Euler-Lagrange prolongations of Maxwell PDEs

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Dedicated to the 70-th anniversary
of Professor Constantin Udriste

Abstract. This paper describes the least squares approximations of the solutions of Maxwell PDEs via their Euler-Lagrange prolongations. We analyze the problem of best approximation point, i.e., point which achieves the minimum distance between a fixed point and a closed convex set in a Hilbert space, in the context of Maxwell theory. Section 1 studies the least squares Lagrangian determined by Maxwell PDEs and the Euclidean metric. Section 2 analizes least squares Lagrangian determined by Maxwell and dielectric relaxation PDEs and the Euclidean metric. Section 3 studies the Lorentzian squares Lagrangian.

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1 Least squares Lagrangian determined by Maxwell PDEs and the Euclidean metric

Ingredients (see [4]): E = electric field strength, B =magnetic field strength, D = electric displacement field, H = magnetic displacement field, j = density of electric current, ρ = electric charge density.

In the context of vectorial internal variables and relaxation phenomena, the Maxwell PDEs are

$$\mathrm{rot}\, H - \frac{1}{c}\frac{\partial D}{\partial t} = \frac{1}{c} j,\ \mathrm{div}\, D = \rho$$

$$\mathrm{rot}\, E - \frac{1}{c}\frac{\partial B}{\partial t} = 0,\ \mathrm{div}\, B = 0$$

(8 first order PDEs with 12 unknowns = components of the vector fields H, B, E, D).

The least squares Lagrangian determined by the Maxwell PDEs and the Euclidean metric is (first order Lagrangian, Udrişte-Maxwell Lagrangian, see [5]-[16])

$$2L = ||\mathrm{rot}\, H - \frac{1}{c}\frac{\partial D}{\partial t} - \frac{1}{c} j||^2 + ||\mathrm{rot}\, E - \frac{1}{c}\frac{\partial B}{\partial t}||^2 + (\mathrm{div}\, D - \rho)^2 + (\mathrm{div}\, B)^2.$$

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Using the coordinates, we can rewrite

$$2L = (H^3_y - H^2_z - \frac{1}{c}D^1_t - \frac{1}{c}j^1)^2 + (H^1_z - H^3_x - \frac{1}{c}D^2_t - \frac{1}{c}j^2)^2 + (H^2_x - H^1_y - \frac{1}{c}D^3_t - \frac{1}{c}j^3)^2$$

$$+(E^3_y - E^2_z + \frac{1}{c}B^1_t)^2 + (E^1_z - E^3_x + \frac{1}{c}B^2_t)^2 + (E^2_x - E^1_y + \frac{1}{c}B^3_t)^2$$

$$+(D^1_x + D^2_y + D^3_z - \rho)^2 + (B^1_x + B^2_y + B^3_z)^2.$$

This Lagrangian has the following partial derivatives

$$\frac{\partial L}{\partial H^1} = 0, \frac{\partial L}{\partial H^1_x} = 0, \frac{\partial L}{\partial H^1_t} = 0$$

$$\frac{\partial L}{\partial H^1_y} = -(H^2_x - H^1_y - \frac{1}{c}D^3_t - \frac{1}{c}j^3), \frac{\partial L}{\partial H^1_z} = H^1_z - H^3_x - \frac{1}{c}D^2_t - \frac{1}{c}j^2;$$

$$\frac{\partial L}{\partial B^1} = 0, \frac{\partial L}{\partial B^1_y} = 0, \frac{\partial L}{\partial B^1_z} = 0$$

$$\frac{\partial L}{\partial B^1_x} = B^1_x + B^2_y + B^3_z, \frac{\partial L}{\partial B^1_t} = \frac{1}{c}(E^3_y - E^2_z + \frac{1}{c}B^1_t);$$

$$\frac{\partial L}{\partial E^1} = 0, \frac{\partial L}{\partial L^1_x} = 0, \frac{\partial L}{\partial E^1_t} = 0$$

$$\frac{\partial L}{\partial E^1_y} = -(E^2_x - E^1_y + \frac{1}{c}B^3_t), \frac{\partial L}{\partial E^1_z} = E^1_z - E^3_x + \frac{1}{c}B^2_t;$$

$$\frac{\partial L}{\partial D^1} = 0, \frac{\partial L}{\partial D^1_y} = 0, \frac{\partial L}{\partial D^1_z} = 0$$

$$\frac{\partial L}{\partial D^1_x} = D^1_x + D^2_y + D^3_z - \rho, \frac{\partial L}{\partial D^1_t} = -\frac{1}{c}(H^3_y - H^2_z - \frac{1}{c}D^1_t - \frac{1}{c}j^1),$$

and other relations obtained by cyclic permutations of $\{1, 2, 3\}$. It follows the Euler-Lagrange PDEs

Theorem *The Euler-Lagrange prolongations of Maxwell PDEs are*

$$\Delta H - grad(div\, H) + \frac{1}{c}\frac{\partial}{\partial t} rot\, D + \frac{1}{c} rot\, j = 0$$

$$grad(div\, B) - \frac{1}{c}\frac{\partial}{\partial t} rot\, E + \frac{1}{c^2}B_{tt} = 0$$

$$\Delta E - grad(div\, E) + \frac{1}{c}\frac{\partial}{\partial t} rot\, B = 0$$

$$grad(div\, B\ \text{-}\ \rho) + \frac{1}{c}\frac{\partial}{\partial t} rot\, H - \frac{1}{c^2}D_{tt} - \frac{1}{c^2}j_t = 0.$$

2 Least squares Lagrangian determined by Maxwell and dielectric relaxation PDEs and the Euclidean metric

In the case that electric currents and electric charges are neglected, the Maxwell PDEs reduce to

$$\mathrm{rot}\, H - \frac{1}{c}\frac{\partial D}{\partial t} = 0,\ \mathrm{div}\, D = 0$$

$$\mathrm{rot}\, E - \frac{1}{c}\frac{\partial B}{\partial t} = 0,\ \mathrm{div}\, B = 0.$$

In case we neglect the magnetic relaxation phenomena, we add $B = \mu H$. Also, the PDE for dielectric relaxation ([3, ?, ?]) is

$$\chi^{(0)}_{(ED)}E + \chi^{(1)}_{(ED)}\frac{\partial E}{\partial t} + \chi^{(2)}\frac{\partial^2 E}{\partial t^2} = \chi^{(0)}_{(DE)}D + \chi^{(1)}_{(DE)}\frac{\partial D}{\partial t} + \chi^{(2)}\frac{\partial^2 D}{\partial t^2}.$$

It follows the least squares Lagrangian (second order Lagrangian, Udrişte-Maxwell Lagrangian)

$$2L = ||\mathrm{rot}\, H - \frac{1}{c}\frac{\partial D}{\partial t}||^2 + ||\mathrm{rot}\, E - \frac{1}{c}\frac{\partial B}{\partial t}||^2$$

$$+(\mathrm{div}\, D)^2 + (\mathrm{div}\, B)^2 + (\mathrm{div}\, H)^2 + ||B - \mu H||^2$$

$$+||\chi^{(0)}_{(ED)}E + \chi^{(1)}_{(ED)}\frac{\partial E}{\partial t} + \chi^{(2)}\frac{\partial^2 E}{\partial t^2} - \chi^{(0)}_{(DE)}D - \chi^{(1)}_{(DE)}\frac{\partial D}{\partial t} - \chi^{(2)}\frac{\partial^2 D}{\partial t^2}||^2.$$

Using the coordinates, we can rewrite

$$2L = (H^3_y - H^2_z - \frac{1}{c}D^1_t)^2 + (H^1_z - H^3_x - \frac{1}{c}D^2_t)^2 + (H^2_x - H^1_y - \frac{1}{c}D^3_t)^2$$

$$+(E^3_y - E^2_z + \frac{1}{c}B^1_t)^2 + (E^1_z - E^3_x + \frac{1}{c}B^2_t)^2 + (E^2_x - E^1_y + \frac{1}{c}B^3_t)^2$$

$$+(D^1_x + D^2_y + D^3_z)^2 + (B^1_x + B^2_y + B^3_z)^2 + (H^1_x + H^2_y + H^3_z)^2$$

$$+(B^1 - \mu H^1)^2 + (B^2 - \mu H^2)^2 + (B^3 - \mu H^3)^2$$

$$+\left(\chi^{(0)}_{(ED)}E^1 + \chi^{(1)}_{(ED)}\frac{\partial E^1}{\partial t} + \chi^{(2)}\frac{\partial^2 E^1}{\partial t^2} - \chi^{(0)}_{(DE)}D^1 - \chi^{(1)}_{(DE)}\frac{\partial D^1}{\partial t} - \chi^{(2)}\frac{\partial^2 D^1}{\partial t^2}\right)^2$$

$$+\left(\chi^{(0)}_{(ED)}E^2 + \chi^{(1)}_{(ED)}\frac{\partial E^2}{\partial t} + \chi^{(2)}\frac{\partial^2 E^2}{\partial t^2} - \chi^{(0)}_{(DE)}D^2 - \chi^{(1)}_{(DE)}\frac{\partial D^2}{\partial t} - \chi^{(2)}\frac{\partial^2 D^2}{\partial t^2}\right)^2$$

$$+\left(\chi^{(0)}_{(ED)}E^3 + \chi^{(1)}_{(ED)}\frac{\partial E^3}{\partial t} + \chi^{(2)}\frac{\partial^2 E^3}{\partial t^2} - \chi^{(0)}_{(DE)}D^3 - \chi^{(1)}_{(DE)}\frac{\partial D^3}{\partial t} - \chi^{(2)}\frac{\partial^2 D^3}{\partial t^2}\right)^2.$$

The Euler-Lagrange PDEs (Euler-Lagrange prolongations of Maxwell and dielectric relaxation PDEs) are of the form

$$\frac{\partial L}{\partial p^i} - D_\alpha \frac{\partial L}{\partial p^i_\alpha} + D^2_{\alpha\beta}\frac{\partial L}{\partial p^i_{\alpha\beta}} = 0,$$

where $p^i, i = 1, ..., 12$, are the components of the point (H, E, D, B). Firstly,

$$\frac{\partial L}{\partial H^1} = -\mu(B^1 - \mu H^1), \ \frac{\partial L}{\partial H^1_x} = H^1_x + H^2_y + H^3_z, \frac{\partial L}{\partial H^1_t} = 0$$

$$\frac{\partial L}{\partial H^1_y} = -(H^2_x - H^1_y - \frac{1}{c}D^3_t), \ \frac{\partial L}{\partial H^1_z} = H^1_z - H^3_x - \frac{1}{c}D^2_t$$

and consequently the first Euler-lagrange PDE is

$$\Delta H + \frac{1}{c}\frac{\partial}{\partial t}\mathrm{rot}\, D = -\mu(B - \mu H).$$

Secondly,

$$\frac{\partial L}{\partial E^1} = \chi^{(0)}_{(ED)}\left(\chi^{(0)}_{(ED)}E^1 + \chi^{(1)}_{(ED)}\frac{\partial E^1}{\partial t} + \chi^{(2)}\frac{\partial^2 E^1}{\partial t^2} - \chi^{(0)}_{(DE)}D^1 - \chi^{(1)}_{(DE)}\frac{\partial D^1}{\partial t} - \chi^{(2)}\frac{\partial^2 D^1}{\partial t^2}\right)$$

$$\frac{\partial L}{\partial E^1_x} = 0, \frac{\partial L}{\partial E^1_y} = -(E^2_x - E^1_y + \frac{1}{c}B^3_t), \frac{\partial L}{\partial E^1_z} = E^1_z - E^3_x - \frac{1}{c}B^2_t$$

$$\frac{\partial L}{\partial E^1_t} = \chi^{(1)}_{(ED)}\left(\chi^{(0)}_{(ED)}E^1 + \chi^{(1)}_{(ED)}\frac{\partial E^1}{\partial t} + \chi^{(2)}\frac{\partial^2 E^1}{\partial t^2} - \chi^{(0)}_{(DE)}D^1 - \chi^{(1)}_{(DE)}\frac{\partial D^1}{\partial t} - \chi^{(2)}\frac{\partial^2 D^1}{\partial t^2}\right)$$

$$\frac{\partial L}{\partial E^1_{tt}} = \chi^{(2)}\left(\chi^{(0)}_{(ED)}E^1 + \chi^{(1)}_{(ED)}\frac{\partial E^1}{\partial t} + \chi^{(2)}\frac{\partial^2 E^1}{\partial t^2} - \chi^{(0)}_{(DE)}D^1 - \chi^{(1)}_{(DE)}\frac{\partial D^1}{\partial t} - \chi^{(2)}\frac{\partial^2 D^1}{\partial t^2}\right)$$

and consequently the second Euler-Lagrange PDE is

$$\mathrm{grad}(\mathrm{div}\, E) - \Delta E + \frac{1}{c}\frac{\partial}{\partial t}\mathrm{rot}\, B$$
$$+\chi^{(1)}_{(ED)}\left(\chi^{(0)}_{(ED)}E + \chi^{(1)}_{(ED)}E_t + \chi^{(2)}E_{t^2} - \chi^{(0)}_{(DE)}D - \chi^{(1)}_{(DE)}D_t - \chi^{(2)}D_{t^2}\right)$$
$$-\chi^{(1)}_{(ED)}\left(\chi^{(0)}_{(ED)}E_t + \chi^{(1)}_{(ED)}E_{t^2} + \chi^{(2)}E_{t^3} - \chi^{(0)}_{(DE)}D_t - \chi^{(1)}_{(DE)}D_{t^2} - \chi^{(2)}D_{t^3}\right)$$
$$+\chi^{(2)}\left(\chi^{(0)}_{(ED)}E_{t^2} + \chi^{(1)}_{(ED)}E_{t^3} + \chi^{(2)}E_{t^4} - \chi^{(0)}_{(DE)}D_{t^2} - \chi^{(1)}_{(DE)}D_{t^3} - \chi^{(2)}D_{t^4}\right) = 0.$$

Similarly, we can write the others equations.

Theorem *In case of dielectric relaxation, the Euler-Lagrange prolongations of Maxwell PDEs are*

$$\Delta H + \frac{1}{c}\frac{\partial}{\partial t}\mathrm{rot}\, D = -\mu(B - \mu H), \qquad \mathrm{grad}(\mathrm{div}\, E) - \Delta E + \frac{1}{c}\frac{\partial}{\partial t}\mathrm{rot}\, B$$
$$+\chi^{(1)}_{(ED)}\left(\chi^{(0)}_{(ED)}E + \chi^{(1)}_{(ED)}E_t + \chi^{(2)}E_{t^2} - \chi^{(0)}_{(DE)}D - \chi^{(1)}_{(DE)}D_t - \chi^{(2)}D_{t^2}\right)$$
$$-\chi^{(1)}_{(ED)}\left(\chi^{(0)}_{(ED)}E_t + \chi^{(1)}_{(ED)}E_{t^2} + \chi^{(2)}E_{t^3} - \chi^{(0)}_{(DE)}D_t - \chi^{(1)}_{(DE)}D_{t^2} - \chi^{(2)}D_{t^3}\right)$$
$$+\chi^{(2)}\left(\chi^{(0)}_{(ED)}E_{t^2} + \chi^{(1)}_{(ED)}E_{t^3} + \chi^{(2)}E_{t^4} - \chi^{(0)}_{(DE)}D_{t^2} - \chi^{(1)}_{(DE)}D_{t^3} - \chi^{(2)}D_{t^4}\right) = 0, \ \ etc.$$

3 Lagrangian determined by Maxwell PDEs and the Lorentzian metric

Let us extend the previous theory to an invariant one with respect the Lorentz transformations. For that we use the covariant formulation of Maxwell PDEs. Ingredients:

- the coordinates $x^\alpha \in \{ct, x, y, z\}$ and the differential operators

$$\left(\frac{\partial}{\partial x^\alpha}\right) = \left(\frac{\partial}{\partial (ct)}, \nabla\right);$$

- the Lorentzian metric

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta, (g_{\alpha\beta}) = \mathrm{diag}\,(-1, 1, 1, 1);$$

- the four-current $(j^\alpha) = (c\rho, \vec{J})$, where ρ is the charge density and $\vec{J}$ is the current density;

- the covariant components of the electromagnetic tensor (a rank-2 covariant antisymmetric tensor combining the electric and magnetic fields)

$$(F_{\alpha\beta}) = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix};$$

- the covariant components of the electromagnetic tensor (the result of raising its indices)

$$(F^{\mu\nu}) = \left(g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}\right) = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}.$$

Properties:

1) $F_{\alpha\beta} = -F_{\beta\alpha}$ (antisymmetry),
2) six independent components,
3) $\det(F_{\alpha\beta}) = \frac{1}{c^2}(\vec{B}, \vec{E})^2$,
4) the Lorentz invariant $F_{\alpha\beta}F^{\alpha\beta} = 2(\vec{B}^2 - c^{-2}\vec{E}^2)$,
5) the pseudoscalar invariant $\epsilon_{\alpha\beta\gamma\delta} F^{\alpha\beta} F^{\gamma\delta} = \frac{4}{c}(\vec{B}, \vec{E})$, where $\epsilon_{\alpha\beta\gamma\delta}$ is the completely antisymmetric unit pseudotensor of the fourth rank or the Levi-Civita symbol, $\epsilon_{0123} = 1$.

In the Lorentzian context, the Maxwell PDEs are written in terms of four-vectors and tensors in the manifestly covariant form,

$$\frac{4\pi}{c} j^\beta = \frac{\partial F^{\alpha\beta}}{\partial x^\alpha}, 0 = \frac{\partial F_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial F_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial F_{\gamma\alpha}}{\partial x^\beta}.$$

These PDEs and the Lorentzian metric determine the Lorentzian squares Lagrangian (Udrişte-Lorentz-Maxwell Lagrangian)

$$L = \frac{1}{2} g_{\beta\delta} \left(\frac{\partial F^{\alpha\beta}}{\partial x^\alpha} - \frac{4\pi}{c} j^\beta\right) \left(\frac{\partial F^{\gamma\delta}}{\partial x^\gamma} - \frac{4\pi}{c} j^\delta\right)$$

$$+\frac{1}{2}g^{\alpha\lambda}g^{\beta\mu}g^{\gamma\nu}\left(\frac{\partial F_{\alpha\beta}}{\partial x^{\gamma}}+\frac{\partial F_{\beta\gamma}}{\partial x^{\alpha}}+\frac{\partial F_{\gamma\alpha}}{\partial x^{\beta}}\right)\left(\frac{\partial F_{\lambda\mu}}{\partial x^{\nu}}+\frac{\partial F_{\mu\nu}}{\partial x^{\lambda}}+\frac{\partial F_{\nu\lambda}}{\partial x^{\mu}}\right)$$

or

$$L=\frac{1}{2}g_{\lambda\mu}\left(g^{\alpha\gamma}g^{\beta\lambda}\frac{\partial F_{\alpha\beta}}{\partial x^{\gamma}}-\frac{4\pi}{c}j^{\lambda}\right)\left(g^{\nu\tau}g^{\sigma\mu}\frac{\partial F_{\tau\sigma}}{\partial x^{\nu}}-\frac{4\pi}{c}j^{\mu}\right)$$
$$+\frac{1}{2}g^{\alpha\lambda}g^{\beta\mu}g^{\gamma\nu}\left(\frac{\partial F_{\alpha\beta}}{\partial x^{\gamma}}+\frac{\partial F_{\beta\gamma}}{\partial x^{\alpha}}+\frac{\partial F_{\gamma\alpha}}{\partial x^{\beta}}\right)\left(\frac{\partial F_{\lambda\mu}}{\partial x^{\nu}}+\frac{\partial F_{\mu\nu}}{\partial x^{\lambda}}+\frac{\partial F_{\nu\lambda}}{\partial x^{\mu}}\right),$$

which is invariant with respect the Lorentz transformations. The Euler-Lagrange PDEs (Euler-Lagrange prolongations of Maxwell PDEs) are of the form

$$D_{\xi}\frac{\partial L}{\partial\dfrac{\partial F_{\epsilon\tau}}{\partial x^{\xi}}}=0,\ \xi=1,2,3,4,\ \ \epsilon\tau\in\{12,13,14,23,24,34\},$$

where

$$\frac{\partial L}{\partial\dfrac{\partial F_{\epsilon\tau}}{\partial x^{\xi}}}=g_{\lambda\mu}g^{\alpha\gamma}g^{\beta\lambda}\delta_{\alpha\epsilon}\delta_{\beta\tau}\delta^{\xi}_{\gamma}\left(g^{\nu\eta}g^{\sigma\mu}\frac{\partial F_{\eta\sigma}}{\partial x^{\nu}}-\frac{4\pi}{c}j^{\mu}\right)$$
$$+g^{\alpha\lambda}g^{\beta\mu}g^{\gamma\nu}\left(\delta_{\alpha\epsilon}\delta_{\beta\tau}\delta^{\xi}_{\gamma}+\delta_{\beta\epsilon}\delta_{\gamma\tau}\delta^{\xi}_{\alpha}+\delta_{\gamma\epsilon}\delta_{\alpha\tau}\delta^{\xi}_{\beta}\right)\left(\frac{\partial F_{\lambda\mu}}{\partial x^{\nu}}+\frac{\partial F_{\mu\nu}}{\partial x^{\lambda}}+\frac{\partial F_{\nu\lambda}}{\partial x^{\mu}}\right).$$

Finally, we find

Theorem. *In case of Lorentzian squares Lagrangian, the Euler-Lagrange prolongations of Maxwell PDEs are the second order PDEs*

$$\delta^{\gamma}_{\epsilon}\delta_{\mu\tau}D_{\gamma}\left(g^{\nu\eta}g^{\sigma\mu}\frac{\partial F_{\eta\sigma}}{\partial x^{\nu}}-\frac{4\pi}{c}j^{\mu}\right)$$
$$+g^{\alpha\lambda}g^{\beta\mu}g^{\gamma\nu}\left(\delta_{\alpha\epsilon}\delta_{\beta\tau}D_{\gamma}+\delta_{\beta\epsilon}\delta_{\gamma\tau}D_{\alpha}+\delta_{\gamma\epsilon}\delta_{\alpha\tau}D_{\beta}\right)\left(\frac{\partial F_{\lambda\mu}}{\partial x^{\nu}}+\frac{\partial F_{\mu\nu}}{\partial x^{\lambda}}+\frac{\partial F_{\nu\lambda}}{\partial x^{\mu}}\right)=0.$$

Remark. The previous theory can be connected to multitime maximum principle, using the ideas in [13]-[16].

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