# Nonholonomic Gibbs hypersurface

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*Dedicated to the 70-th anniversary*
*of Professor Constantin Udriste*

**Abstract.** The paper presents three different geometries of the Gibbs nonholonomic hypersurface: Vrănceanu geometry (Section 2), Dobrescu-Stamin-Udrişte geometry (Section 3) and Udrişte-Dogaru geometry (Section 4). All require to introduce the first and the second fundamental forms, and to use the derived geometrical objects. The main results include: the scalar curvature of Gibbs-Vrănceanu nonholonomic hypersurface reflects a saddle behavior, the Gibbs-Dobrescu-Stamin-Udrişte noholonomic hypersurface has positive Gauss-Kronecker curvature and vanishing mean curvature, the Gibbs-Udrişte-Dogaru nonholonomic hypersurface has a rational scalar curvature.

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## 1 Gibbs contact structure

The Gibbs contact structure means the triplet

$$(\mathbf{R}^5, \delta_{ab}, \theta), \ \mathbf{R}^5 = \{(U, T, S, P, V)\}, \ \theta = dU - TdS + PdV,$$

where we preserve the names $U$ - *internal energy*, $S$ - *entropy*, $T$ - *temperature*, $V$ - *volume* and $P$ - *pressure* for the independent variables, but none is restricted to positive values as in thermodynamics.

## 2 Gibbs-Vrănceanu nonholonomic hypersurface

The nonholonomic spaces submersed into a Riemannian space has been introduced by Vrănceanu ([7]) and independently by Horak ([2]) in 1927, with the purpose to obtain a geometric interpretation of nonholonomic mechanical systems.

Here, we follow the theory of nonholonomic subspaces, as it is exposed in [7], [6]. The Gibbs-Vrănceanu (GV) hypersurface is defined by the Pfaff equation

$$V_5^4 : dU - TdS + PdV = 0$$

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and by a co-frame that will be introduced in this Section. For mathematical convenience, we denote the coordinates by

$$x^1 = T,\ x^2 = S,\ x^3 = P,\ x^4 = V,\ x^5 = U \tag{2.1}$$

and rewrite the GV equation as

$$V_5^4 : ds^5 = dx^5 - x^1 dx^2 + x^3 dx^4 = 0.$$

The transformation group of $V_5^4$ is given by the following formula

$$\begin{gathered} d\overline{s}^h = c_k^h ds^k + c_5^h ds^5,\ (h,k = \overline{1,4}) \\ d\overline{s}^5 = c_5^5 ds^5, \\ c_k^5 = 0,\ c_k^h c_l^h = \delta_l^k,\ c_5^5 = 1. \end{gathered} \tag{2.2}$$

The intrinsic invariants of nonholonomic hypersurface $V_5^4$ are ([6], p. 243-244): the *Gibbs-Pfaff equation*

$$ds^5 = 0 \tag{2.3}$$

and the *Riemannian metric*

$$\varphi = \left(ds^1\right)^2 + \left(ds^2\right)^2 + \left(ds^3\right)^2 + \left(ds^4\right)^2 \mod \left(ds^5\right). \tag{2.4}$$

The equation (2.3) can be written

$$dx^5 + a_i^5 dx^i = 0,\ (i = \overline{1,4}). \tag{2.5}$$

Thus, by a transformation

$$d\overline{s}^h = ds^h + c_5^h ds^5,\ h = \overline{1,4}, i = \overline{1,4},$$

of the group (2.2), we get

$$d\overline{s}^h = \lambda_i^h dx^i + \lambda_5^h dx^5 + c_5^h \left(dx^5 + a_i^5 dx^i\right).$$

Choosing $c_5^h = -\lambda_5^h$, one can suppose that

$$ds^h = (\lambda_i^h - \lambda_5^h a_i^5) dx^i,\ h = \overline{1,4}. \tag{2.6}$$

Being given the Riemannian metric $ds^2 = \delta_{ab} ds^a ds^b$ of the space $\mathbb{R}^5$ and the equation $ds^5 = dx^5 + \lambda_i^5 dx^i = 0$, $i = \overline{1,4}$, the induced Riemannian metric of $V_5^4 : ds^5 = 0$ is completely determined

$$ds^2 = \delta_{\alpha\beta} ds^\alpha ds^\beta,\ \alpha, \beta = \overline{1,4},$$

$$\begin{aligned} ds^2 &= \delta_{\alpha\beta}\lambda_i^\alpha \lambda_j^\beta dx^i dx^j \\ &= \delta_{\alpha\beta}\lambda_\gamma^\alpha \lambda_\eta^\beta dx^\gamma dx^\eta + 2\delta_{\alpha\beta}\lambda_5^\alpha \lambda_\gamma^\beta dx^5 dx^\gamma + \delta_{\alpha\beta}\lambda_5^\alpha \lambda_5^\beta dx^5 dx^5 \\ &= \delta_{\alpha\beta}\lambda_\gamma^\alpha \lambda_\eta^\beta dx^\gamma dx^\eta + 2\delta_{\alpha\beta}\lambda_5^\alpha \lambda_\gamma^\beta \left(-\lambda_\varepsilon^5 dx^\varepsilon\right) dx^\gamma + \\ &\quad +\delta_{\alpha\beta}\lambda_5^\alpha \lambda_5^\beta \left(-\lambda_\varepsilon^5 dx^\varepsilon\right)\left(-\lambda_5^5 dx^5\right) \\ &= \left(\delta_{\alpha\beta}\lambda_\gamma^\alpha \lambda_\eta^\beta - 2\delta_{\alpha\beta}\lambda_5^\alpha \lambda_\gamma^\beta \lambda_\eta^5 + \delta_{\alpha\beta}\lambda_5^\alpha \lambda_5^\beta \lambda_\eta^5 \lambda_\gamma^5\right) dx^\gamma dx^\eta. \end{aligned} \tag{2.7}$$

To introduce a suitable co-frame, we select the 1-forms 2.6 as

$$\begin{aligned} ds^1 &= x^1 dx^2, \\ ds^2 &= x^3 dx^4, \\ ds^3 &= dx^5 + x^4 dx^3 + x^3 dx^4, \\ ds^4 &= dx^5 - x^2 dx^1 - x^1 dx^2, \end{aligned}$$

or in matrix language

$$\lambda = \begin{pmatrix} 0 & x^1 & 0 & 0 \\ 0 & 0 & 0 & x^3 \\ 0 & x^1 & x^4 & 0 \\ -x^2 & 0 & 0 & -x^3 \end{pmatrix}.$$

Then the Riemannian metric of the nonholonomic hypersurface $V_5^4$ is

$$\begin{aligned} ds^2 &= \left(ds^1\right)^2 + \left(ds^2\right)^2 + \left(ds^3\right)^2 + \left(ds^4\right)^2 \overset{ds^5=0}{=} \\ &= \left(x^1 dx^2\right)^2 + \left(x^3 dx^4\right)^2 + \left(x^1 dx^2 + x^4 dx^3\right)^2 + \left(x^3 dx^4 + x^2 dx^1\right)^2 \\ &= x^{2^2} dx^{1^2} + 2x^{1^2} dx^{2^2} + x^{4^2} dx^{3^2} + 2x^{3^2} dx^{4^2} + 2x^1 x^4 dx^2 dx^3 + \\ &\phantom{=} +2x^2 x^3 dx^1 dx^4 \\ &= g_{ij} dx^i dx^j, \ \ i,j = \overline{1,4}, \end{aligned}$$

$$(g_{ij}) = \begin{pmatrix} x^{2^2} & 0 & 0 & x^2 x^3 \\ 0 & 2x^{1^2} & x^1 x^4 & 0 \\ 0 & x^1 x^4 & x^{4^2} & 0 \\ x^2 x^3 & 0 & 0 & 2x^{3^2} \end{pmatrix}.$$

If we operate the substitutions (2.1), we recognize that the components $g_{ij}$, $i,j = \overline{1,4}$ are the *thermodynamic energies* $T^2$, $S^2$, $P^2$, $V^2$, $TV$, $SP$.

Reciprocally, starting from the Riemannian metric $(g_{ij})$, $i,j = \overline{1,4}$, of $V_5^4 : ds^5 = 0$, we can build an infinity of Riemannian metrics of $\mathbb{R}^5$,

$$\begin{aligned} ds^2 &= g_{ij} dx^i dx^j + 2g_{i5} dx^i ds^5 + g_{55}\left(ds^5\right)^2 \\ &= x^{2^2} dx^{1^2} + 2x^{1^2} dx^{2^2} + x^{4^2} dx^{3^2} + 2x^{3^2} dx^{4^2} + 2x^1 x^4 dx^2 dx^3 + \\ &\phantom{=} +2x^2 x^3 dx^1 dx^4 + 2g_{i5} dx^i \left(dx^5 - x^1 dx^2 + x^3 dx^4\right) + \\ &\phantom{=} +g_{55}\left(dx^5 - x^1 dx^2 + x^3 dx^4\right)^2, \end{aligned}$$

where $g_{i5}, g_{55} : \mathbb{R}^5 \to R$ are arbitrary $C^\infty$ functions.

Taking now the Lagrange function

$$L = \frac{1}{2}\left(g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} + 2g_{i5}\frac{dx^i}{dt}\frac{ds^5}{dt} + g_{55}\frac{ds^5}{dt}\frac{ds^5}{dt}\right),$$

on $\mathbb{R} \times \mathbb{R}^5 \times \mathbb{R}^5$, where $g_{ij} = \delta_{hk}\lambda_i^h \lambda_j^k$, the equations of the geodesics of the space $V_5^4 : ds^5 = dx^5 + \lambda_i^5 dx^i = 0$ (the Euler-Lagrange equations induced on $V_5^4 : ds^5 = 0$)

are

$$\frac{d}{dt}\left(g_{ij}\frac{dx^j}{dt}+g_{j5}\frac{dx^j}{dt}\lambda_i^5\right)-\frac{1}{2}\frac{\partial g_{jk}}{\partial x^i}\frac{dx^j}{dt}\frac{dx^k}{dt}-g_{j5}\frac{dx^j}{dt}\frac{\partial\lambda_k^5}{\partial x^i}\frac{dx^k}{dt}=0,$$
$$\frac{d}{dt}\left(g_{i5}\frac{dx^i}{dt}\right)-\frac{1}{2}\frac{\partial g_{jk}}{\partial x^5}\frac{dx^j}{dt}\frac{dx^k}{dt}-g_{j5}\frac{dx^j}{dt}\frac{\partial\lambda_k^5}{\partial x^5}\frac{dx^k}{dt}=0.$$

Using the second order Christoffel symbols

$$\Gamma_{jk}^i=\frac{1}{2}g^{il}\left(\frac{\partial g_{kl}}{\partial x^j}+\frac{\partial g_{jl}}{\partial x^k}-\frac{\partial g_{jk}}{\partial x^l}\right),$$

we compute the Riemannian tensor

$$R_{jkl}^i=\Gamma_{jl,k}^i-\Gamma_{jk,l}^i+\Gamma_{jl}^m\Gamma_{mk}^i-\Gamma_{jk}^m\Gamma_{ml}^i,\ \ i,j,k,l=\overline{1,4},$$

where the comma indicates the derivation with respect to the variable whose index follows the comma ([3]). Finally, the Ricci tensor has the components $R_{jl}=R_{jil}^i$ explicitly given by

$$R_{11}=-\frac{x^2\left(x^1x^2+2x^3x^4\right)}{x^1x^{3^2}x^{4^2}},$$
$$R_{12}=R_{21}=\frac{30x^{3^2}x^{4^2}+6x^1x^2x^3x^4+2x^{1^2}x^{2^2}}{2x^1x^2x^{3^2}x^{4^2}},$$
$$R_{13}=R_{31}=\frac{\left(x^1x^2+x^3x^4\right)^2}{6x^{1^2}x^2x^{3^2}x^4},$$
$$R_{14}=R_{41}=-\frac{2x^{3^2}x^{4^2}+7x^1x^2x^3x^4+4x^{1^2}x^{2^2}}{2x^{1^2}x^2x^3x^{4^2}},$$
$$R_{22}=-\frac{3x^{3^2}x^{4^2}+8x^1x^2x^3x^4+2x^{1^2}x^{2^2}}{2x^{2^2}x^{3^2}x^{4^2}},$$
$$R_{23}=R_{32}=-\frac{4x^{3^2}x^{4^2}+7x^1x^2x^3x^4+2x^{1^2}x^{2^2}}{2x^1x^{2^2}x^{3^2}x^4},$$
$$R_{24}=R_{42}=\frac{4x^{3^2}x^{4^2}+7x^1x^2x^3x^4+4x^{1^2}x^{2^2}}{2x^1x^{2^2}x^3x^{4^2}},$$
$$R_{33}=-\frac{x^4\left(2x^1x^2+x^3x^4\right)}{x^{1^2}x^{2^2}x^{3^2}},$$
$$R_{34}=R_{43}=\frac{2x^{3^2}x^{4^2}+6x^1x^2x^3x^4+3x^{1^2}x^{2^2}}{2x^{1^2}x^{2^2}x^3x^4},$$
$$R_{44}=-\frac{2x^{3^2}x^{4^2}+8x^1x^2x^3x^4+3x^{1^2}x^{2^2}}{2x^{1^2}x^{2^2}x^{4^2}}.$$

The scalar curvature

$$c=g^{ij}R_{ij},\ \ c=\frac{3x^{3^2}x^{4^2}+3x^{1^2}x^{2^2}-4x^1x^2x^3x^4}{2x^{1^2}x^{2^2}x^{3^2}x^{4^2}}$$

reflects a saddle behavior.

## 3 Gibbs-Dobrescu-Stamin-Udrişte noholonomic hypersurface

Consider again the nonholonomic hypersurface described by the Pfaff equation

$$E_5^4: dx^5-x^1dx^2+x^3dx^4=0,$$

in the notations (2.1). To apply the congruences method, as it is described in [1] (p. 398-403), we use the equivalent equation (no singular point)

$$ds^5 = \frac{1}{Q}dx^5 - \frac{x^1}{Q}dx^2 + \frac{x^3}{Q}dx^4 = 0,\ Q = \sqrt{1 + {x^1}^2 + {x^3}^3}.$$

We add now our co-frame, by associating the Pfaff forms

$$\begin{aligned}
ds^1 &= TdS,\\
ds^2 &= PdV,\\
ds^3 &= dU + VdP + PdV,\\
ds^4 &= dU - SdT - TdS,
\end{aligned}$$

whose thermodynamic sense is obvious. In general notations, we have

$$\begin{aligned}
ds^1 &= x^1dx^2,\\
ds^2 &= x^3dx^4,\\
ds^3 &= dx^5 + x^4dx^3 + x^3dx^4,\\
ds^4 &= dx^5 - x^2dx^1 - x^1dx^2,
\end{aligned}$$

with the associated matrix

$$\lambda = \begin{pmatrix} 0 & x^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x^3 & 0 \\ 0 & 0 & x^4 & x^3 & 1 \\ -x^2 & -x^1 & 0 & 0 & 1 \\ 0 & -\frac{x^1}{Q} & 0 & \frac{x^3}{Q} & \frac{1}{Q} \end{pmatrix}$$

and its inverse

$$\mu = \begin{pmatrix} 0 & -\frac{1}{x^2} & 0 & -\frac{1}{x^2} & \frac{Q}{x^2} \\ \frac{1}{x^1} & 0 & 0 & 0 & 0 \\ -\frac{1}{x^4} & 0 & \frac{1}{x^4} & 0 & -\frac{Q}{x^4} \\ 0 & \frac{1}{x^3} & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & Q \end{pmatrix}.$$

The first fundamental form of the nonholonomic hypersurface $E_5^4$ ([1], p. 401) is given by

$$I = ds^2 = \delta_{ij}dx^idx^j = E_{\alpha\beta}ds^\alpha ds^\beta,\ E_{\alpha\beta} = \delta_{ij}\mu^i_\alpha\mu^j_\beta,\ i,j = \overline{1,5},\ \alpha,\beta = \overline{1,4}.$$

Explicitly,

$$E_{\alpha\beta} = \begin{pmatrix} \frac{1}{{x^1}^2} + \frac{1}{{x^4}^2} + 1 & -1 & -\frac{1}{{x^4}^2} & 0 \\ -1 & \frac{1}{{x^2}^2} + \frac{1}{{x^3}^2} + 1 & 0 & \frac{1}{{x^2}^2} \\ -\frac{1}{{x^4}^2} & 0 & \frac{1}{{x^4}^2} & 0 \\ 0 & \frac{1}{{x^2}^2} & 0 & \frac{1}{{x^2}^2} \end{pmatrix},$$

respectively

$$\begin{aligned}
ds^2 \quad = \quad & \left(\frac{1}{{x^1}^2} + \frac{1}{{x^4}^2} + 1\right)\left(ds^1\right)^2 + \left(\frac{1}{{x^2}^2} + \frac{1}{{x^3}^2} + 1\right)\left(ds^2\right)^2 + \frac{1}{{x^4}^2}\left(ds^3\right)^2 + \\
& + \frac{1}{{x^2}^2}\left(ds^4\right)^2 - ds^1ds^2 - \frac{2}{{x^4}^2}ds^1ds^3 + \frac{2}{{x^2}^2}ds^2ds^4.
\end{aligned}$$

The second fundamental form of the nonholonomic hypersurface $E_5^4$ ([1], p. 403) is

$$II = D_{\alpha\beta} ds^\alpha ds^\beta,\ D_{\alpha\beta} = -\frac{\partial \lambda_i^5}{\partial x^j} \mu_\alpha^i \mu_\beta^j,\ i,j = \overline{1,5},\ \alpha,\beta = \overline{1,4}.$$

We obtain

$$D_{\alpha\beta} = \begin{pmatrix} 0 & -\frac{1}{x^1x^2Q} & 0 & -\frac{1}{x^1x^2Q} \\ \frac{1}{x^3x^4Q} & 0 & -\frac{1}{x^3x^4Q} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and thus

$$II = -\frac{1}{x^1x^2x^3x^4Q}\left[\left(x^3x^4 - x^1x^2\right) ds^1 ds^2 - x^3x^4 ds^1 ds^4 + x^1x^2 ds^2 ds^3\right].$$

The symmetric part is

$$\begin{aligned} D^s_{\alpha\beta} &= \frac{D_{\alpha\beta} + D_{\beta\alpha}}{2} = \\ &= \begin{pmatrix} 0 & \frac{1}{4Qx^3x^4} - \frac{1}{2Qx^1x^2} & 0 & -\frac{1}{2Qx^1x^2} \\ \frac{1}{2Qx^3x^4} - \frac{1}{4Qx^1x^2} & 0 & -\frac{1}{2Qx^3x^4} & 0 \\ 0 & -\frac{1}{4Qx^3x^4} & 0 & 0 \\ -\frac{1}{4Qx^1x^2} & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

and

$$II_s = -\frac{\left(3x^3x^4 - 3x^1x^2\right) dx^1 dx^2 + x^3x^4 dx^1 dx^3 + 3x^1x^2 dx^2 dx^3 + 2x^3x^4 dx^1 dx^4}{4Qx^1x^2x^3x^4}.$$

The curvature of normal section, tangent to a curve $C$ of the nonholonomic hypersurface, is given by ([1], p. 403):

$$\frac{1}{\rho} = \frac{D_{\alpha\beta} ds^\alpha ds^\beta}{E_{\alpha\beta} ds^\alpha ds^\beta}.$$

It follows the *Gauss-Kronecker curvature*

$$K = \frac{\det II_s}{\det I} = \frac{1}{64Q^4\left(x_1^2 + x_3^2 + 1\right)} > 0$$

and the mean curvature $H = 0$. The nonholonomic hypersurface $E_5^4$ has no umbilical point since the system

$$\begin{aligned} &\frac{E_{11}}{D_{11}} = \frac{E_{22}}{D_{22}} = \frac{E_{33}}{D_{33}} = \frac{E_{44}}{D_{44}} = \\ &= \frac{E_{12}}{D^s_{12}} = \frac{E_{13}}{D^s_{13}} = \frac{E_{14}}{D^s_{14}} = \frac{E_{21}}{D^s_{21}} = \frac{E_{23}}{D^s_{23}} = \frac{E_{24}}{D^s_{24}} \\ &= \frac{E_{31}}{D^s_{31}} = \frac{E_{32}}{D^s_{32}} = \frac{E_{34}}{D^s_{34}} = \frac{E_{41}}{D^s_{41}} = \frac{E_{42}}{D^s_{42}} = \frac{E_{43}}{D^s_{43}}, \end{aligned}$$

has no solution.

## 4 Gibbs-Udrişte-Dogaru nonholonomic hypersurface

We follow the ideas in [4]. Let $D$ be an open subset in $\mathbb{R}^5$ and $I$ a compact interval in $\mathbb{R}^m$, $m \in \{1, 2\}$. Consider the Pfaff equation $\omega = \omega_i(x)\, dx^i$, $i = \overline{1, 5}$, where $\omega_1 = 0$, $\omega_2 = -x^1$, $\omega_3 = 0$, $\omega_4 = x^3$, $\omega_5 = 1$.

We say that the integral manifold $r : I \to D$ of the Pfaff equation passes through the point $x_0 \in D$ if $\exists u_0 \in I$ such that $r(u_0) = x_0$. We observe that there is always an infinity of integral curves (the case $m = 1$) passing through $x_0$. Because the Pfaff equation is not completely integrable, through $x_0$ there can pass integral manifolds of dimension 2 which have in common only the point $x_0$. We denote by $M_{x_0}$ the image of an integral manifold through $x_0$ and by $\Sigma_{x_0}$ the family of all the images of integral manifolds through $x_0$. The pair $(\Sigma, D)$, where we denoted $\Sigma = \{\Sigma_{x_0}\}_{x_0 \in D}$, is called *nonholonomic hypersurface on $D$ attached to the Pfaff equation.*

We can attach to the Pfaff equation and to the point $x_0 \in D$, the tangent hyperplane

$$Q_{x_0} = \left\{x \in \mathbf{R}^5 | - x_0^1(x^2 - x_0^2) + x_0^3(x^4 - x_0^4) + x^5 - x_0^5 = 0\right\}.$$

**Remark 4.1.** *The correspondence $x_0 \to Q_{x_0} \subset T_{x_0}D$ defines a function $Q$ on $D$ which is a $4-$dimensional distribution. A vector field $Y$ belongs to the distribution $Q$ if $Y(x_0) \in Q_{x_0}$, $\forall x_0 \in D$ or, equivalently $\omega(Y) = 0$ on $D$. Obviously, for every $x_0$ there exists a neighborhood $U$ of $x_0$ and $4$ vector fields $Y_1, ..., Y_4$ of class $C^1$ on $U$ such that $\{Y_1(x_0), ..., Y_4(x_0)\}$ is a basis of $H_{x_0}$. The set $\{Y_1, ..., Y_4\}$ is called local basis of distribution $Q$. The integral manifolds of Pfaff equation are called integral manifolds of the distribution $Q$.*

The distribution $Q$ is not involutive (i.e. $Y, Z \in Q$ and $Y, Z$ of class $C^1$ do not imply $[Y, Z] \in Q$) since the Pfaff equation is not completely integrable.

The quadratic form associated to the identity on $Q_{x_0}$, i.e., the real function $v \to (v, v)$, $v \in Q_{x_0}$ is called *the first fundamental form of the nonholonomic hypersurface $(\Sigma, D)$ at the point $x_0$ and is denoted by $g_{x_0}$*. This function is a scalar product (the restriction to $Q_{x_0}$ of the scalar product on $\mathbb{R}^5$). The function $x_0 \to g_{x_0}$, $x_0 \in D$ is called *the first fundamental form of* $(\Sigma, D)$. Taking into account the Pfaff equation, it appears the Riemannian metric

$$\begin{aligned} g &= \delta_{ij} dx^i dx^j \overset{ds^5=0}{=} \\ &= \left(dx^1\right)^2 + \left(1 + x^{1^2}\right)\left(dx^2\right)^2 + \left(dx^3\right)^2 + \left(1 + x^{3^2}\right)\left(dx^4\right)^2 - 2x^1x^3dx^2dx^4. \end{aligned}$$

Let $\xi$ be the unit vector field on $D$ given by $\frac{1}{\|\omega\|}(\omega_1, ..., \omega_5)$, where

$$\|\omega\| = \sqrt{\omega_1^2 + ... + \omega_5^2}.$$

Let $M_{x_0}$ be the image of a bidimensional integral manifold through $x_0$, admitting the orthonormal vector fields $\xi_1$, $\xi_2$, $\xi_3$. One of these fields coincides with the restriction of $\xi$ to $M_{x_0}$ if and only if $M_{x_0} \subset H_{x_0}$. If we put the condition $\sum\limits_{i=1}^{5} \omega_i^2(x) = 1$, then $\xi = (\omega_1, ..., \omega_5)$. Let $v \in H_{x_0}$. By differentiating the identity $(\xi, \xi) = 1$ with respect to $v$ we find $(D_v\xi, \xi)(x_0) = 0$ and hence $D_v\xi \in H_{x_0}$. The linear map

$$S_{x_0} : H_{x_0} \to H_{x_0},\ S_{x_0} = -D_v\xi$$

is called the *Weingarten map of* $(\Sigma, D)$ *at* $x_0$. The geometric meaning of $S_{x_0}$ can be deduced from the formula

$$D_v\xi = \frac{d}{dt}\xi(\alpha(t))\Big|_{t=t_0},$$

where $\alpha : I \to D$ is an integral curve passing through $x_0$ at the moment $t_0$ with the velocity vector $\alpha'(t_0) = v$. Therefore $S_{x_0}$ measures the rate of change of the direction of $\xi$ while passing through $x_0$ along $\alpha(I)$. Since $Q_{x_0}$ is identified to $\xi(\alpha(t_0))^{\perp}$, $Q_{x_0}$ turns as the normal $\xi$ turns and thus $S_{x_0}(v)$ can be interpreted as a measure of the turning of $Q_{x_0}$ when passing through $x_0$ along $\alpha(I)$. Thus $S_{x_0}$ contains information about the shape of $(\Sigma, D)$ in $x_0$.

The most important properties of Weingarten map of a nonholonomic hypersurface are included in the following two theorems ([4], [5]).

**Theorem 4.1.** *Let* $D$ *be an open set in* $\mathbb{R}^n$, $n \geq 3$ *and the Pfaff equation*

$$\omega(x) = \sum_{i=1}^{n} \omega_i(x)\, dx^i = 0,\ x = \left(x^1, ..., x^n\right),$$

*where* $\omega_i : D \to \mathbb{R}$, $i = \overline{1,n}$ *are* $C^1$ *functions. Let* $(\Sigma, D)$ *be the nonholonomic hypersurface attached to the Pfaff equation, with* $\sum\limits_{i=1}^{n} \omega_i^2(x) = 1$. *Let* $x_0 \in D$ *and* $v \in H_{x_0}$. *Then, for every integral curve* $\alpha : I \to D$ *with* $\alpha(t_0) = x_0$, $\alpha'(t_0) = v$ *the following relation holds*

$$(\alpha''(t_0), \xi(x_0)) = (S_{x_0}(v), v).$$

**Theorem 4.2.** *The Weingarten map* $S_{x_0}$ *is not self-adjoint, i.e., there exist* $v, w \in Q_{x_0}$, *such that*

$$(S_{x_0}(v), w) \neq (v, S_{x_0}(w)).$$

Let $S^*_{x_0} : Q_{x_0} \to Q_{x_0}$ be the adjoint of $S_{x_0}$, i.e., the linear map (uniquely) defined by $\left(S^*_{x_0}(v), w\right) = (v, S_{x_0}(w))$, $\forall v, w \in Q_{x_0}$. The matrix of $S^*_{x_0}$ with respect to an orthonormal basis of $Q_{x_0}$ is the transpose of the matrix of $S_{x_0}$ with respect to that basis. The linear map $\frac{1}{2}\left(S_{x_0} + S^*_{x_0}\right)$ is self-adjoint. The quadratic form associated to this map, i.e., the function

$$\Omega_{x_0} : Q_{x_0} \to \mathbf{R},\ \Omega_{x_0}(v) = \frac{1}{2}\left(S_{x_0}(v) + S^*_{x_0}(v), v\right),$$

is called the *second fundamental form of nonholonomic hypersurface* $(\Sigma, D)$ *at* $x_0$.

We observe that $\Omega_{x_0}(v) = (\alpha''(t_0), \xi(x_0))$, where $\alpha : I \to D$ is an integral curve, with $\alpha(t_0) = x_0$ şi $\alpha'(t_0) = v$. This quadratic form can be obtained by usual differentiation of the Pfaff form

$$-\sum_{i=1}^{n} \omega_i(x) dx^i \quad \left(\sum_{i=1}^{n} \omega_i^2(x) = 1\right)$$

and then taking the symmetric part, i.e.

$$\Omega = -\frac{1}{2}\sum_{i,j=1}^{n} \left(\frac{\partial \omega_i}{\partial x^j} + \frac{\partial \omega_j}{\partial x^i}\right)(x) dx^i dx^j.$$

In general, the second fundamental form of $(\Sigma, D)$ is given by

$$\Omega = -\frac{1}{2\left\|\omega\right\|} \sum_{i,j=1}^{n} \left( \frac{\partial \omega_i}{\partial x^j} + \frac{\partial \omega_j}{\partial x^i} \right) (x) dx^i dx^j.$$

The second fundamental form of the Pfaff nonholonomic hypersurface is $(\Sigma, D)$ is

$$\Omega = \frac{1}{2\sqrt{1 + x^{1^2} + x^{3^2}}} \left( dx^1 dx^2 - dx^3 dx^4 \right).$$

Particularly, if $v \in Q_{x_0}$, $\|v\| = 1$, then $k(v) = \Omega_{x_0}(v)$ is called *the normal curvature of* $(\Sigma, D)$ at $x_0$ in the direction $v$. The extrema of normal curvature are proper values of the linear map $\frac{1}{2}\left(S_{x_0} + S^*_{x_0}\right)$. This, being a self-adjoint linear map, admits real proper values $k_1(x_0), ..., k_4(x_0)$ and orthonormal proper vectors $e_1, ..., e_4$, which constitute a basis in $Q_{x_0}$. The proper values $k_1(x_0), ..., k_4(x_0)$ are called ıprincipal curvatures of $(\Sigma, D)$ at $x_0$, and the proper vectors $e_1, ..., e_4$ are called *principal directions.* If the principal curvatures are such ordered that $k_1(x_0) \leq k_2(x_0) \leq k_3(x_0) \leq k_4(x_0)$, then

$$\begin{aligned} k_4(x_0) &= \max_{\|v\|=1} k(v) \\ k_3(x_0) &= \max_{\|v\|=1, v \perp e_4} k(v) \\ k_2(x_0) &= \max_{\|v\|=1, v \perp \{e_3, e_4\}} k(v) \\ k_1(x_0) &= \min_{\|v\|=1} k(v). \end{aligned}$$

Since $v = \sum\limits_{i=1}^{4} (v, e_i) e_i = \sum\limits_{i=1}^{4} (\cos\theta) e_i$, we find the Euler formula

$$k(v) = \sum_{i=1}^{4} k_i(x_0) \cos^2 \theta_i.$$

The number

$$K(v) = \det \frac{1}{2} c = k_1(x_0) k_2(x_0) k_3(x_0) k_4(x_0)$$

is called *the Gauss-Kronecker curvature of* $(\Sigma, D)$ *at* $x_0$. The function $x_0 \to K(x_0)$, $x_0 \in D$ is called *Gauss-Kronecker curvature of* $(\Sigma, D)$.

The number

$$H(x_0) = \frac{1}{4} tr \frac{1}{2} (S_{x_0} + S^*_{x_0}) = \frac{1}{4} \sum_{i=1}^{4} k_i(x_0)$$

is called *the mean curvature of* $(\Sigma, D)$ *at* $x_0$. The function $x_0 \to H(x_0)$, $x_0 \in D$ is called *the mean curvature of* $(\Sigma, D)$. The Gauss-Kronecker curvature and the mean curvature of Gibbs nonholonomic hypersurface $(\Sigma, D)$ vanishes.

In this context, using the Christoffel symbol of second order, we calculate the Riemann tensor and finally we get the Ricci tensor

$$\begin{aligned}
&R_{11} = -\tfrac{2+3x^{3^2}+x^{3^4}+x^{1^2}x^{3^2}}{2\left(1+x^{1^2}+x^{3^2}\right)^2},\\
&R_{12} = R_{21} = 0,\\
&R_{13} = R_{31} = -\tfrac{x^1x^3\left(3+x^{1^2}+x^{3^2}\right)}{2\left(1+x^{1^2}+x^{3^2}\right)^2},\\
&R_{14} = R_{41} = 0,\\
&R_{22} = \tfrac{-2+x^{1^2}-x^{3^2}+x^{1^2}x^{3^2}+x^{1^4}}{2\left(1+x^{1^2}+x^{3^2}\right)},\\
&R_{23} = R_{32} = 0,\\
&R_{24} = R_{42} = -\tfrac{x^1x^3\left(2+x^{1^2}+x^{3^2}\right)}{2\left(1+x^{1^2}+x^{3^2}\right)},\\
&R_{33} = -\tfrac{2+3x^{1^2}+x^{1^4}+x^{1^2}x^{3^2}}{2\left(1+x^{1^2}+x^{3^2}\right)^2},\\
&R_{34} = R_{43} = 0,\\
&R_{44} = \tfrac{-2-x^{1^2}+x^{3^2}+x^{1^2}x^{3^2}+x^{3^4}}{2\left(1+x^{1^2}+x^{3^2}\right)}.
\end{aligned}$$

and the scalar curvature

$$c = \frac{-10+2x^{1^2}x^{3^4}-x^{1^2}x^{3^2}+x^{3^6}-2x^{1^4}-6x^{3^2}-8x^{1^2}+x^{3^4}+x^{1^4}x^{3^2}}{2\left(1+x^{1^2}+x^{3^2}\right)^2}.$$

# 5 Conclusions

The paper studies three different geometries of Gibbs nonholonomic hypersurface, namely: Vrănceanu geometry ([7], [6]), Udrişte-Dogaru geometry ([4], [5]) and a third geometry introduced by the authors, combining Dobrescu's theory ([1]) with Udrişte's theory ([5]). These include the first and second fundamental forms and their derived geometric objects.

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