Two characterizations of the Chern connection

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Dedicated to the 70-th anniversary
of Professor Constantin Udriste

Abstract. Since its introduction in [6, 7, 12], the Chern connection associated to a second order differential system on a smooth manifold M, has been studied by several authors; *e.g.* see [4, 5, 8, 14]. In this work the Chern connection is presented in a similar way as the Levi-Civita connection is introduced in Riemannian Geometry, by following the next points: i) first, a second-order ordinary differential equations system on M is defined as a section σ of the canonical projection $p^{21}\colon J^2(\mathbb{R},M)\to J^1(\mathbb{R},M)$, ii) the notion of a linear frame of $J^1(\mathbb{R},M)$ adapted to σ is given and the set of such frames is seen to be a G-structure P^σ of the linear frames of $J^1(\mathbb{R},M)$, iii) two characterizations of the Chern connection are given: the first one as a derivation law on the tangent bundle of $J^1(\mathbb{R},M)$ and the second one as a principal connection on P^σ. Below, the statements of the main results of this point of view, are presented, the proofs of which will be published elsewhere.

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1 Notations

Let $p\colon \mathbb{R}\times M\to\mathbb{R}$, $p(t,x)=t$ and $p'\colon\mathbb{R}\times M\to M$, $p'(t,x)=x$, be the natural projections, M being a connected manifold of class C^∞ and $\dim M=n$.

We denote by $p^r\colon J^r(\mathbb{R},M)\to\mathbb{R}$ the r-jet bundle of C^∞ maps $\mathbb{R}\to M$ and the r-jet prolongation of a curve $\gamma\colon\mathbb{R}\to M$ is denoted by $j^r\gamma\colon\mathbb{R}\to J^r(\mathbb{R},M)$.

Moreover, $p^{rs}\colon J^r(\mathbb{R},M)\to J^s(\mathbb{R},M)$ for $r>s$ denote the natural projections, i.e., $p^{rs}(j^r_t\gamma)=j^s_t\gamma$.

Every coordinate system (x^i), $1\le i\le n=\dim M$, on M induces a coordinate system (t,x^i,x^i_k), $1\le k\le r$, $x^i_0=x^i$, on $J^r(\mathbb{R},M)$ by setting

$$x^i_k\left(j^r_{t_0}\gamma\right)=\frac{d^k\left(x^i\circ\gamma\right)}{dt^k}(t_0),\quad 1\le k\le r.$$

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Nevertheless, for the sake of simplicity, in what follows we also write $\dot{x}^i$ (resp. $\ddot{x}^i$) instead of x_1^i (resp. x_2^i).

A second-order ordinary differential system

$$\ddot{x}^i = F^i\left(t, x^i, \dot{x}^i\right), \quad F^i \in C^\infty\left(J^1(\mathbb{R}, M)\right), \tag{1.1}$$

can be viewed as a section $\sigma\colon J^1(\mathbb{R}, M) \to J^2(\mathbb{R}, M)$ of the projection p^{21} by simply setting $\ddot{x}^i \circ \sigma = F^i$.

The second-order ordinary differential system considered in the original paper by Chern [7] (also see [6], [12]) is $\ddot{x}^i + F^i\left(t, x^i, \dot{x}^i\right) = 0$, $1 \leq i \leq n$, instead of (1.1). Hence, in all the formulas below, F^i should be replaced by $-F^i$ in order to compare with the formulas in [7].

Every second-order SODE σ defines a vector field $X^\sigma \in \mathfrak{X}(J^1(\mathbb{R}, M))$, called 'the dynamical flow' associated to σ (cf. [14]), as follows:

$$(X^\sigma)_\xi = (j^1\gamma)_*\left(\frac{d}{dt}\right)_{t_0}, \quad \forall \xi \in (p^1)^{-1}(t_0),$$

where $\gamma^i = x^i \circ \gamma$ is the only solution to (1.1) satisfying the initial conditions

$$\begin{aligned} \gamma^i(t_0) &= x^i(\xi), \\ \frac{d\gamma^i}{dt}(t_0) &= \dot{x}^i(\xi). \end{aligned}$$

The local expression of the dynamical flow is

$$X^\sigma = \frac{\partial}{\partial t} + \dot{x}^i \frac{\partial}{\partial x^i} + F^i \frac{\partial}{\partial \dot{x}^i}.$$

As is well known (e.g., see [15]), $p^{10}\colon J^1(\mathbb{R}, M) \to \mathbb{R} \times M$ is an affine bundle modelled over p'^*TM; in fact, given $v \in T_{x_0}M$ and $j^1_{t_0}\gamma \in J^1(\mathbb{R}, M)$, with $\gamma(t_0) = x_0$, then $v + j^1_{t_0}\gamma = j^1_{t_0}\gamma'$ is defined as follows:

1. $\gamma'(t_0) = x_0$,
2. $\gamma'_*\left(\frac{d}{dt}\right)_{t_0} = v + \gamma_*\left(\frac{d}{dt}\right)_{t_0}$.

Hence, the following exact sequence of vector bundles over $J^1(\mathbb{R}, M)$ holds:

$$0 \to \left(p' \circ p^{10}\right)^* TM \stackrel{\varepsilon}{\longrightarrow} V\left(p^{10}\right) \to T\left(J^1(\mathbb{R}, M)\right) \stackrel{(p^{10})_*}{\longrightarrow} (p^{10})^* T(\mathbb{R} \times M) \to 0, \tag{1.2}$$

where $V\left(p^{10}\right)$ denotes the vector subbundle of p^{10}-vertical vectors and ε is locally determined by the equation

$$\varepsilon\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial \dot{x}^i}. \tag{1.3}$$

Given a second-order SODE σ, the Lie derivative $L_{X^\sigma}J$ of the fundamental tensor

$$J = \omega^i \otimes \frac{\partial}{\partial \dot{x}^i}, \quad \omega^i = dx^i - \dot{x}^i dt \tag{1.4}$$

of $J^1(\mathbb{R}, M)$ (see [14, formula (1.13)], [4, p. 6620]) with respect to X^σ is diagonalizable with eigenvalues 0, +1, −1, and multiplicities 1, n, n, respectively.

If T^0, T^+, T^- are the corresponding vector subbundles of eigenvectors, then

$$T(J^1(\mathbb{R}, M)) = T^0 \oplus T^- \oplus T^+,$$

$$T^0 = \langle X^\sigma \rangle, \tag{1.5}$$

$$T^- = \langle X_i^\sigma \rangle, \; X_i^\sigma = \frac{\partial}{\partial x^i} + \tfrac{1}{2}\frac{\partial F^j}{\partial \dot{x}^i}\frac{\partial}{\partial \dot{x}^j}, \tag{1.6}$$

$$T^+ = V\left(p^{10}\right) = \left\langle \frac{\partial}{\partial \dot{x}^1}, \dots, \frac{\partial}{\partial \dot{x}^n} \right\rangle. \tag{1.7}$$

Hence, the epimorphism $(p^{10})_*$ in (1.2) induces an isomorphism

$$(p^{10})_* \colon T^0 \oplus T^- \xrightarrow{\cong} (p^{10})^* T(\mathbb{R} \times M),$$

whose inverse mapping determines a section

$$H^\sigma \colon (p^{10})^* T(\mathbb{R} \times M) \to T(J^1(\mathbb{R}, M))$$

of $(p^{10})_*$, defined as follows:

$$H^\sigma = dt \otimes X^\sigma + \omega^i \otimes X_i^\sigma, \tag{1.8}$$

and consequently, the exact sequence (1.2) splits; i.e., every tangent vector X in $T(J^1(\mathbb{R}, M))$ can uniquely be written as $X = X^v + X^h$.

In coordinates,

$$\begin{aligned}
\left(\frac{\partial}{\partial t}\right)^v &= \left(\tfrac{1}{2}\dot{x}^i \frac{\partial F^j}{\partial \dot{x}^i} - F^j\right)\frac{\partial}{\partial \dot{x}^j}, &
\left(\frac{\partial}{\partial t}\right)^h &= \frac{\partial}{\partial t} + \left(F^j - \tfrac{1}{2}\dot{x}^i \frac{\partial F^j}{\partial \dot{x}^i}\right)\frac{\partial}{\partial \dot{x}^j}, \\
\left(\frac{\partial}{\partial x^i}\right)^v &= -\tfrac{1}{2}\frac{\partial F^j}{\partial \dot{x}^i}\frac{\partial}{\partial \dot{x}^j}, &
\left(\frac{\partial}{\partial x^i}\right)^h &= X_i^\sigma, \\
\left(\frac{\partial}{\partial \dot{x}^i}\right)^v &= \frac{\partial}{\partial \dot{x}^i}, &
\left(\frac{\partial}{\partial \dot{x}^i}\right)^h &= 0.
\end{aligned}$$

The explicit expression for the curvature form of the splitting H^σ, i.e.,

$$\begin{aligned}
&K^\sigma \in \textstyle\bigwedge^2 T^*\left(J^1(\mathbb{R}, M)\right) \otimes V(p^{10}), \\
&K^\sigma(X, Y) = \left[X^h, Y^h\right]^v, \quad \forall X, Y \in \mathfrak{X}(J^1(\mathbb{R}, M)),
\end{aligned} \tag{1.9}$$

is given by

$$K^\sigma = -\Big(P_j^h dt \wedge \omega^j + \sum_{i<j} T_{ij}^h \omega^i \wedge \omega^j\Big) \otimes \frac{\partial}{\partial \dot{x}^h},$$

where ω^j is as in (1.4) and the functions T_{ij}^k, P_j^i are the following expressions (cf. [6, 7, 12]):

$$\begin{aligned}
T_{ij}^k &= \tfrac{1}{2}\left(\frac{\partial^2 F^k}{\partial x^i \partial \dot{x}^j} - \frac{\partial^2 F^k}{\partial x^j \partial \dot{x}^i} + \tfrac{1}{2}\left(\frac{\partial F^h}{\partial \dot{x}^i}\frac{\partial^2 F^k}{\partial \dot{x}^h \partial \dot{x}^j} - \frac{\partial F^h}{\partial \dot{x}^j}\frac{\partial^2 F^k}{\partial \dot{x}^h \partial \dot{x}^i}\right)\right), \\
P_j^i &= \tfrac{1}{2}X^\sigma\left(\frac{\partial F^i}{\partial \dot{x}^j}\right) - \frac{\partial F^i}{\partial x^j} - \tfrac{1}{4}\frac{\partial F^k}{\partial \dot{x}^j}\frac{\partial F^i}{\partial \dot{x}^k}.
\end{aligned}$$

For a geometric interpretation of such functions see [1], [2] and [3].

2 The reduction P^σ defined

Proposition 2.1. *Let G be the image of the Lie-group monomorphism*

$$\iota\colon Gl(n,\mathbb{R}) \to Gl(2n+1,\mathbb{R}),$$
$$\iota(\Lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & \Lambda \end{pmatrix}, \quad \forall \Lambda \in Gl(n,\mathbb{R}).$$

Let $\pi\colon F(J^1(\mathbb{R},M)) \to J^1(\mathbb{R},M)$ be the bundle of linear frames of the manifold $J^1(\mathbb{R},M)$. Given second-order SODE σ on M, the bundle $\pi\colon P^\sigma \to J^1(\mathbb{R},M)$ of all linear frames

$$(X_0, X_1, \dots, X_{2n}) \in F_\xi(J^1(\mathbb{R},M)), \quad \xi \in J^1(\mathbb{R},M),$$

such that,

(i) $X_0 = (X^\sigma)_\xi$,

(ii) The tangent vector X_{n+i} is p^{10}-vertical for $1 \le i \le n$,

(iii) $X_i = \left(H^\sigma \circ \varepsilon^{-1}\right)(X_{n+i})$ *for* $1 \le i \le n$,

is a G-structure. Moreover, if $(U;x^i)$ is a coordinate open domain in M, then the linear frame

$$s\colon J^1(\mathbb{R},U) \to F(J^1(\mathbb{R},U)),$$
$$s(\xi) = \left((X^\sigma)_\xi, (X_i^\sigma)_\xi, \left(\tfrac{\partial}{\partial \dot{x}^i}\right)_\xi\right), \quad 1 \le i \le n, \ \xi \in J^1(\mathbb{R},U),$$

defines a section of P^σ, with dual coframe $\left(dt, \omega^i, \varpi^i\right)$, $1 \le i \le n$, where

$$\varpi^i = d\dot{x}^i - F^i dt - \tfrac{1}{2}\frac{\partial F^i}{\partial \dot{x}^j}\left(dx^j - \dot{x}^j dt\right).$$

2.1 Remarks

1. Each G-structure $P \subset F(J^1(\mathbb{R},M))$ determines a vector field X_P on $J^1(\mathbb{R},M)$ by, $(X_P)_\xi = X_0 \in T_\xi\left(J^1(\mathbb{R},M)\right)$ for every $\xi \in J^1(\mathbb{R},M)$, where $(X_0, X_1, \dots, X_{2n})$ is an arbitrary linear frame in P at ξ. The definition makes sense as X_0 is kept invariant under all the elements in G. A G-structure P is the G-structure associated with a second-order SODE if and only if X_P is a dynamical flow.

2. The adjoint bundle of the G-structure $\pi\colon P^\sigma \to J^1(\mathbb{R},M)$ corresponding to a SODE σ on M can be identified to the vector subbundle

$$\mathrm{ad}P^\sigma \subseteq T^*(J^1(\mathbb{R},M)) \otimes T(J^1(\mathbb{R},M))$$

of all endomorphisms $E\colon T(J^1(\mathbb{R},M)) \to T(J^1(\mathbb{R},M))$ such that,

$$\begin{aligned} E(T^0) &= \{0\}, \\ E(T^-) &\subseteq T^-, \\ E(T^+) &\subseteq T^+, \end{aligned}$$

and the composition map $H^\sigma \circ \varepsilon^{-1}$ conjugates $E|_{T^-}$ and $E|_{T^+}$, i.e.,

$$E|_{T^-} \circ H^\sigma \circ \varepsilon^{-1} = H^\sigma \circ \varepsilon^{-1} \circ E|_{T^+},$$

where ε, T^0, T^-, T^+, and H^σ are given in the formulas (1.3), (1.5), (1.6), (1.7), and (1.8), respectively.

2.2 Reduction to P^σ

Proposition 2.2. *The Chern connection as defined in* [7] *and* [14] *is reducible (cf.* [11, p. 81]*) to the G-structure P^σ.*

The proof uses the expression of the Chern connection ∇^σ attached to a SODE σ in the frame

$$(X_\alpha)_{\alpha=0}^{2n} = \left(X^\sigma, X_i^\sigma, \partial/\partial\dot{x}^j\right), i,j = 1,\dots,n,$$

as given in (see [14, Corollary 2.1]), namely,

$$\begin{array}{lll} \nabla^\sigma_{X^\sigma} X^\sigma = 0, & \nabla^\sigma_{X^\sigma} X_i^\sigma = -\frac{1}{2}\frac{\partial F^j}{\partial \dot{x}^i} X_j^\sigma, & \nabla^\sigma_{X^\sigma} \frac{\partial}{\partial \dot{x}^i} = -\frac{1}{2}\frac{\partial F^j}{\partial \dot{x}^i}\frac{\partial}{\partial \dot{x}^j}, \\ \nabla^\sigma_{X_i^\sigma} X^\sigma = 0, & \nabla^\sigma_{X_j^\sigma} X_i^\sigma = -\frac{1}{2}\frac{\partial^2 F^k}{\partial \dot{x}^i \partial \dot{x}^j} X_k^\sigma, & \nabla^\sigma_{X_i^\sigma} \frac{\partial}{\partial \dot{x}^j} = -\frac{1}{2}\frac{\partial^2 F^k}{\partial \dot{x}^i \partial \dot{x}^j}\frac{\partial}{\partial \dot{x}^k}, \\ \nabla^\sigma_{\frac{\partial}{\partial \dot{x}^i}} X^\sigma = 0, & \nabla^\sigma_{\frac{\partial}{\partial \dot{x}^i}} X_j^\sigma = 0, & \nabla^\sigma_{\frac{\partial}{\partial \dot{x}^i}} \frac{\partial}{\partial \dot{x}^j} = 0. \end{array}$$

and the criterion given in ([9, Proposition 5.4]).

3 Characterizations of the Chern connection

3.1 First characterization

Theorem 3.1. *Given a SODE σ on M, there is a unique linear connection ∇^σ on $J^1(\mathbb{R}, M)$ such that,*

(1) $\nabla^\sigma X^\sigma = 0$.

(2) $\nabla^\sigma L_{X^\sigma} J = 0$.

(3) *If $E^\sigma \colon T(J^1(\mathbb{R}, M)) \to T(J^1(\mathbb{R}, M))$ is $E^\sigma = J + H^\sigma \circ \varepsilon^{-1} \circ (L_{X^\sigma} J)^v$, then $\nabla^\sigma E^\sigma = 0$.*

(4) *The torsion of ∇^σ is the tensor field T^σ given by,*

$$T^\sigma = K^\sigma + dt \wedge \left(H^\sigma \circ \varepsilon^{-1} \circ (L_{X^\sigma} J)^v\right), \tag{3.1}$$

where K^σ is the curvature form of the splitting H^σ induced by σ as defined in the formula (1.9).

This connection coincides with that defined in [7] *and* [14].

3.2 Remarks

1. From the characterization of the adjoint bundle of $\pi\colon P^\sigma \to J^1(\mathbb{R},M)$ given in the second remark of the subsection 2.1 it follows that the first structure tensor of P^σ, i.e. (e.g. see [10], [13]),

$$T^\sigma \operatorname{mod} \operatorname{alt}\left(T^*\left(J^1(\mathbb{R},M)\right)\otimes \operatorname{ad}P^\sigma\right)$$

   in $\wedge^2 T^*\left(J^1(\mathbb{R},M)\right)\otimes T\left(J^1(\mathbb{R},M)\right)/\operatorname{alt}\left(T^*\left(J^1(\mathbb{R},M)\right)\otimes \operatorname{ad}P^\sigma\right)$ never vanishes and that P^σ is not 1-integrable.

2. The item (4) in the theorem can be replaced by the following weaker conditions:

$$\begin{aligned} \operatorname{Tor}\nabla^\sigma|_{T^-\times T^+} &= 0,\\ i_{X^\sigma}\operatorname{Tor}\nabla^\sigma|_{T^+} &= H^\sigma\circ\varepsilon^{-1}. \end{aligned}$$

3.3 Second characterization

Theorem 3.2. *The Chern connection ∇^σ attached to a SODE σ on M is the only linear connection on $J^1(\mathbb{R},M)$ reducible to the bundle $\pi\colon P^\sigma \to J^1(\mathbb{R},M)$ introduced in* Proposition 2.1 *whose torsion tensor field is given in the formula* (3.1).

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