A new approach to electromagnetism in anisotropic spaces

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Dedicated to the 70-th anniversary of Professor Constantin Udriste

Abstract. Anisotropy of a space naturally leads to direction dependent electromagnetic tensors and electromagnetic potentials. Starting from this idea and using variational approaches and exterior derivative formalism, we extend some of the classical equations of electromagnetism to anisotropic (Finslerian) spaces. The results differ from the ones obtained by means of the known approach in [5], [7].

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1 Introduction

In anisotropic spaces, where the fundamental metric tensor depends on the directional variables, the electromagnetic-type tensor F and accordingly, the electromagnetic potential A, may also depend on these.

Starting from this idea, we propose a generalization of the electromagnetic tensor, of the notion of current and of the corresponding Maxwell equations - based on variational methods and exterior derivative formalism. We chose those anisotropic spaces which provide the simplest equations, namely, pseudo-Finslerian ones. A similar approach for a particular class of pseudo-Finsler spaces was already considered by the authors in [2]. New perspectives of solving problems in modern astrophysics by means of Finsler-Lagrange geometry have been pointed out by the second author in [9].

When dealing with the equations of electromagnetism, one can either: 1) consider as a fundamental object the electromagnetic tensor F satisfying the homogeneous Maxwell equations (written in a condensed manner as $dF = 0$) and deduce by Poincaré lemma the existence of a potential 1-form A such that $F = dA$, or, conversely: 2) consider the potential 1-form A as a fundamental object and define the electromagnetic tensor as its exterior derivative, thus getting the homogeneous Maxwell equations as identities. It appeared as more convenient to use for the beginning the second approach, and then point out (Theorem 5.2) that using the first one, we are led to similar results.

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The theory we are going to develop stems from considering a *direction dependent potential* 4-covector field $A = A(x, y)$, (where, $x = (x^i)$ are the space-time coordinates and $y = (y^i)$, the directional ones) as arising from a Lagrangian. Namely, it appears as reasonable to consider the following Lagrangian $\mathcal{L}$, providing the Lorentz force in pseudo-Finsler spaces:

$$\mathcal{L} = \frac{1}{2} g_{ij}(x, y) y^i y^j + \frac{q}{c} L_1(x, y), \; y^i = \dot{x}^i,$$

where L_1 is a 1-homogeneous function in y, $(L_1(x, \lambda y) = \lambda L_1(x, y), \lambda \in \mathbb{R})$ and g is the Finslerian metric tensor. Then, the Liouville (canonical) 1-form $\theta = \frac{\partial \mathcal{L}}{\partial y^i} dx^i$ and the Poincaré 2-form $\omega = d\theta$ (where d denotes exterior derivative) attached to $\mathcal{L}$ carry information on both the metric of the space and on electromagnetic properties. The potential 1-form A can be defined (similarly to the idea in [7]) as $A = \theta - y_i dx^i$, which is,

$$A = A_i(x, y) dx^i, \quad A_i = \frac{\partial L_1}{\partial y^i}.$$

and the electromagnetic tensor, as $F = \omega - g_{ij} \delta y^j \wedge dx^i$, which is nothing but the exterior derivative of A :

$$F = dA = \frac{1}{2}(A_{j|i} - A_{i|j}) dx^i \wedge dx^j - \frac{\partial A_i}{\partial y^a} dx^i \wedge \delta y^a,$$

(where bars denote Chern type covariant derivatives) and has hh- and hv- components. The equations of motion of charged particles are then elegantly expressed in terms of the electromagnetic tensor, (4.2).

Maxwell equations in pseudo-Finslerian spaces are obtained as

$$dF = 0 \tag{1.1}$$

$$d * F = \frac{\beta}{4\alpha}(*\mathcal{J}). \tag{1.2}$$

where $*$ denotes Hodge star operator, the *current* $\mathcal{J}$ is a vector field on TM, and α, β are constants.

From a physical point of view, we notice the appearance of an additional term (reminding inertial forces) in the expression of Lorentz force (4.2), as well as the appearance of a correction to the usual expression of currents.

The generalized current $\mathcal{J} = J^i \delta_i + \tilde{J}^a \dot{\partial}_a$ obeys the continuity equation $div \mathcal{J} = 0$. The horizontal component J^i is equal to the regular current plus a correction due to anisotropy, while the vertical one $\tilde{J}^a$ plays the role of compensating quantity which ensures the satisfying of the continuity equation.

2 Pseudo-Finsler spaces

Let M be a 4-dimensional differentiable manifold of class $\mathcal{C}^\infty$, thought of as spacetime manifold, (TM, π, M) its tangent bundle and $(x^i, y^i)_{i=\overline{1,4}}$ the coordinates in a local chart on TM. By "smooth" we shall always mean $\mathcal{C}^\infty$-differentiable. Also, we denote partial derivation with respect to x^i by $_{,i}$ and partial derivation with respect to y^i, by a dot: $_{\cdot i}$.

A *pseudo-Finslerian function* on M, is a function $\mathcal{F} : TM \to \mathbb{R}$ with the properties, [10]:

1. $\mathcal{F} = \mathcal{F}(x, y)$ is smooth for $y \neq 0$;
2. $\mathcal{F}$ is positively homogeneous of degree 1, i.e., $\mathcal{F}(x, \lambda y) = \lambda \mathcal{F}(x, y)$ for all $\lambda > 0$;
3. The *pseudo-Finslerian metric tensor*:

(2.1) $$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 \mathcal{F}^2}{\partial y^i \partial y^j}$$

is nondegenerate: $\det(g_{ij}(x, y)) \neq 0,\ \forall x \in M,\ y \in T_x M \backslash \{0\}$.

Particularly, we shall consider that the metric has signature $(+, -, -, -)$.

The equations of geodesics $s \mapsto (x^i(s))$ of a pseudo-Finsler space $(M, \mathcal{F})$ are

$$\frac{dy^i}{ds} + 2G^i(x, y) = 0, \quad y^i = \dot{x}^i.$$

These equations give rise to the *Cartan nonlinear connection* on TM, [5], [1], of local coefficients $N^i{}_j = \dfrac{\partial G^i}{\partial y^j}$. Let

$$\delta_i = \frac{\partial}{\partial x^i} - N^a{}_i \frac{\partial}{\partial y^a}, \quad \dot{\partial}_a = \frac{\partial}{\partial y^a}$$

be the adapted basis corresponding to the Cartan nonlinear connection and $(dx^i,\ \delta y^a = dy^a + N^a{}_i dx^i)$, its dual basis. We will also denote by semicolons adapted derivatives:

$$f_{;i} = \delta_i f, \quad \forall f \in \mathcal{F}(TM).$$

Any vector field V on TM can be written as $V = V^i \delta_i + \tilde{V}^a \dot{\partial}_a$; the component $hV = V^i \delta_i$ is a vector field, called the *horizontal* component of V, while $vV = \tilde{V}^a \dot{\partial}_a$ is its *vertical* component. Similarly, a 1-form ω on TM can be decomposed as $\omega = \omega_i dx^i + \tilde{\omega}_a \delta y^a$, with $h\omega = \omega_i dx^i$ called the *horizontal* component, and $v\omega = \tilde{\omega}_a \delta y^a$ the *vertical* one, [5].

In terms of the Cartan nonlinear connection, the divergence of a vector field $V = V^i \delta_i + \tilde{V}^a \dot{\partial}_a \in \mathcal{X}(TM)$, is obtained, [13], (from $d(*V^\flat) = divV\sqrt{G}dx^1 \wedge ... \wedge \delta y^4$) as

$$divV = \frac{1}{\sqrt{G}} \delta_i (\tilde{V}^i \sqrt{G}) - N^a{}_{i \cdot a} V^i + \frac{1}{\sqrt{G}} (\tilde{V}^a \sqrt{G})_{\cdot a}.$$

where $G = \det(G_{\alpha\beta})$ is the determinant of the Sasaki lift of g_{ij} :

(2.2) $$G_{\alpha\beta}(x, y) = g_{ij}(x, y) dx^i \otimes dx^j + g_{ij}(x, y) \delta y^i \otimes \delta y^j.$$

Also, it is convenient to express the electromagnetic tensor in terms of the *Chern linear connection* $C\Gamma(N) = (L^i{}_{jk}, 0)$ of local coefficients:

$$L^i{}_{jk} = \frac{1}{2} g^{ih}(g_{hj;k} + g_{hk;j} - g_{jk;h}).$$

We denote by $_{|i}$ and $_{\cdot i}$ the corresponding covariant derivations

$$X^j{}_{|i} = \delta_i X^j + L^j{}_{ki} X^i, \quad X^j{}_{\cdot i} = \frac{\partial X^j}{\partial y^i},$$

(where X^j are local coordinates of a vector field X on TM).

The Chern connection is h-metrical: $g_{ij|k} = 0$. Also, there hold the equalities

$$y_{i|j} = 0. \tag{2.3}$$

The *vertical endomorphism* or *almost tangent structure* of TTM, [10], is the $\mathcal{F}(TM)$-linear function $\mathbf{J} : TTM \to TTM$, which acts on the elements of the adapted basis as

$$\mathbf{J}(\delta_i) = \dot{\partial}_i, \quad \mathbf{J}(\dot{\partial}_i) = 0,$$

where $\mathcal{F}(TM)$ denotes the set of smooth real valued functions defined on TM.

For a smooth function $f : TM \to \mathbb{R}$, the *vertical differential* $d_{\mathbf{J}} f$, [10], is defined by $d_{\mathbf{J}} f = df \circ \mathbf{J}$; in local writing,

$$d_{\mathbf{J}} f = \frac{\partial f}{\partial y^j} dx^j.$$

Whenever convenient or necessary to make a clear distinction, we shall denote by $i, j, k, ...$ indices corresponding to horizontal geometrical objects, and by $a, b, c, ...$ indices corresponding to vertical ones.

3 Direction dependent electromagnetic potential. Electromagnetic tensor

In anisotropic spaces and particularly, in pseudo-Finsler spaces, the components of an electromagnetic-type tensor F_{ij}, $F^i{}_j$, F^{ij} and accordingly, of the electromagnetic potential 1-form A basically depend on the directional variables y^i, $i = 1, ..., 4$. In order to make sure of this, let us notice the following simple example. In isotropic (pseudo-Riemannian) spaces with vanishing Ricci tensor, under some simplifying assumptions, the components of the free electromagnetic potential 4-vector $A^i = A^i(x)$ obey Maxwell- de Rham equations, [3]:

$$g^{ij}(x) \nabla_i \nabla_j (A^k) = 0,$$

where $\nabla_k = \nabla_{\frac{\partial}{\partial x^k}}$ denotes covariant derivative with respect to Levi-Civita connection.

When passing to *anisotropic* spaces with metric $g_{ij} = g_{ij}(x, y)$, the solution of such an equation would generally depend on the directional variables y^i (not to mention that the equation itself could become more complicated). So, it is meaningful to consider that *the potential 4-vector (and, accordingly, the corresponding 1-form A) also depends on the directional variables* $y = (y^i)$.

In order to obtain the expression for Lorentz force in pseudo-Finsler spaces, let us consider a Lagrangian of the form

$$\mathcal{L} = \frac{1}{2} g_{ij}(x, y) y^i y^j + \frac{q}{c} L_1, \tag{3.1}$$

where $L_1 = L_1(x, y)$ is a scalar function which is 1-homogeneous in the directional variables (from a physical point of view, this comes to the fact that we will allow the electromagnetic potential A to depend on the directional variable y, but not on the magnitude of y).

Let

$$\theta = d_{\mathbf{J}} \mathcal{L},$$

be the *Liouville (canonical) 1-form* attached to $\mathcal{L}$. In local coordinates,

$$\theta = \frac{\partial \mathcal{L}}{\partial y^i} dx^i = (y_i + \frac{q}{c} \frac{\partial L_1}{\partial y^i}) dx^i.$$

Definition 3.1. We call *potential 1-form* A, the 1-form given by

$$\frac{q}{c} A = \theta - y_i dx^i.$$

In local writing, we have

$$A = A_i(x, y) dx^i, \qquad A_i(x, y) = \frac{\partial L_1}{\partial y^i}. \tag{3.2}$$

By the 1-homogeneity of L_1, there holds $L_1 = A_j(x, y) y^j$, hence the Lorentz force Lagrangian in the pseudo-Finsler space $(M, \mathcal{F})$ is

$$\mathcal{L}(x, y) = \frac{1}{2} g_{ij}(x, y) y^i y^j + \frac{q}{c} A_i(x, y) y^i. \tag{3.3}$$

The quantities $A_j = A_j(x, y)$, thus, become the components of a direction dependent electromagnetic potential. They have the property

$$A_{i \cdot k} y^k = 0; \;\; A_{i \cdot k} y^i = 0. \tag{3.4}$$

Remark 3.2. In *isotropic* spaces, there exists only one potential 4-vector providing a given $L_1 = A_i(x) y^i$ (which is $A_i = \frac{\partial L_1}{\partial y^i}$). In anisotropic spaces, there exist infinitely many covector fields $A_i = A_i(x, y)$ with $A_i y^i = L_1$ for a fixed L_1. Among them, (3.2) is the one which gives the simplest equations of motion.

Taking (2.3) into account, the exterior derivative of the 1-form θ yields the following *"gravito-electromagnetic" 2-form*:

$$\omega = d\theta = \frac{1}{2} (A_{j|i} - A_{i|j}) dx^i \wedge dx^j - (g_{ia} + A_{i \cdot a}) dx^i \wedge \delta y^a, \tag{3.5}$$

which contains information both on the metric structure of the space and on the electromagnetic field.

Particular cases.

1. If $A_i = A_i(x)$ is *isotropic,* then $\tilde{F}_{ia} = 0$ and the 2-form ω is simply

$$\omega = \frac{1}{2}(A_{j|i} - A_{i|j})dx^i \wedge dx^j - g_{ia}dx^i \wedge \delta y^a,$$

which is similar to the expression in [7].

2. If $A_i = 0$ (*no electromagnetic potential*), then θ is the Hilbert 1-form of the (pseudo)-Finsler space,

$$\theta = y_i dx^i$$

and ω, the fundamental 2-form of $(M, \mathcal{F})$, [10]: $\omega = g_{ia}\delta y^a \wedge dx^i$.

Definition 3.3. We call *electromagnetic tensor* in the Finslerian space, $(M, \mathcal{F})$, the following 2-form on TM :

$$F = \omega + g_{ia}dx^i \wedge \delta y^a,$$

The above definition is equivalent to

$$F = dA. \tag{3.6}$$

In local coordinates, we have

$$F := \frac{1}{2}F_{ij}dx^i \wedge dx^j + \tilde{F}_{ia}dx^i \wedge \delta y^a, \tag{3.7}$$

where

$$F_{ij} = A_{j|i} - A_{i|j}, \quad \tilde{F}_{ia} = -A_{i\cdot a}, \ \tilde{F}_{ai} = A_{i\cdot a}. \tag{3.8}$$

In relation (3.8) we denoted indices corresponding to vertical geometric objects by different letters $a, b, c...$, in order to point out the antisymmetry of F.

The new component, $\tilde{F}_{ia}$, will play an important role in the equations of motion of charged particles (see Section 4).

Remark. The electromagnetic tensor F remains invariant under transformations

$$A(x, y) \ \mapsto A(x, y) + d\lambda(x),$$

where $\lambda : M \to \mathbb{R}$ is a scalar function, since $d(A + d\lambda) = dA + d(d\lambda) = dA$.

Particular case. If $A = A(x)$ does not depend on the directional variables, we get $\tilde{F}_{ia} = 0$ and

$$F = \frac{1}{2}(A_{j|i} - A_{i|j})dx^i \wedge dx^j,$$

which is similar to the expression in [5], [7].

4 Lorentz force

The equations of motion of a charged particle in an electromagnetic field can be obtained by the variational procedure applied to the Lagrangian (3.3). The corresponding Euler-Lagrange equations $\frac{\partial \mathcal{L}}{\partial x^i} - \frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial y^i}) = 0$ lead to

$$(4.1) \qquad g_{ij}\left(\frac{dy^j}{dt} + 2G^j\right) + \frac{q}{c}\left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}\right) y^j + \frac{q}{c} A_{i\cdot j}\frac{dy^j}{dt} = 0, \quad y^i = \dot{x}^i.$$

Writing the second term above, in terms of covariant derivatives and taking into account (3.8) and the equality $\frac{\delta y^i}{dt} = \frac{dy^i}{dt} + N^i{}_j y^j = \frac{dy^i}{dt} + 2G^i$, we get

Proposition 4.1. (Lorentz force law): *The extremal curves* $t \mapsto (x^i(t)) : [0,1] \to \mathbb{R}^4$ *of the Lagrangian (3.3) are given by*

$$(4.2) \qquad \frac{\delta y^i}{dt} = \frac{q}{c} F^i{}_j y^j + \frac{q}{c}\tilde{F}^i{}_a \frac{\delta y^a}{dt},$$

An elegant equivalent writing of the above can be obtained in terms of the gravito-electromagnetic 2-form ω in (3.5). With this aim, let us write

$$\omega = \frac{1}{2}\omega_{ij} dx^i \wedge dx^j + \tilde{\omega}_{ia} dx^i \wedge \delta y^a,$$

where $\omega_{ij} = \frac{q}{c} F_{ij}$, $\quad \tilde{\omega}_{ia} = \left(\frac{q}{c}\tilde{F}_{ia} - g_{ia}\right)$. We are led to

Proposition 4.2. *The equations of motion of a charged particle in electromagnetic field in pseudo-Finslerian spaces are*

$$(4.3) \qquad \omega_{ij}(x,y)\frac{dx^j}{dt} + \tilde{\omega}_{ia}(x,y)\frac{\delta y^a}{dt} = 0, \quad y = \dot{x}.$$

Remark 4.3. 1. In the case of an anisotropic potential A, there appears an additional term

$$(4.4) \qquad \tilde{F}^i(x,y) := \tilde{F}^i{}_a \frac{\delta y^a}{dt}$$

in the equations of motion, in comparison to the isotropic case.

2. Both the "traditional" Lorentz force (given by $F^i = F^i{}_h y^h$) and the correction $\tilde{F}$ are orthogonal to the velocity 4-vector $y = \dot{x}$:

$$(4.5) \qquad g_{ij} F^i y^j = 0, g_{ij}\tilde{F}^i y^j = 0.$$

3. The above defined F_{kh}, $\tilde{F}_{ka}, F^\iota, \tilde{F}^i$ are components of *distinguished tensor fields*, [5].

Physical interpretation. The expression in the right hand side of (4.2) presents the Lorentz force in pseudo-Finslerian spaces. We see that its first term which is common with the isotropic case is proportional to velocity, while the second term is proportional to acceleration which brings to mind the idea of an "inertial force" in accelerated reference frames.

5 Homogeneous Maxwell equations

Taking into account that $F = dA$, we immediately get the identity $dF = d(dA) = 0$. In other words:

Proposition 5.1. *There holds the generalized homogeneous Maxwell equation:*

$$dF = 0, \tag{5.1}$$

where F is the electromagnetic tensor (3.7), (3.8), and d denotes exterior derivative.

In local coordinates, the homogeneous Maxwell equation is read as:

$$\begin{aligned} &F_{ij|k} + F_{ki|j} + F_{jk|i} = -\sum_{(i,j,k)} R^b{}_{jk}\tilde{F}_{ib}; \\ &\tilde{F}_{aj|k} + \tilde{F}_{ka|j} + F_{jk\cdot a} = 0 \\ &\tilde{F}_{ka\cdot b} + \tilde{F}_{bk\cdot a} = 0, \end{aligned}$$

where $R^b{}_{jk} = N^b{}_{j;k} - N^b{}_{k;j}$. The first set in the above is the analogue (in adapted coordinates) of the regular homogeneous (sourceless) Maxwell equations.

Taking into account the 1-homogeneity of L_1 and the fact that $A_i = \dfrac{\partial L_1}{\partial y^i}$, there also hold the relations

$$\tilde{F}_{ia}y^i = 0, \ \tilde{F}_{ia}y^a = 0. \tag{5.2}$$

Conversely, on a topologically "nice enough" domain, we have

Theorem 5.2. *If on a contractible subset of $T\mathbb{R}^4$ it is given a 2-form*

$$F := \frac{1}{2}F_{ij}dx^i \wedge dx^j + \tilde{F}_{ia}dx^i \wedge \delta y^a,$$

satisfying $dF = 0$, then there exists a horizontal *1-form $A = A_i(x,y)dx^i$, such that $F = dA$. Moreover, if $\tilde{F}_{ia}y^i = 0$ and $\tilde{F}_{ia}y^a = 0$, then $A_i = \dfrac{\partial L_1}{\partial y^i}$ for some 1-homogeneous in y scalar function $L_1(x,y)$.*

Proof. By Poincaré lemma, we deduce that there exists a 1-form

$$\bar{A} = \phi_i(x,y)dx^i + \psi_a(x,y)\delta y^a$$

such that $F = d\bar{A}$. By computing $d\bar{A}$ and equating its components with those of F, we get

$$F_{ij} = \phi_{j|i} - \phi_{i|j} - R^a{}_{ij}\psi_a; \quad \tilde{F}_{ia} = -\phi_{i\cdot a} - \psi_{a\cdot i}, \quad 0 = \psi_{a\cdot b} - \psi_{b\cdot a}.$$

From this relation, we get that there exists a scalar function $\psi = \psi(x,y)$ such that $\psi_a = \psi_{\cdot a}$, $a = \overline{1,4}$. Then, by direct computation, it follows that, if we build the following *horizontal* 1-form:

$$A := A_i dx^i, \quad A_i := \phi_i + \delta_i\psi,$$

then our 2-form F is none but $F = dA$. Further, from $\tilde{F}_{ia}y^i = 0$, we get $A_{i\cdot a}y^i = 0$, which is, $(A_i y^i)_{\cdot a} = A_{\cdot a}$. By setting $L_1 = A_i y^i$ and re-denoting indices, we have $A_i = \dfrac{\partial L_1}{\partial y^i}$. 1-homogeneity of L_1 now follows from $\tilde{F}_{ia}y^a = 0$, q.e.d.

6 Currents in Finslerian spaces

In the classical Riemannian case, *the inhomogeneous Maxwell equation* $d(*F) = 4\pi * J$ can be obtained by means of the variational principle applied to $\int(\alpha F_{ij}F^{ij} - \beta J^k A_k)\sqrt{-g}d\Omega$, [8], where J denotes the current 4-vector, α and β are constants, $g = \det(g_{ij})$ and $d\Omega = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$, and the integral is taken on some domain in M.

In our case, the quantities F_{ij}, F^{ij}, A_k depend on y, hence it is natural to look for a generalization of the above Lagrangian to TM. Also, it is reasonable to think of the current as of a vector field on TM :

$$\mathcal{J} = J^i(x,y)\delta_i + \tilde{J}^a(x,y)\dot{\partial}_a. \tag{6.1}$$

The meaning of the quantities $\tilde{J}^a$ will reveal itself later. Let us consider $A_i = A_i(x,y)$ and the following integral of action on some domain in TM :

$$I = \int(\alpha(F_{ij}F^{ij} + 2\tilde{F}_{ia}\tilde{F}^{ia} - \beta J^k A_k)\sqrt{G}d\Omega, \tag{6.2}$$

where $d\Omega = dx^1 \wedge ...dx^4 \wedge \delta y^1 \wedge ..\delta y^4$, $G = \det(G_{\alpha\beta})$, and $G_{\alpha\beta}$ denotes the Sasaki lift of g.

In order to make physical sense for the above, we need to adjust measurement units so as to have $[F_{ij}] = [\tilde{F}_{ia}]$. Hence, let $u^a = \frac{1}{H}y^a$, be the fibre coordinates on TM, where, H is a constant (ex: $[H] = \frac{1}{\sec}$) meant to have the same measurement units for x^i and u^a : $[x^i] = [u^a]$, (consequently, also $[F_{ij}(x,u)] = [\tilde{F}_{ia}(x,u)]$). Also, let $\dot{\partial}_a = \frac{\partial}{\partial u^a}$. The integral (6.2) only involves the horizontal part $h\mathcal{J} = J^i(x,u)\delta_i$ of the extended current $\mathcal{J}$, let us denote it by $J = J^i(x,u)\delta_i$. By varying in (6.2) the potentials A_k and considering variations δA_i which vanish on the boundary of the integration domain, we get

Proposition 6.1. *There hold the generalized inhomogeneous Maxwell equations:*

$$\frac{1}{\sqrt{G}}\{(F^{ij}\sqrt{G})_{;j} - F^{ij}N^a_{j\cdot a}\sqrt{G}\} + \frac{1}{\sqrt{G}}(\tilde{F}^{ia}\sqrt{G})_{\cdot a} = \frac{\beta}{4\alpha}J^i, \tag{6.3}$$

Particular case. If the space is pseudo-Riemannian, then $A_i = A_i(x)$ and equations (6.3) reduce to the usual ones, written in terms of the Levi-Civita connection as:

$$\nabla_j F^{ij} = \frac{\beta}{4\alpha}J^i.$$

Conclusion. In comparison to the case of isotropic spaces, there appears the new term $(\tilde{F}^{ia}\sqrt{G})_{\cdot a}/\sqrt{G}$ in the expression of the current. This means that in an anisotropic space the measured fields would correspond to an effective current consisting of two terms: one is the current provided by the experimental environment, the other is the current corresponding to the anisotropy of space. The presence of the current ζ^i in experimental situations could be noticed if $\left|(\tilde{F}^{ia}\sqrt{G})_{\cdot a}\right| \approx \left|(F^{ij}\sqrt{G})_{;j}\right|$. Particularly, if the space is isotropic, then $A_i = A_i(x)$, and $\zeta^k = 0$.

Relation (6.3) above does not involve the vertical components $\tilde{J}^a$ of the current. Still, a formal approach using exterior derivative would emphasize them, and they appear as necessary as "compensating" quantities in order to obtain the continuity equation.

If we formally generalize the inhomogeneous Maxwell equation as

$$d(*F) = \frac{\beta}{4\alpha} * \mathcal{J}, \tag{6.4}$$

where $*$ denotes the Hodge star operator on the manifold TM, then we obtain by direct computation

$$\begin{aligned} \frac{1}{\sqrt{G}}\{(F^{ij}\sqrt{G})_{;j} + (\tilde{F}^{ia}\sqrt{G})_{\cdot a}\} - F^{ij}N^a_{j\cdot a} &= \frac{\beta}{4\alpha}J^i \\ \frac{1}{\sqrt{G}}(\tilde{F}^{ai}\sqrt{G})_{;i} &= \frac{\beta}{4\alpha}\tilde{J}^a, \end{aligned}$$

where $\mathcal{J} = J^i\delta_i + \tilde{J}^a\dot{\partial}_a$ is as in (6.1). The first set of equations is nothing but (6.3) obtained by means of variational methods, while the second one is new. We notice the appearance of the vertical components $\tilde{J}^a := \frac{4\alpha}{\beta}\frac{1}{\sqrt{G}}(\tilde{F}^{ai}\sqrt{G})_{;i}$.

With the above expression of $\mathcal{J}$, there holds the generalized continuity equation: $d(*\mathcal{J}) = \frac{4\alpha}{\beta}d(d(*F)) = 0$, which is, $div\mathcal{J} = 0$.

Comparison to existent results.

A previous approach for the equations of electromagnetism in anisotropic spaces was made by R. Miron and collaborators, [5], [7], [6], where the definition of the electromagnetic tensor is given by means of deflection tensors of metrical linear connections. There, it is proposed an *electromagnetic tensor* with *hh*- and *vv*- components defined by means of covariant derivatives attached to a certain (metrical) linear connection. In the respective works, only position dependent potentials $A(x)$ are considered.

Here, we propose an alternative definition of the electromagnetic tensor (3.7),(3.8) (with horizontal *hh*- and mixed *hv*- components, arising from exterior differentiation and variational calculus), based on the idea that in anisotropic spaces, the electromagnetic potential A is generally direction dependent, which corresponds to the physically testable situation. Maxwell equations are obtained here as solutions of a variational problem and in terms of exterior derivatives and they differ from the ones obtained in [5], [7]. The new components $\tilde{F}_{ia}$ are tightly related to Lorentz force. Also, newly appearing currents have a precise role in making the continuity equation fulfilled.

In terms of deflection tensors [5], if we consider the Lorentz nonlinear connection, [4], [5], of coefficients $\bar{N}^i{}_j = N^i{}_j - \frac{q}{c}F^i{}_j$ and the linear connection given by $\bar{D}\Gamma(N) = (\bar{L}^i{}_{jk}, \bar{C}^i{}_{jk} = -g^{il}A_{l\cdot jk})$, where $\bar{L}^i{}_{jk} = \frac{1}{2}g^{ih}(g_{hj;k} + g_{hk;j} - g_{jk;h})$, then, F can be described by

$$F_{ij} = \frac{1}{2}(y_{j||i} - y_{i||j}), \quad \tilde{F}_{ia} = g_{ia} - y_i||_a$$

(where $_{||i}$, $||_a$ denote the associated covariant derivations).

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