s)\mathrm{d}g(s)\right]$ exists, and in this case*

$$\int_a^b h(t)f(t)\mathrm{d}g(t) = \int_a^b h(t)\mathrm{d}\left[\int_a^t f(s)\mathrm{d}g(s)\right].$$

Theorem 2.5. (Dirichlet formula, [20, Theorem I.4.32]). *If $h : [a,b]\times[a,b] \to \mathbf{R}$ is a bounded function and $var_a^b h(s,\cdot) + var_a^b h(\cdot,t) < \infty$, $\forall t, s \in [a,b]$, then for any*

$f, g \in BV(a, b)$

$$\int_a^b \mathrm{d}g(t) \left(\int_a^t h(s,t)\mathrm{d}f(s) \right) = \int_a^b \left(\int_s^b \mathrm{d}g(t)h(s,t) \right) \mathrm{d}f(s) +$$
$$+ \sum_t [\Delta^- g(t)h(t,t)\Delta^- f(t) - \Delta^+ g(t)h(t,t)\Delta^+ f(t)].$$

The symbol

$$\mathrm{d}x = \mathrm{d}[A]x + \mathrm{d}g \tag{2.1}$$

where $A \in BV^{n\times n}$ and $g \in G^n(a, b)$ is said to be a *generalized linear differential equation* (GLDE) *in the space of regulated functions.*

Definition 2.6. A function $x : [a, b] \to \mathbf{R}^n$ is said to be a *solution* of GLDE (2.1) if for any $t, t_0 \in [a, b]$ it verifies the equality

$$x(t) = x(t_0) + \int_{t_0}^t \mathrm{d}[A(s)]x(s) + g(t) - g(t_0). \tag{2.2}$$

If x satisfies the initial condition

$$x(t_0) = x_0 \tag{2.3}$$

for given $t_0 \in [a, b]$ and $x_0 \in \mathbf{R}^n$ then x is called the *solution of the initial value problem* (2.1), (2.3).

Theorem 2.7. ([20, Theorem III.2.10]). *Assume that for any $t \in [a, b]$ the matrix $A \in BV^{n\times n}$ verifies the condition*

$$\det[I + \Delta^+ A(t)] \det[I - \Delta^- A(t)] \neq 0. \tag{2.4}$$

Then there exists a unique matrix valued function $U : [a, b] \times [a, b] \to \mathbf{R}^{n\times n}$ such that, for any $(t, s) \in [a, b] \times [a, b]$

$$U(t, s) = I + \int_s^t d[A(\tau)]U(\tau, s). \tag{2.5}$$

$U(t, s)$ is called the fundamental matrix solution of the homogeneous equation

$$\mathrm{d}x = \mathrm{d}[A]x \tag{2.6}$$

(or the fundamental matrix of A) and it has the following properties, for any $\tau, t, s \in [a, b]$:

$$U(t, s) = U(t, \tau)U(\tau, s); \tag{2.7}$$

$$U(t, t) = I; \tag{2.8}$$

$$\begin{aligned} &U(t+, s) = [I + \Delta^+ A(t)]U(t, s),\ U(t-, s) = [I - \Delta^- A(t)]U(t, s); \\ &U(t, s+) = U(t, s)[I + \Delta^+ A(s)]^{-1},\ U(t, s-) = U(t, s)[I - \Delta^- A(s)]^{-1}; \end{aligned} \tag{2.9}$$

$$U(t,s)^{-1} = U(s,t); \tag{2.10}$$

there exists a constant $M > 0$ *such that*

$$|U(t,s)| + \mathrm{var}_a^b U(t,\cdot) + \mathrm{var}_a^b U(\cdot,s) + \mathrm{v}(U) < M \tag{2.11}$$

where $\mathrm{v}(U)$ *is the twodimensional Vitali variation of* U *on* $[a,b]\times[a,b]$ ([20, *Definition* I.6.1]).

Some methods for the calculus of the fundamental matrix $U(t,s)$ were provided in [16].

From [20, Theorem III.3.1] and [22, Proposition 2.5], one obtains

Theorem 2.8. (Variation-of-parameters formula). *If* $A \in BV^{n\times n}$ *satisfies the condition* (2.4), *then the initial value problem* (2.1), (2.3) *has a unique solution given by*

$$x(t) = U(t,t_0)x_0 + g(t) - g(t_0) - \int_{t_0}^{t} \mathrm{d}_s[U(t,s)](g(s) - g(t_0)). \tag{2.12}$$

If $g \in G^n$ ($g \in BV^n$) *then* $x \in G^n$ ($x \in BV^n$).

3 Input-output maps of 2D generalized hybrid systems

The linear spaces $X = G_1^n$, $U = G_1^m$ and $Y = G_1^p$ are called respectively the *state*, *input* and *output* spaces. The *time set* is $T = [a_1,b_1] \times \{a_2, a_2+1, \ldots, b_2\}$, where $[a_1,b_1] \subset \mathbf{R}$ and $a_2, b_2 \in \mathbf{Z}$.

Definition 3.1. A 2D *generalized hybrid system* (2Dgh) is an ensemble

$$\Sigma = (A_1(t,k), A_2(t,k), B(t,k), C(t,k), D(t,k), N_1, N_2, M_1, M_2) \in$$
$$\in BV_1^{n\times n} \times G_1^{n\times n} \times G_1^{n\times m} \times G_1^{p\times n} \times G_1^{p\times m} \times \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times n}$$

where $A_1(t,k)A_2(t,k) = A_2(t,k)A_1(t,k)$, $\forall (t,k) \in T$, with the following state equation, output equation, boundary condition and output vector equation:

$$\begin{aligned} \mathrm{d}x(t,k+1) &= \mathrm{d}[A_1(t,k+1)]x(t,k+1) + A_2(t,k)\mathrm{d}x(t,k) - \\ &- \mathrm{d}[A_1(t,k)]A_2(t,k)x(t,k) + B(t,k)\mathrm{d}u(t,k), \end{aligned} \tag{3.1}$$

$$y(t,k) = C(t,k)x(t,k) + D(t,k)u(t,k), \tag{3.2}$$

$$N_1 x(a_1,a_2) + N_2 x(b_1,b_2) = v. \tag{3.3}$$

n is called the *dimension* of the system Σ and it is denoted $\dim\Sigma$.

Let $U(t,t_0;k)$ be the fundamental matrix of $A_1(t,k)$, $k \in \{a_2, a_2+1, \ldots, b_2\}$ and $F(t;k,k_0)$ the discrete fundamental matrix of $A_2(t,k)$, $t \in [a,b]$, i.e.

$$F(t;k,k_0) = \begin{cases} A_2(t,k-1)A_2(t,k-2)\cdots A_2(t,k_0) & \text{for} \quad k > k_0 \\ I_n & \text{for} \quad k = k_0. \end{cases}$$

Since $A_1(t,k)$ and $A_2(t,k)$ are commutative matrices for any $(t,k) \in T$, by the Peano-Baker type formula for U [16] and by the definition of F it results that $U(t,t_0;k)$ and $F(t;k,k_0)$ are commutative matrices too. We shall use the following notations: $\Delta^+ f(s,l) = f(s+,l) - f(s,l)$, $\Delta_s^+ U(t,s;k) = U(t,s+;k) - U(t,s;k)$ and similarly we define $\Delta^- f(s,l)$ and $\Delta_s^- U(t,s;k)$.

Definition 3.2. A vector $x_0 \in X$ is called the *initial state* of Σ at the moment (a_1,a_2) if $\forall (t,k) \in T$ with $(t,k) \geq (a_1,a_2)$

$$x(t,a_2) = U(t,a_1;a_2)x_0 \quad \text{and} \quad x(a_1,k) = F(a_1;k,a_2)x_0. \tag{3.4}$$

Proposition 3.3. (2D *generalized variation of parameters formula). If*

$$\det[(I - \Delta^- A_i(t,k))(I + \Delta^+ A_i(t,k))] \neq 0,\ i = 1,2,\ \forall t \in [a,b],\ k \in \mathbf{Z}, \tag{3.5}$$

then the solution of the generalized differential-difference equation

$$\begin{aligned} \mathrm{d}x(t,k+1) &= \mathrm{d}[A_1(t,k+1)]x(t,k+1) + A_2(t,k)\mathrm{d}x(t,k) - \\ &- \mathrm{d}[A_1(t,k)]A_2(t,k)x(t,k) + \mathrm{d}f(t,k) \end{aligned} \tag{3.6}$$

with the initial conditions (3.4) *is*

$$\begin{aligned} x(t,k) &= U(t,a_1;k)F(a_1;k,a_2)x_0 + \\ &+ \int_{a_1}^{t} \sum_{l=a_2}^{k-1} U(t,s;k)F(s;k,l+1)\mathrm{d}f(s,l) + \\ &+ \sum_{a \leq s < t} \Delta_s^+ U(t,s;k) \sum_{l=a_2}^{k-1} F(s;k,l+1)\Delta^+ f(s,l) - \\ &- \sum_{a < s \leq t} \Delta_s^- U(t,s;k) \sum_{l=a_2}^{k-1} F(s;k,l+1)\Delta^- f(s,l). \end{aligned} \tag{3.7}$$

Proof. We shall use the notation

$$\mathrm{d}g(t,k) = \mathrm{d}x(t,k) - \mathrm{d}[A_1(t,k)]x(t,k). \tag{3.8}$$

The equation (3.6) becomes

$$\mathrm{d}g(t,k+1) = A_2(t,k)\mathrm{d}g(t,k) + \mathrm{d}f(t,k). \tag{3.9}$$

Then

$$\begin{aligned} \mathrm{d}g(t,a_2+1) &= A_2(t,a_2)\mathrm{d}g(t,a_2) + \mathrm{d}f(t,a_2) = \\ &= F(t;a_2+1,a_2)\mathrm{d}g(t,a_2) + F(t;a_2+1,a_2+1)\mathrm{d}f(t,a_2). \end{aligned}$$

Let us assume that

$$\mathrm{d}g(t,k) = F(t;k,a_2)\mathrm{d}g(t,a_2) + \sum_{l=a_2}^{k-1} F(t;k,l+1)\mathrm{d}f(t,l). \tag{3.10}$$

Then, by (3.9), (3.10) and by the definition of $F(t;k,a_2)$, we get

$$\begin{aligned} \mathrm{d}g(t,k+1) &= A_2(t,k)F(t;k,a_2)\mathrm{d}g(t,a_2)+ \\ &+ \sum_{l=a_2}^{k-1} A_2(t,k)F(t;k,l+1)\mathrm{d}f(t,l)+\mathrm{d}f(t,k)= \\ &= F(t;k+1,a_2)\mathrm{d}g(t,a_2)+\sum_{l=a_2}^{k} F(t;k+1,l+1)\mathrm{d}f(t,l) \end{aligned}$$

hence (3.10) is true $\forall k > a_2$. Moreover, from (3.4), (3.9) and (2.7) one obtains

$$\begin{aligned} \mathrm{d}g(t,a_2) &= \mathrm{d}x(t,a_2)-\mathrm{d}[A_1(t,a_2)]x(t,a_2)= \\ &= \mathrm{d}[U(t,a_1;a_2)]x_0-\mathrm{d}[A_1(t,a_2)]x(t,a_2)= \\ &= \mathrm{d}[A_1(t,a_2)]U(t,a_1;a_2)x_0-\mathrm{d}[A_1(t,a_2)]U(t,a_1;a_2)x_0=0 \end{aligned}$$

hence (3.10) becomes

$$\mathrm{d}g(t,k)=\sum_{l=a_2}^{k-1} F(t;k,l+1)\mathrm{d}f(t,l). \tag{3.11}$$

Equation (3.8) is equivalent to the generalized differential equation $\mathrm{d}x(t,k) = \mathrm{d}[A_1(t,k)]x(t,k)+\mathrm{d}g(t,k)$ with the solution given by Theorem 2.8

$$\begin{aligned} x(t,k) &= U(t,a_1;k)x(a_1,k)-\int_{a_1}^{t}\mathrm{d}_s[U(t,s;k)]\int_{a_1}^{s}\mathrm{d}g(\tau,k)+ \\ &+ \int_{a_1}^{t}\mathrm{d}g(s,k). \end{aligned} \tag{3.12}$$

By Theorem 2.8, (3.12) becomes

$$\begin{aligned} x(t,k) &= U(t,a_1;k)x(a_1,k)+\int_{a_1}^{t}U(t,s;k)\mathrm{d}\int_{a_1}^{s}\mathrm{d}g(\tau,k)+ \\ &+ \sum_{a\le s<t}\Delta_s^{+}U(t,s;k)\Delta^{+}\int_{a_1}^{s}\mathrm{d}g(\tau,k)- \\ &- \sum_{a<s\le t}\Delta_s^{-}U(t,s;k)\Delta^{-}\int_{a_1}^{s}\mathrm{d}g(\tau,k). \end{aligned} \tag{3.13}$$

We replace (3.10) in (3.12). One obtains (3.7) from (3.14) taking into account the equality

$$\int_{a_1}^{t}\mathrm{d}g(s,k)=\sum_{l=a_2}^{k-1}\int_{a_1}^{t}F(s;k,l+1)\mathrm{d}f(s,l)$$

and also (3.4) and Theorems 2.3, 2.4 and 2.5. □

Proposition 3.4. *If* (3.4) *holds, then the state of the system at the moment* $(t,k) \in T$ *determined by the initial state* x_0 *at the moment* (a_1, a_2) *and the control* $u : [a_1, t] \times \{a_2, a_2+1, \ldots, k-1\} \to \mathbf{R}^m$ *is*

$$\begin{aligned} x(t,k) &= U(t,a_1;k)F(a_1;k,a_2)x_0 + \\ &+ \int_{a_1}^{t} \sum_{l=a_2}^{k-1} U(t,s;k)F(s;k,l+1)B(s,l)\mathrm{d}u(s,l) + \\ &+ \sum_{a \le s < t} \Delta_s^+ U(t,s;k) \sum_{l=a_2}^{k-1} F(s;k,l+1)B(s,l)\Delta^+ u(s,l) - \\ &- \sum_{a < s \le t} \Delta_s^- U(t,s;k) \sum_{l=a_2}^{k-1} F(s;k,l+1)B(s,l)\Delta^- u(s,l). \end{aligned} \tag{3.14}$$

Proof. The state equation (3.1) can be obtained from (3.3) by replacing $f(t,k) = \int_{a_1}^{t} B(s,k)\mathrm{d}u(s,k)$. Then (3.14) results from (3.6) and (**??**). □

Definition 3.5. The boundary condition (3.3) is said to be *well-posed* if the homogeneous problem corresponding to (3.1) and (3.3) (i.e. with $u \equiv 0$ and $v = 0$) has the unique solution $x = 0$.

Proposition 3.6. *The boundary condition* (3.3) *is well-posed if and only if the matrix* $R = N_1 + N_2 U(b_1, a_1; b_2)F(a_1; b_2, a_2)$ *is nonsingular.*

Proof. By (3.14) with $u \equiv 0$ we get

$$x(b_1, b_2) = U(b_1, a_1; b_2)F(a_1; b_2, a_2)x(a_1, a_2);$$

we replace $x(b_1, b_2)$ and $v = 0$ in (3.3). It results that (3.3) is well-posed if and only if the equation $[N_1 + N_2 U(b_1, a_1; b_2)F(a_1; b_2, a_2)]x(a_1, a_2) = 0$ has the unique solution $x(a_1, a_2) = 0$, condition which is equivalent to R nonsingular.

In the sequel we shall consider systems with well-posed boundary condition (**??**3) and which verify (3.5). Moreover, the discrete-time character of Σ with respect to the variable k imposes the following assumption: the matrices A_2 depend only on k and $A_2(k)$ are nonsingular for any $k \in \{a_2, a_2+1, \ldots, b_2\}$.

Then the discrete fundamental matrix of A_2 becomes $F(k,l)$ and we can define, for $k < l$,

$$F(k,l) = [A_2(l-1)A_2(l-2)\cdots A_2(k+1)A_2(k)]^{-1}.$$

In this case the semigroup property $F(k,l)F(l,i) = F(k,i)$ is true for any $k, l, i \in \{a_2, a_2+1, \ldots, b_2\}$.

Definition 3.7. The matrix $P = P_\Sigma = R^{-1}N_2 U(b_1, a_1; b_2)F(b_2, a_2)$ is called the *canonical boundary value operator* of the system with well-posed boundary condition Σ.

Theorem 3.8. *The state of the system* Σ *determined by the control* $u : T \to \mathbf{R}^m$ *and* *by the input vector* $v \in \mathbf{R}^n$ *is*

(3.15)

$$x(t,k) = U(t,a_1;k)F(k,a_2)R^{-1}v-$$
$$-\int_{a_1}^{b_1}\sum_{l=a_2}^{b_2-1} U(t,a_1;k)F(k,a_2)PU(a_1,s;b_2)F(a_2,l+1)B(s,l)\mathrm{d}u(s,l)+$$
$$+\int_{a_1}^{t}\sum_{l=a_2}^{k-1} U(t,s;k)F(k,l+1)B(s,l)\mathrm{d}u(s,l)-$$
$$-U(t,a_1;k)F(k,a_2)P\left(\sum_{a_1\le s<b_1}\Delta_s^+U(a_1,s;b_2)\sum_{l=a_2}^{b_2-1}F(a_2,l+1)B(s,l)\Delta^+u(s,l)-\right.$$
$$\left.-\sum_{a_1<s\le b_1}\Delta_s^-U(a_1,s;b_2)\sum_{l=a_2}^{b_2-1}F(a_2,l+1)B(s,l)\Delta^-u(s,l)\right)+$$
$$+\sum_{a\le s<t}\Delta_s^+U(t,s;k)\sum_{l=a_2}^{k-1}F(k,l+1)B(s,l)\Delta^+u(s,l)-$$
$$-\sum_{a<s\le t}\Delta_s^-U(t,s;k)\sum_{l=a_2}^{k-1}F(k,l+1)B(s,l)\Delta^-u(s,l).$$

Proof. We replace $x(b_1,b_2)$ given by (3.14) in the boundary condition (3.3). We get

$$[N_1+N_2U(b_1,a_1;b_2)F(b_2,a_2)]x_0+$$
$$+N_2\int_{a_1}^{b_1}\sum_{l=a_2}^{b_2-1}U(b_1,s;b_2)F(b_2,l+1)B(s,l)\mathrm{d}u(s,l)+$$
$$+\sum_{a_1\le s<b_1}\Delta_s^+U(b_1,s;b_2)\sum_{l=a_2}^{b_2-1}F(b_2,l+1)B(s,l)\Delta^+u(s,l)-$$
$$-\sum_{a_1<s\le b_1}\Delta_s^-U(b_1,s;b_2)\sum_{l=a_2}^{b_2-1}F(b_2,l+1)B(s,l)\Delta^-u(s,l)=v$$

hence, by the semigroup properties of the fundamental matrices U and F, we obtain

$$x_0 = R^{-1}v - P\int_{a_1}^{b_1}\sum_{l=a_2}^{b_2-1}U(a_1,s;b_2)F(a_2,l+1)B(s,l)\mathrm{d}u(s,l)-$$
$$-P\sum_{a_1\le s<b_1}\Delta_s^+U(a_1,s;b_2)\sum_{l=a_2}^{b_2-1}F(a_2,l+1)B(s,l)\Delta^+u(s,l)+ \tag{3.16}$$
$$+P\sum_{a_1<s\le b_1}\Delta_s^-U(a_1,s;b_2)\sum_{l=a_2}^{b_2-1}F(a_2,l+1)B(s,l)\Delta^-u(s,l).$$

We replace $x_0 = x(a_1, a_2)$ in (3.14); then (3.15) results by using the semigroup property, i.e. $U(b_1, s; b_2) = U(b_1, a_1; b_2)U(a_1, s; b_2)$ and $F(b_2, l+1) = F(b_2, a_2)F(a_2, l+1)$. $\square$

Theorem 3.9. *The input-output map of the* 2Dgcd *system* Σ *is* $H : G_1^m \times \mathbf{R}^n \to G_1^p \times \mathbf{R}^n$, $H(u, v) = y$ *where*

(3.17)

$$\begin{aligned}
&y(t,k) = C(t,k)U(t,a_1;k)F(k,a_2)R^{-1}v - \\
&- \int_{a_1}^{b_1} \sum_{l=a_2}^{b_2-1} C(t,k)U(t,a_1;k)F(k,a_2)PU(a_1,s;b_2)F(a_2,l+1)B(s,l)\mathrm{d}u(s,l) + \\
&+ \int_{a_1}^{t} \sum_{l=a_2}^{k-1} C(t,k)U(t,s;k)F(k,l+1)B(s,l)\mathrm{d}u(s,l) + D(t,k)u(t,k) - \\
&-C(t,k)U(t,a_1;k)F(k,a_2)P \left(\sum_{a_1 \le s < b_1} \Delta_s^+ U(a_1,s;b_2) \sum_{l=a_2}^{b_2-1} F(a_2,l+1)B(s,l)\Delta^+ u(s,l) - \right. \\
&\left. - \sum_{a_1 < s \le b_1} \Delta_s^- U(a_1,s;b_2) \sum_{l=a_2}^{b_2-1} F(a_2,l+1)B(s,l)\Delta^- u(s,l) \right) + \\
&+ \sum_{a \le s < t} C(t,k)\Delta_s^+ U(t,s;k) \sum_{l=a_2}^{k-1} F(k,l+1)B(s,l)\Delta^+ u(s,l) - \\
&- \sum_{a < s \le t} C(t,k)\Delta_s^- U(t,s;k) \sum_{l=a_2}^{k-1} F(k,l+1)B(s,l)\Delta^- u(s,l)
\end{aligned}$$

Proof. We obtain (3.17) by replacing the state $x(t, k)$ given by (3.15) in the output equation (3.2). $\square$

Corollary 3.10. *If* $u \in G_1^m$ *then* $x \in G_1^n$ *and* $y \in G_1^p$. *If* $A_2 \in BV_1^{n\times n}, B \in BV_1^{n\times m}, C \in BV_1^{p\times n}$, $D \in BV_1^{n\times m}$ *and* $u \in BV_1^m$ *then* $x \in BV_1^n$ *and* $y \in BV_1^p$.

Proof. We apply Theorems 2.8, 3.8 and 3.9. $\square$

Remark 3.11. The continuous-discrete 2D systems [17] with the state equation

$$\begin{aligned}
&\frac{\partial x}{\partial t}(t,k+1) = \tilde{A}_1(t,k+1)x(t,k+1) + \\
&+\tilde{A}_2(t,k)\frac{\partial x}{\partial t}(t,k) - \tilde{A}_1(t,k)\tilde{A}_2(t,k)x(t,k) + \tilde{B}(t,k)\tilde{u}(t,k)
\end{aligned}$$

represent particular cases of 2Dgh (3.1) with absolutely continuous matrices $A_i(t,k) = \int_a^t \tilde{A}_i(s,k)\mathrm{d}s$, $i = 1, 2$ and controls $u(t,k) = \int_a^t \tilde{u}(s,k)\mathrm{d}s$.

4 2D Generalized hybrid semiseparable kernels

Let us consider the 2D time sets $T = [a_1, b_1] \times \{a_2, a_2 + 1, \dots, b_2\}$, where $a_1, b_1 \in \mathbf{R}$, $a_2, b_2 \in \mathbf{Z}$, $a_1 < b_1$, $a_2 < b_2$, $T^* = [a_1, b_1] \times \{a_2, a_2 + 1, \dots, b_2 - 1\}$, $T(t, k) =$

$[a_1, t) \times \{a_2, a_2+1, \ldots, k-1\}$, $t \in \mathbf{R}$, $a_1 < t \leq b_1$, $k \in \mathbf{Z}$, $a_2 < k \leq b_2$, and $T^*(t,k) = T^* \setminus T(t,k)$.

We shall use the following notations:

$$BV_{1,2}^{p\times m} = \{K : [a_1, b_1] \times [a_1, b_1] \times \{a_2, a_2+1, \ldots, b_2\} \times \{a_2, a_2+1, \ldots, b_2\} \to \mathbf{R}^{p\times m} |$$
$$K(t,s;k,l) \text{ function of bounded variation w.r.t. } t \text{ and } s\};$$
$$\Delta_s^+ K(t,s;k,l) = K(t,s+;k,l) - K(t,s;k,l);\ \Delta^- K(t,s;k,l) =$$
$$= K(t,s;k,l) - K(t,s-;k,l).$$

Definition 4.1. A function $K(t,s;k,l) \in BV_{1,2}^{p\times m}$ is called the 2D *generalized hybrid kernel* of a 2D generalized hybrid system Σ if the input-output map of Σ has the form

$$
\begin{aligned}
y(t,k) &= \int_{a_1}^{b_1} \sum_{l=a_2}^{b_2-1} K(t,s;k,l)du(s,l) + \\
&+ \sum_{a_1 \leq s < b_1} \sum_{l=a_2}^{k-1} \Delta_s^+ K(t,s;k,l) \Delta_s^+ u(s,l) - \\
&- \sum_{a_1 < s \leq b_1} \sum_{l=a_2}^{b_2-1} \Delta_s^- K(t,s;k,l) \Delta_s^- u(s,l) + D(t,k)u(t,k)
\end{aligned}
\tag{4.1}
$$

where $D(t,k) \in G_1^{p\times m}$, $u \in G_1^m$, $y \in G_1^P$.

Definition 4.2. The 2D generalized hybrid kernel $K(t,s;k,l)$ is said to be *semiseparable* if it has the form

$$
K(t,s;k,l) = \begin{cases} E_1(t,k)G_1(s,l) & \text{if} \quad (s,l) \in T(t,k) \\ -E_2(t,k)G_2(s,l) & \text{if} \quad (s,l) \in T^*(t,k). \end{cases}
\tag{4.2}
$$

In [9] the authors state that they have come to systems with boundary conditions via an analysis of Wiener-Hopf integral equations and related convolution equations of the form $y(t) = u(t) - \int_0^\tau k(t-s)u(s)ds$, $0 \leq t \leq \tau$.

These equations can be extended to 2D hybrid ones of the form

$$y(t,k) = \int_{a_1}^{b_1} \sum_{l=a_2}^{b_2-1} K(t-s;k-l)du(s,l) + D(t,k)u(t,k)$$

or of the form (4.1) in the case of spaces of regulated functions. In order to obtain realizations of such representations it is necessary to connect them with 2D generalized hybrid systems with boundary conditions.

5 Realizations of generalized 2D semiseparable kernels

In this section we shall consider systems Σ with $A_1 : [a_1, b_1] \to \mathbf{R}^{n\times n}$, $A_2 : \{a_2, a_2+1, \ldots, b_2\} \to \mathbf{R}^{n\times n}$ and B continuous w.r.t. s. In this case the fundamental matrices of A_1 and A_2 become respectively $U(t,s)$ and $F(k,l)$. We shall use the following notations:

$$\diamond(t,k)\int_{a_1}^{b_1}\sum_{l=a_2}^{b_2-1}=\int_{a_1}^{b_1}\sum_{l=a_2}^{b_2-1}-\int_{a_1}^{t}\sum_{l=a_2}^{k-1},$$

$$\diamond(t,k)\sum_{a_1\le s<b_1}\sum_{l=a_2}^{b_2-1}=\sum_{a_1\le s<b_1}\sum_{l=a_2}^{b_2-1}-\sum_{a_1\le s<t}\sum_{l=a_2}^{k-1}$$

and similarly

$$\diamond(t,k)=\sum_{a_1<s\le b_1}\sum_{l=a_2}^{b_2-1}.$$

By Theorem 3.9 we obtain the following result.

Theorem 5.1. *The output of the system Σ determined by the control u and by $v=0$ is*

(5.1)

$$\begin{aligned}
&y(t,k)=\int_{a_1}^{t}\sum_{l=a_2}^{k-1}C(t,k)U(t,a_1)F(k,a_2)(I-P)U(a_1,s)F(a_2,l+1)B(s,l)du(s,l)-\\
&-\diamond(t,k)\int_{a_1}^{b_1}\sum_{l=a_2}^{b_2-1}C(t,k)U(t,a_1)F(k,a_2)PU(a_1,s)F(a_2,l+1)B(s,l)du(s,l)+\\
&+\sum_{a_1\le s<t}\sum_{l=a_2}^{k-1}C(t,k)U(t,a_1)F(k,a_2)(I-P)\Delta_s^+U(a_1,s)F(a_2,l+1)B(s,l)\Delta^+u(s,l)-\\
&-\diamond(t,k)\sum_{a_1\le s<t}\sum_{l=a_2}^{b_2-1}C(t,k)U(t,a_1)F(k,a_2)P\Delta_s^+U(a_1,s)F(a_2,l+1)B(s,l)\Delta^+u(s,l)-\\
&-\sum_{a_1<s\le t}\sum_{l=a_2}^{k-1}C(t,k)U(t,a_1)F(k,a_2)(I-P)\Delta_s^-U(a_1,s)F(a_2,l+1)B(s,l)\Delta^-u(s,l)+\\
&+\diamond(t,k)\sum_{a_1<s\le t}\sum_{l=a_2}^{b_2-1}C(t,k)U(t,a_1)F(k,a_2)P\Delta_s^-U(a_1,s)F(a_2,l+1)B(s,l)\Delta^-u(s,l)+\\
&+D(t,k)u(t,k).
\end{aligned}$$

Proposition 5.2. The kernel K_Σ of a 2D generalized hybrid system Σ is semiseparable.

Proof. By equation (5.1) the input-output map of a system Σ can be written as

$$
\begin{aligned}
y(t,k) &= \int_{a_1}^{b_1} \sum_{l=a_2}^{b_2-1} K_\Sigma(t,s;k,l)du(s,l)+ \\
&+ \sum_{a_1 \le s < b_1} \sum_{l=a_2}^{b_2-1} \Delta_s^+ K(t,s;k,l)\Delta^+ u(s,l)- \\
&- \sum_{a_1 < s \le b_1} \sum_{l=a_2}^{b_2-1} \Delta_s^- K(t,s;k,l)\Delta^- u(s,l),
\end{aligned} \tag{5.2}
$$

where the kernel K_Σ of the system Σ has the form

$$
K_\Sigma(t,s;k,l) = \begin{cases} E_1(t,k)G_1(s,l) & \text{if} \quad (s,l) \in T(t,k) \\ -E_2(t,k)G_2(s,l) & \text{if} \quad (s,l) \in T^*(t,k) \end{cases} \tag{5.3}
$$

where

$$
E_1(t,k) = C(t,k)U(t,a_1)F(k,a_2)(I-P), \quad E_2(t,k) = C(t,k)U(t,a_1)F(k,a_2)P,
$$

$$
G_1(s,l) = G_2(s,l) = U(a_1,s)F(a_2,l+1)B(s,l), \tag{5.4}
$$

hence by (4.2) the kernel K_Σ is semiseparable. $\square$

Definition 5.3. Given a 2D generalized hybrid kernel $\mathcal{K}$, a system Σ is said to be a *realization* of $\mathcal{K}$ if $K = K_\Sigma$.

Theorem 5.4. *Any* 2D *generalized semiseparable hybrid kernel has a realization.*

Proof. Assume that the kernel K has the form (4.2). Then a realization of K is the system $\hat{\Sigma}$ given by $\hat{A}_1 = 0_n$ on $[a_1, b_1]$, $\hat{A}_2 = I_n$,

$$
\hat{B}(t,k) = \begin{bmatrix} G_1(t,k) \\ G_2(t,k) \end{bmatrix}, \quad \hat{C}(t,k) = [E_1(t,k) \quad E_2(t,k)], \quad \hat{D} = 0_p^m \in \mathbf{R}^{p \times n},
$$

$$
\hat{N}_1 = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{N}_2 = \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix} \text{ with } n_1, n_2 \ge 0,\ n_1 + n_2 = n.
$$

Indeed, $U(t,s) = I_n$, $F(k,l) = I_n$, hence $R = N_1 + N_2 U(b_1,a_1)F(b_2,a_2) = N_1 + N_2 = I_n$ which means that the boundary condition (3.3) is well-posed and the canonical operator of Σ is $P = P_{\hat{\Sigma}} = R^{-1} N_2 U(b_1,a_1)F(b_2,a_2) = N_2$ and $I - P = I - N_2 = N_1$. Then, by (5.4)

$$
\hat{E}_1(t,k) = \hat{C}(t,k)U(t,a_1)F(k,a_2)(I-P) = [E_2(t,k)\ E_2(t,k)]I_n I_n N_1 = [E_1(t,k)\ 0]
$$

$$
\hat{E}_2(t,k) = \hat{C}(t,k)U(t,a_1)F(k,a_2)P = [0\ E_2(t,k)],
$$

$$
\hat{G}_i(s,l) = U(a_1,s)F(a_2,l+1) \begin{bmatrix} G_1(s,l) \\ G_2(s,l) \end{bmatrix} = \begin{bmatrix} G_1(s,l) \\ G_2(s,l) \end{bmatrix}, \quad i = 1,2
$$

hence

$$K_\Sigma(t,s,k,l) = \begin{cases} \hat{E}_1(t,k)\hat{G}_1(s,l) \\ -\hat{E}_2(t,k)\hat{G}_2(s,l) \end{cases} = \begin{cases} E_1(t,k)G_1(s,l) & \text{if} \quad (s,l)\in T(t,k) \\ -E_2(t,k)G_2(s,l) & \text{if} \quad (s,l)\in T^*(t,k), \end{cases}$$

i.e. $K_{\hat{\Sigma}} = K$. □

Conclusion

A class of 2D hybrid systems is studied in the general framework of spaces of functions of bounded variation and regulated functions. The input-output maps of the causal and acausal systems is obtained and the description of the input-output behavior by 2D semiseparable kernels is emphasized. The realization problem is discussed. This study can be continued with other topics, such as controllability, observability, stability or optimal control.

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