Lagrange-Hamilton conjugations

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Dedicated to the 70-th anniversary
of Professor Constantin Udriste

Abstract. The general theory of almost symplectic ω-conjugation was elaborated, for the first time, by P. Stavre ([7]). Its applications to the total space E of a vector bundle $\xi = (E, \pi, M)$ are already studied ([3]). In the cases of the tangent bundle and of the cotangent bundle, the two authors obtained new results, with applications in Hamiltonian models ([9], [4]). Using these results and the theory of the G-conjugation ([8]), in the framework of the theory of Lagrange-Hamilton models, the authors obtain the case when the theory of the almost symplectic ω-conjugation and the theory of the G-conjugation are equivalent.

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1 General notions. Theorems

Let us consider a p-dimensional paracompact differentiable manifold E_p, endowed with an almost symplectic structure.

Definition 1.1. ([7]) Let us consider the structure $\overset{(1)}{L}_p = (E_p, \omega, \overset{(1)}{D})$ where ω is an almost symplectic one and $\overset{(1)}{D}$ a linear connection on E and analogously $\overset{(2)}{L}_p = (E_p, \omega, \overset{(2)}{D})$. Let us consider two one-dimensional distributions $\underset{(1)}{\overset{(1)}{\mathcal{D}}}$, $\underset{(1)}{\overset{(2)}{\mathcal{D}}}$:

$$\underset{(1)}{\overset{(1)}{\mathcal{D}}} : u \to \underset{(1)}{\overset{(1)}{\mathcal{D}}}_u \subset T_u E; \quad \underset{(1)}{\overset{(2)}{\mathcal{D}}} : u \to \underset{(1)}{\overset{(2)}{\mathcal{D}}}_u \subset T_u E, \quad u \in E. \tag{1.1}$$

If we have

$$\omega(\underset{(1)}{V}, \underset{(2)}{V}) = 0, \ \forall \ V_1 \in \underset{(1)}{\overset{(1)}{\mathcal{D}}}_u, \ V_2 \in \underset{(1)}{\overset{(2)}{\mathcal{D}}}_u, \quad u \in E \tag{1.2}$$

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at the parallel transport of the distributions $\underset{(1)}{\overset{(1)}{\mathcal{D}}}, \underset{(1)}{\overset{(2)}{\mathcal{D}}}$, with respect to $\overset{(1)}{D}, \overset{(2)}{D}$, then we will say that the linear connections $\overset{(1)}{D}, \overset{(2)}{D}$ are ω-conjugated connections or that the spaces $\overset{(1)}{L}_p, \overset{(2)}{L}_p$ are ω-conjugated-spaces. We will write: $\overset{(1)}{L}_p \overset{\omega}{\sim} \overset{(2)}{L}_p$ or $\overset{(1)}{D} \overset{\omega}{\sim} \overset{(2)}{D}$.

In the general case we have:

Proposition 1.1. *In the general case the relation "* $\overset{\omega}{\sim}$ *" is not an equivalence one on the set* $\underset{(\omega)}{D} = \{D, \omega, \overset{\omega}{\sim}\}$ *of the ω-conjugated connections.*

We further denote:

$$\underset{(\omega)}{C}_{\overline{D}} = \{(E_p, \omega, D) \mid D \overset{\omega}{\sim} \overline{D},\ \text{with } \overline{D}\omega = 0\}; \quad \underset{(\omega)}{M}_1(D) = \bigcup_{\overline{D}} \underset{(\omega)}{C}_{\overline{D}}. \tag{1.3}$$

Let us consider $\underset{(\omega)}{M}_2(D)$ the set of the linear connections which are not ω-conjugated with any ω-compatible connection.

Partition Theorem 1.2. ([9]) *One has the partitions:*

$$\underset{(\omega)}{M}(D) = \underset{(\omega)}{M}_1(D) \cup \underset{(\omega)}{M}_2(D); \quad \underset{(\omega)}{M}_1(D) \cap \underset{(\omega)}{M}_2(D) = \varnothing \tag{1.4}$$

$$\underset{(\omega)}{C}_{\overset{(1)}{D}} \cap \underset{(\omega)}{C}_{\overset{(2)}{D}} = \varnothing\ ,\ \forall\ \underset{(\omega)}{C}_{\overset{(1)}{D}}, \underset{(\omega)}{C}_{\overset{(2)}{D}} \subset \underset{(\omega)}{M}_1(D).$$

As well, $\{\underset{(\omega)}{C}_{\overline{D}}\}$ *are equivalence classes.*

It is has already been established that:

$$(d\omega)(XYZ) = \sum_{XYZ} \{(D_X\omega)(Y, Z) + \omega T(XY, Z)\}. \tag{1.5}$$

The following result holds true:

Proposition 1.3. *If ω is a symplectic structure, i.e. $d\omega = 0$, then there exist linear connections ∇, which are ω-compatible ($(\nabla\omega = 0)$) and torsion free ($(\overset{\nabla}{T} = 0)$).*

Starting from the above relations and Proposition 1.3, we infer:

Proposition 1.4. *If ω is a symplectic structure, then there exist at least one class,* $\underset{(\omega)}{C}_{\nabla}$ *in the partition (1.4).*

Proposition 1.5. *If a linear connection, D, is ω-compatible, then D is not ω-conjugated with ∇.*

Proposition 1.6. *If D is a linear and symmetrical connection* ($\overset{D}{T} = 0$) *then D is not ω-conjugated with ∇.*

In [7] P. Stavre gave a general theory of ω-conjugation, based on the operator $\overset{(12)}{D}_X$ defined by:

$$(\overset{(12)}{D}_X\omega)(YZ) = (\overset{(1)}{D}_X\omega)(YZ) - \omega(Y, \overset{(2)}{\tau}(XZ)) \tag{1.6}$$

where

$$\overset{(2)}{\tau}(XZ) = \overset{(2)}{D}_X Z - \overset{(1)}{D}_X Z \tag{1.7}$$

will be called the covariant mixed ω-derivative. The already established general relations between $\overset{(12)}{D}_X\omega$ and $\overset{(21)}{D}_X\omega$ allow us to state

Proposition 1.7. *The necessary and sufficient condition for* $\overset{(1)}{D} \overset{\omega}{\sim} \overset{(2)}{D}$ *is:*

$$\overset{(12)}{D}_X\omega = \alpha(X)\omega, \ \forall \ X \in \mathcal{X}(E) \ \textit{where} \ \alpha \in \Lambda_1(E). \tag{1.8}$$

As a corollary we have $\overset{(21)}{D}_X\omega = \alpha(X)\omega$ and the converse, as well. From (1.6), (1.8), it results

Proposition 1.8. *The set of linear connection transformations induced by ω-conjugation relation does not, generally, form a group of transformations.*

The cases of subsets of transformations induced by " $\overset{\omega}{\sim}$ ", which form a group of transformations, are already studied. We further present several consequences of the above results.

2 Natural almost symplectic structures

The almost symplectic structures and the integrable almost symplectic structures play an important role in the theory of the geometrical models of the Hamiltonian mechanical systems ([2]).

Let us consider a real, n-dimensional C^∞ differentiable manifold $M_n = (M, [A], R^n)$, which is paracompact, connected. Let $\xi = (E = TM, \omega, M)$ be the tangent bundle and let $\xi^* = (E^* = T^*M, \pi^*, M)$ be the cotangent bundle. In Hamiltonian mechanics, M is called the configuration space, TM is the speed space and T^*M is the phase space.

Generally speaking, a nondegenerate differential 2 form on a differentiable manifold is an almost symplectic structure. The manifold dimension must be an even number, so that the nondegenerate differential 2-form could exist. For ω to be integrable, from topological point of view, the manifold must be orientable. We have $\dim TM = 2n$ and $\dim T^*M = 2n$. Moreover, we can state the following result:

Proposition 2.1. *On the phase space T^*M, there exists an integrable, almost symplectic structure ω.*

Because TM is paracompact, connected and C^∞-differentiable manifold and $\dim TM = 2n$, there exists a metric structure, g and an almost complex structure F ($F^2 = -I$). We define:

$$G(X,Y) = g(X,Y) + g(FX,FY), \quad \forall X,Y \in \mathcal{X}(TM). \tag{2.1}$$

It results:

$$G(FX,FY) = G(X,Y) \tag{2.2}$$

hence this represents an almost Hermitian G structure. We define ω by

$$\omega(X,Y) = G(X,FY); \tag{2.3}$$

and we infer $\omega(X,Y) = -\omega(Y,X)$ hence ω is a nondegenerate, differential 2-form. It will be called natural almost symplectic structure, associated to (G,F). Let us take a natural, almost symplectic structure ω, defined by (2.3). We obtain:

Proposition 2.2. *We have, for any linear connection D on $E = TM$:*

$$(D_X\omega)(Y,Z) = (D_XG)(Y,FZ) + G(Y,(D_XF)(Z)). \tag{2.4}$$

Definition 2.1.A linear connection, D, on TM with the property $DF = 0$ will be called F-connection.

We will denote by $D(F)$ the set of the F-connections on $E = TM$. We have ([4]):

$$(D_X\omega)(Y,Z) = (D_XG)(Y,FZ),\ \forall\ D \in D(F). \tag{2.5}$$

Let us associate the Nijenhuis tensor to the almost complex structure F:

$$N_F(X,Y) = [FX,FY] - F[FX,Y] - F[X,FY] + F^2[X,Y], \quad (F^2 = -I). \tag{2.6}$$

Proposition 2.3. *We have:*

$$\begin{aligned} N_F(X,Y) = \quad & (D_{FX}F)(Y) - (D_{FY}F)(X) - F(D_XF)(Y) + F(D_YF)(X) \\ & -\{T(FX,FY) - FT(FX,Y) - FT(X,FY) - T(X,Y)\}. \end{aligned} \tag{2.7}$$

Proposition 2.4. *If there exists a linear torsion-free (symmetrical) connection, $D \in D(F)$, then the almost complex structure F is integrable. So F is a complex structure.*

Proof. From (2.8) with $T = 0$ and $DF = 0$, it results $N_F = 0$. So F is a complex structure and, therefore, (G,F) is a Hermitian structure. □

Proposition 2.5. *If $D \in D(F)$ and $T = 0$ then:*
a) The structure (G,F) is a Hermitian structure.
b) We have:

$$(d\omega)(X,Y,Z) = \sum_{(XYZ)} (D_XG)(Y,FZ).$$

Definition 2.2. A linear connection D will be called ω-compatible if we have:

$$D_X\omega = 0,\ \forall\ X. \tag{2.8}$$

It results:

Proposition 2.6. *If D is ω-compatible and torsion free, then ω is a symplectic structure.*

Proof. From (2.8) it follows $d\omega = 0$, so ω is a symplectic structure. □

(G,F,ω) is an almost Kählerian structure.

Proposition 2.7. *The Levi-Civita connection, ∇, is an F-connection if and only if the structure (G,F,ω) is a Kählerian structure.*

3 Natural almost symplectic conjugations and ω-compatible models

Any geometric model of an Hamiltonian mechanical system contains a differentiable manifold E, an almost symplectic structure, ω, and a linear connection, D, on E. So it is a space with linear connection, $\overset{(1)}{D}$, denoted by $(E, \overset{(1)}{D})$ endowed with an almost symplectic structure ω. This model will be denoted by $\overset{(1)}{L} = (E, \omega, \overset{(1)}{D})$.

If we change $\overset{(1)}{D}$ with $\overset{(2)}{D}$ we will obtain another model, $\overset{(2)}{L} = (E, \omega, \overset{(2)}{D})$.

The linear and symmetrical connections D on E play an important role in mathematical modelling. There exist such connections. But a linear connection D of the model can be chosen such that D is ω-compatible, i.e., $D\omega = 0$. In Lagrange models, based on a Riemannian metric G and on a relativistic (pseudo-Riemannian) metric G, respectively, there always exists a linear connection ∇, which is symmetrical and G-compatible ($\nabla G = 0$). This is the beforementioned Levi-Civita connection.

Proposition 3.1. ([5]). *Let us consider (E, ω), where ω is an almost symplectic structure.*

a) There exist symmetrical linear connections D.

b) There exist ω-compatible, linear connections D.

c) There exist no linear connections D, on E, which have both properties a) and b).

But generally speaking, we have $d\omega \neq 0$ for almost symplectic structures.

Example 3.1. If ω is natural, i.e., it is defined by (2.3), then the Levi-Civita connection ∇, is torsion free: ($\overset{\nabla}{T} = 0$) but is not ω-compatible, provided that ω is not integrable.

Having two models $\overset{(1)}{L} = (E, \omega, \overset{(1)}{D})$, $\overset{(2)}{L} = (E, \omega, \overset{(2)}{D})$ of a Hamiltonian mechanical systems, there should exist a natural criterion of comparison (compatibility) for these systems.

Definition 3.1. We say that the two models $\overset{(1)}{L} = (E, \omega, \overset{(1)}{D})$, $\overset{(2)}{L} = (E, \omega, \overset{(2)}{D})$ are ω-compatible if $\overset{(1)}{D}$, $\overset{(2)}{D}$ are ω-conjugated. Then we shall write $\overset{(1)}{L} \overset{\omega}{\sim} \overset{(2)}{L}$ or $\overset{(1)}{D} \overset{\omega}{\sim} \overset{(2)}{D}$.

In [7] the first author gave, for the first time, a general theory of geometries of spaces with ω-conjugated linear connections. If we apply this theory to the above case, it will result:

Proposition 3.2. *Two models $\overset{(1)}{L}$, $\overset{(2)}{L}$ are not ω-compatible if $\overset{(1)}{D}$, $\overset{(2)}{D}$ are ω-compatible ($\overset{(1)}{D}\omega = 0$; $\overset{(2)}{D}\omega = 0$).*

Proposition 3.3. *Let us consider $\overset{(1)}{L} = (E, \omega, \overset{(1)}{D})$, $\overset{(2)}{L} = (E, \omega, \overset{(2)}{D})$ such that $\overset{(1)}{T} = \overset{(2)}{T}$. Then $\overset{(1)}{L}$, $\overset{(2)}{L}$ are not ω-compatible if $\overset{(1)}{D}$ is ω-compatible or $\overset{(2)}{D}$ is ω-compatible.*

As a consequence we obtain:

Proposition 3.4. *Let us consider two Hamiltonian mechanical models, with the symplectic structure ω ($d\omega = 0$). These two models are not ω-compatible if $\overset{(1)}{T} = \overset{(2)}{T} = 0$ and $\overset{(1)}{D}$ is ω-compatible.*

Example 3.2. Let us consider the natural, almost symplectic structure ω defined by (2.3). Generally speaking, we have $d\omega \neq 0$. So it does not exist a symmetrical, ω-compatible connection D, but there exist ω-compatible connections D. If we apply Proposition 3.2 and Proposition 3.3 we will obtain:

Proposition 3.5. *Let us consider $\overset{(1)}{L} = (E, \omega, \overset{(1)}{D})$, $\overset{(2)}{L} = (E, \omega, \overset{(2)}{D})$ two models with $\overset{(1)}{T} = 0$, $\overset{(2)}{T} = 0$. They can be ω-compatible and $\overset{(1)}{D}$, $\overset{(2)}{D}$ are mutually determined.*

In the following we will emphasize the cases of the spaces TM, T^*M, endowed with such structures. Let us consider an almost Hermitian structure G, on $E = TM$ and a natural, almost symplectic structure ω, which is defined by (2.3). Let us consider the nonlinear connection N, given by the condition ([5]):

$$G(hX, vY) = 0, \ \forall \ X, Y \in \mathcal{X}(TM). \tag{3.1}$$

Then we have:

$$T_u E = H_u E \oplus V_u E, \ \forall \ u \in E \ (E = TM), \tag{3.2}$$

and it results $G = hG + vG$.

A general theory of the linear connections $\overset{(1)}{D} \overset{G}{\sim} \overset{(2)}{D}$ (or G-conjugated connections) was given by P. Stavre in [8]. Afterwards the theory was applied to the total space E of a vector bundle and, as a particular case, to the case of the total space of the tangent bundle, with numerous applications, when G is a pseudo-Riemannian one. For $n = 4$, the Einstein model (with torsion, which was elaborated by Einstein) is included in the general theory of the G-conjugated models. In the above conditions we investigated the equivalence between the G-conjugated models and the natural ω-conjugated Hamiltonian models.

If we define, in the same way, the G-conjugation of the linear connections $\overset{(1)}{D}$, $\overset{(2)}{D}$ ([8]), we obtain:

Proposition 3.6. *We have:*

a) $(\overset{(12)}{D}_X \omega)(Y, Z) = (\overset{(12)}{D}_X G)(Y, FZ) + G(Y, (\overset{(2)}{D}_X F)(Z))$;

b) $(\overset{(21)}{D}_X \omega)(Y, Z) = (\overset{(21)}{D}_X G)(Y, FZ) + G(Y, (\overset{(1)}{D}_X F)(Z))$.

Using these properties we can solve the important problem of giving the conditions for an equivalence between the G-conjugation and the ω-conjugation. The normal linear d-connections play an important role in the theory of the tangent bundle.

It is already known that, if D is a normal linear d-connection then we have:

$$D_X F = 0. \tag{3.3}$$

From Proposition 3.6 and (3.2), it results:

Proposition 3.7. *In the above conditions we have:*

$$(\overset{(12)}{D}_X\omega)(Y,Z) = (\overset{(12)}{D}_X G)(Y,FZ) \text{ if and only if } \overset{(2)}{D}_X F = 0 \tag{3.4}$$

$$(\overset{(21)}{D}_X\omega)(Y,Z) = (\overset{(21)}{D}_X G)(Y,FZ) \text{ if and only if } \overset{(1)}{D}_X F = 0. \tag{3.5}$$

From the relation (2.3) and from Proposition 3.7 it results:

The equivalence theorem. *Let us consider two Hamiltonian models* $\overset{(1)}{L}(\omega) = (TM, G, \omega, \overset{(1)}{D})$, $\overset{(2)}{L}(\omega) = (TM, G, \omega, \overset{(2)}{D})$, *where* $\overset{(1)}{D}, \overset{(2)}{D}$ *are normal, linear d-connections on* $E = TM$. *Let us consider the corresponding G-models* $\overset{(1)}{L}(G) = (TM, G, \overset{(1)}{D})$, $\overset{(2)}{L}(G) = (TM, G, \overset{(2)}{D})$. *We have* $\overset{(1)}{L} \overset{\omega}{\sim} \overset{(2)}{L}$ *if and only if* $\overset{(1)}{L} \overset{G}{\sim} \overset{(2)}{L}$.

Therefore the relation of ω-conjugation is equivalent with the G-conjugation for *normal linear d-connections.* This result is very important for the Lagrange-Hamilton duality and also in applications, especially for the case when G is defined by a Lagrangian. Some extensions and the curvature invariants of the G-conjugated, respectively ω-conjugated connections have recently been obtained.

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