About the solution of a battery mathematical model

A. Aron, G. Gîrban, C. Pop

Abstract. In this paper we analyze the properties of a differential system derived from the battery electrochemical model. We study the stability and prime integrals of the controlled system denoted by separator equations from electrochemical model of Lithium-ion batteries.

M.S.C. 2010: 53D17, 37J25.
Key words: Hamilton-Poisson realization; Kahan integrator; battery model; energy monitoring.

1 Introduction

One of the common problems in electronic mobile devices is the accuracy of available energy estimation. Even if the battery modeling is considered a mature research area, to implement a run-time monitoring system for battery state-of-charge implies application specific constraints which can not be satisfied by existing models without a significant impact on their accuracy.

In case of mobile phones, an inaccurate battery state-of-charge determination may lead to user discomfort but the precision of available energy estimation can be improved through a combination of battery voltage measurement in open circuit and, while there is load, by means of coulomb counting as it is proposed in [5]. The cost of integrating an electronic device for measuring the battery voltage is not feasible for the smart sensors that are part of wireless sensor networks (WSN) as we can have hundreds or thousands of nodes in these networks. Because these sensors are usually considered as being disposable, the energy available in their batteries will determine the time while the wireless sensor network is operational and the limitation of the energy consumption is critical for prolonging the network life. In this context, some strategies used to reduce de energy consumption are based on state-of-charge monitoring, and hence the need for a battery model that implies a small computational effort and a limited energy consumption due to memory access and execution time.

Difficulties in obtaining a battery model that satisfies the constraints just mentioned, arises from nonlinear battery behavior illustrated through rate capacity and relaxation effects, capacity fading, self discharge and temperature influence. Therefore

---

BSG Proceedings 18. The Int. Conf. of Diff. Geom. and Dynamical Systems (DGDS-2010), October 8-11, 2010, Bucharest Romania, pp. 1-10.

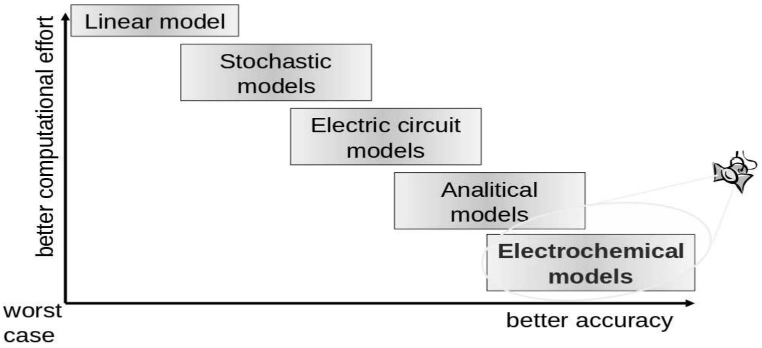

Figure 1: Accuracy and computational effort of battery models.

we have two extrema, the widely used linear battery model and the electrochemical models, as well as some types of battery models based on a trade off between accuracy and required computational effort (summarized in Figure 1). The linear battery model is assuming that amount of available energy is a sum of harvested energy and the negative amount of consumed energy, limited to positive values with the battery nominal charge as an upper limit. Due to its small computational effort, this model is preferred in application where the accuracy of available charge estimation is not critical, as it will not take into account the real battery behavior.

Opposed to linear models we have the electrochemical battery models that are based on chemical reduction-oxidation reactions at the battery physical components (the two electrodes and the separator). These electrochemical models are the most accurate but are based on systems of differential equations with tens of parameters and are not proper for real-time computations even if some efforts have been performed to reduce the computational effort as in [8].

The electric circuit models are used in circuit design as they describe the battery behavior based on an equivalent electric schema while for software implementation some models based on stochastic simulations are proposed due to the fact that they are usually faster than electric circuit models but are less accurate. Such a stochastic model taking into account the recovery and rate capacity effects is proposed in [4].

There is another class of analytical battery models, derived from electrochemical reactions and nearly as accurate as the electrochemical models but not so computational intensive. The model proposed by D. Rakhmatov and S. Vrudhula in [7] is taking into account the ions diffusion in the electrolyte while in [3] a Kinetic Battery Model is presented.

To obtain a model of batteries behavior that can be used for energy management inside the nodes of wireless sensor networks, we are working on a novel approach by taking the differential equations of an electrochemical model (the ones for Lithium-ion batteries used as reference in [8]) and study the dynamics in order to calibrate some look-up tables that will be based on a Hamilton-Poisson realization.

Through this paper, we present some partial results of our work that consist in a study of the separator equations from the electrochemical model related to Li-ion batteries. In the next section the separator equations are reduced to a three dimensional differential system. Further on, in section 3, we perform the numerical integration of the analyzed system through Runge-Kuta and a defined Kahan integrators, in order to compare them with the equilibrium states and the system prime integrals obtained in section 4. Finally, the Hamilton-Poisson realization is provided for two cases.

2 The mathematical model of a battery separator

Starting from the mathematical model of the Li-ion battery, that is taken as reference in [8], we focus on the separator equations and use the following notations $x(\xi) = c(\xi)$, $y(\xi) = c'(\xi)$ and $z(\xi) = \Phi_2(\xi)$, where $\xi = ux + t$, $u=$ ct. Through substitution of the constants, we obtain the system of differential equations below:

$$\begin{cases} x'(t) = y(t), \\ y'(t) = by(t), \\ z'(t) = e\dfrac{y(t)}{x(t)} - g, \end{cases} \tag{2.1}$$

where $b, d, e, g \in \mathbb{R}$.

By adding a control $u(t) = ax(t) + cz(t) + dg$ to the system (2.1) one gets:

$$\begin{cases} x'(t) = y(t), \\ y'(t) = ax(t) + by(t) + cz(t) + dg, \\ z'(t) = e\dfrac{y(t)}{x(t)} - g, \end{cases} \tag{2.2}$$

$a, b, c, d, e, g \in \mathbb{R}$, which is the system that will be further analyzed through this paper.

3 Numerical integration

In this section, we build the Kahan integrator for the system (2.2) and compare it with related Runge-Kutta integrator (4 steps). The Kahan integrator has an advantage as it can be much easier implemented. Let us now point out some properties of the Kahan integrator related with this dynamical system.

The Kahan integrator of the system (2.2) is given by:

$$\begin{cases} x^{(n+1)} - x^{(n)} = \dfrac{h}{2}((y^{(n)} + y^{(n+1)})), \\[2ex] y^{(n+1)} - y^{(n)} = \dfrac{h}{2}(a(x^{(n)} + x^{(n+1)}) + b(y^{(n)} + y^{(n+1)}) \\[2ex] \qquad\qquad +c(z^{(n)} + z^{(n+1)}) + 2dg), \\[2ex] z^{(n+1)} - z^{(n)} = \dfrac{h}{2}\left(e\left(\dfrac{y^{(n)}}{x^{(n+1)}} + \dfrac{y^{(n+1)}}{x^{(n)}}\right) - 2g\right). \end{cases} \tag{3.1}$$

In the next section we study the invariance of the Kahan integrator (3.1) related to the prime integrals of the system (2.2).

4 The equilibrium states of the system (2.2)

In this section we analyze the stability properties of the equilibrium states related to the system (2.2). Then one get the following characterization for the equilibrium states:

Theorem 4.1. *For the system* (2.2) *we have the following cases for equilibrium states:*

i) If $g = 0$*, and* $c \neq 0$*, then* (2.2) *has the equilibrium* $e_1 = \left(M, 0, -\frac{aM}{c}\right)$. *In this case, if* $Re\left(-\frac{-bM-\sqrt{M(4ce+4aM+b^2M)}}{2M}\right) > 0$ *or* $Re\left(-\frac{-bM+\sqrt{M(4ce+4aM+b^2M)}}{2M}\right) > 0$*, then the equilibrium* e_1 *is instable.*[1]

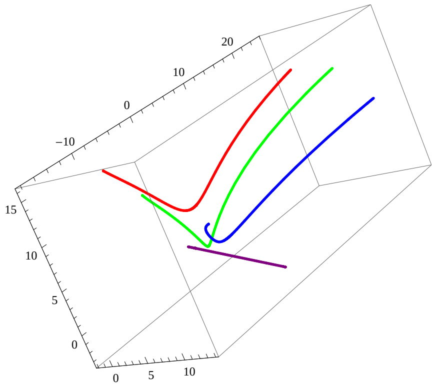

Figure 2: Equilibrium line and different trajectories of the RK integrator

ii) If $g = 0$*,* $a = 0$*,* $c = 0$*, then the system* (2.2) *has the equilibrium* $e_2 = (M, 0, N)$. *In this case, if* $b > 0$*, then the equilibrium* e_2 *is instable.*

Proof. The conclusion directly follows as consequence of Lyapunov theorem ([2]). □

5 The Poisson geometry associated to the Li-ion separator equations

Proposition 5.1. *The following smooth real functions* H *are two degree polynomial constants of the motion defined by the system* (2.2)*:*

(i) If $a = c = 0$*, then the system becomes:*

$$\begin{cases} x'(t) = y(t), \\ y'(t) = by(t) + dg, \\ z'(t) = e\dfrac{y(t)}{x(t)} - g \end{cases} \tag{5.1}$$

---

[1] In Figure 2, the equilibrium line is represented together with the Runge-Kutta integrator for different trajectories of the system (2.2). In this context, the Kahan's integrator fails.

and $H(x,y,z) = deLn(x) + bx - y - dz$, $b, d, e \in \mathbb{R}$, *where we denote by* Ln *the natural logarithm.*

If $g = 0$, *this corresponds to case (ii) from the Theorem 4.1 and we have the equilibrium plane along with the integral surface* $H(x,y,z) = ct$.

If $g \neq 0$, *then the plane of equilibria is missing.*

(ii) *If* $a = e = 0$, *then the system becomes:*

$$\left\{\begin{array}{l} x'(t) = y(t), \\ y'(t) = by(t) + cz(t) + dg, \\ z'(t) = -g \end{array}\right. \tag{5.2}$$

and $H(x,y,z) = -2bgx + cz^2 + 2g(y + dz)$, $b, d, g \in \mathbb{R}$.
In this case, from Theorem 4.1, we have a line of equilibria.

(iii) *If* $c = 0$, $a = 2b^2$ *the system becomes:*

$$\left\{\begin{array}{l} x'(t) = y(t), \\ y'(t) = 2b^2x(t) + by(t) + dg, \\ z'(t) = e\dfrac{y(t)}{x(t)} - g \end{array}\right. \tag{5.3}$$

and $H(x,y,z) = 6d^2g^2x + 2b(-2bx + y)^2(bx + y) + 3dg(4b^2x^2 - y^2)$, $b, d, g \in \mathbb{R}$. *As a consequence of the Theorem 4.1, the system* (2.2) *does not admit equilibrium points in this case.*

(iv) *If* $e = 0$, $a = 2b^2$ *the system becomes:*

$$\left\{\begin{array}{l} x'(t) = y(t), \\ y'(t) = 2b^2x(t) + by(t) + cz(t) + dg, \\ z'(t) = -g \end{array}\right. \tag{5.4}$$

and

$$\begin{aligned} H(x,y,z) = \; & 16b^7x^3 + 4b^4(3cgx^2 + y^3) + 12b^5x(2dgx - y^2 + 2cxz) \\ & + 3b^2cg(4dgx - 3y^2 + 4cxz) + 3c^2g(cz^2 + 2g(y + dz)) \\ & + 2bc(3d^2g^2z + 3dg(gy + cz^2) + cz(3gy + cz^2)) \\ & + 6b^3(2d^2g^2x - dg(y^2 - 4cxz) + c(2gxy - y^2z + 2cxz^2)), \end{aligned}$$

with b, c, d, g *real. If, in addition,* $g = 0$, *then with the Theorem 4.1 (i) we have a line of equilibria represented in Figure 3 along with the integral surface* $H(x,y,z) = ct$ *and Runge-Kutta integrator.*

If $g \neq 0$, *then the line of equilibria is missing.*

Proof. It is easy to see that $dH = 0$ for each case mentioned above. □

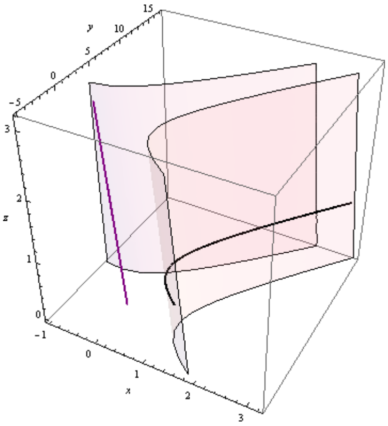

Figure 3: The line of equilibria (gray), the integral surface $H(x, y, z) = ct$ and the RK integrator (black line) for the system 5.4, when $g = 0$.

A long but straightforward computation enhanced by MATHEMATICA leads us to:

Remark 5.1. The Kahan integrator (3.1) has the following properties:

It preseserves the constant of motion $H(x, y, z) = de\mathrm{Ln}(x) + bx - y - dz$, $b, d, e \in \mathbb{R}$ of the system (5.1) if and only if $de = 0$.

(ii) It preseserves the constant of motion $H(x, y, z) = -2bgx + cz^2 + 2g(y + dz)$, $b, d, g \in \mathbb{R}$ of the system (5.2) if and only if $bg = 0$.

(iii) It does not preseserve the constant of motion $H(x, y, z) = 6d^2g^2x + 2b(-2bx + y)^2(bx + y) + 3dg(4b^2x^2 - y^2)$, $b, d, g \in \mathbb{R}$ of the system (5.3).

(iv) It preseserves the constant of motion of the system (5.4)

$$\begin{aligned} H(x,y,z) = \quad & 16b^7x^3 + 4b^4(3cgx^2 + y^3) + 12b^5x(2dgx - y^2 + 2cxz) \\ & +3b^2cg(4dgx - 3y^2 + 4cxz) + 3c^2g(cz^2 + 2g(y + dz)) \\ & +2bc(3d^2g^2z + 3dg(gy + cz^2) + cz(3gy + cz^2)) \\ & +6b^3(2d^2g^2x - dg(y^2 - 4cxz) + c(2gxy - y^2z + 2cxz^2)), \end{aligned}$$

with $b, c, d, g \in \mathbb{R}$, if and only if $b = 0$.

Proof. Indeed, it is easy to see that the factorizations of $H_{n+1} - H_n$ contain the factor de in the case (i), bg in the case (ii) respectively b in the case (iv), and then our result follows immediately. □

The goal of this section is to find a Hamilton-Poisson structure for system (2.2). For this, let us consider the skew-symmetric matrix given by:

$$\Pi := \begin{pmatrix} 0 & p_1(x,y,z) & p_2(x,y,z) \\ -p_1(x,y,z) & 0 & p_3(x,y,z) \\ -p_2(x,y,z) & -p_3(x,y,z) & 0 \end{pmatrix}.$$

We have to find the real smooth functions $p_1, p_2, p_3 : \mathbb{R}^3 \to \mathbb{R}$ such that:

$$\left(\begin{array}{c} \dot{x} \\ \dot{y} \\ \dot{z} \end{array}\right) = \Pi \cdot \nabla H \tag{5.5}$$

and

$$-p_3\left(\frac{\partial p_2}{\partial z} + \frac{\partial p_1}{\partial y}\right) + p_2\left(\frac{\partial p_3}{\partial z} - \frac{\partial p_1}{\partial x}\right) + p_1\left(\frac{\partial p_3}{\partial y} + \frac{\partial p_2}{\partial x}\right) = 0, \tag{5.6}$$

that is the Jacobi identity should be satisfied.

Let us consider first the system (5.1) from the case (i) of the Proposition (5.1). The condition (5.6) lead us to the following differential equation:

$$-bxp_2 + (-gx + ey)\frac{\partial p_2}{\partial z} + x\left(g + (dg + by)\frac{\partial p_2}{\partial y} + y\frac{\partial p_2}{\partial x}\right) = 0. \tag{5.7}$$

Then, (5.7) and (5.5) have the following solutions:

$$\begin{aligned} p_1(x,y,z) &= -\frac{dg + by}{b}\left(1 + bd\varphi[u,v]\right), \\ p_2(x,y,z) &= \frac{g}{b} + (dg + by)\varphi[u,v], \\ p_3(x,y,z) &= \frac{dg + by}{bx}\left(e + b(de + bx)\varphi[u,v]\right), \end{aligned}$$

with $\varphi \in C^1(\mathbb{R}^2)$ is an arbitrary real function and

$$\begin{aligned} u &= -\frac{dg + b^2x + dg\mathrm{Ln}\left[(dg + by)\mathrm{Exp}(-1 - \frac{by}{dg})\right]}{b^2} \\ v &= z - e\mathrm{Ln}(x) + \frac{g\mathrm{Ln}\left[\mathrm{ProductLog}\left[-\frac{(dg+by)\mathrm{Exp}\left(-1-\frac{by}{dg}\right)}{dg}\right]\right]}{b}. \end{aligned} \tag{5.8}$$

where ProductLog is the product logarithm (Lambert W function). As a consequence, we obtain the following result:

Proposition 5.2. *The system* (5.1) *has the Hamilton-Poisson realization:*

$$(\mathbb{R}^3, \Pi := \left(\Pi^{ij}\right), H),$$

where

$$\Pi = \left(\begin{array}{ccc} 0 & -\frac{dg+by}{b}\left(1 + bd\varphi\right) & \frac{g}{b} + (dg + by)\varphi \\ -\frac{dg+by}{b}\left(1 + bd\varphi\right) & 0 & \frac{dg+by}{bx}\left(e + b(de + bx)\varphi\right) \\ -\frac{g}{b} + (dg + by)\varphi & -\frac{dg+by}{bx}\left(e + b(de + bx)\varphi\right) & 0 \end{array}\right),$$

$\varphi \in C^1(\mathbb{R}^2)$, $\varphi = \varphi\left(u, v\right)$, *where u and v are given in* (5.8), *is an arbitrary real function, and*

$$H(x,y,z) = de\mathrm{Ln}(x) + bx - y - dz,\ \ b, d, e \in \mathbb{R}.$$

Let us consider the system (5.2) from the second case (ii) of the Proposition (5.1). The condition (5.6) lead us to the following differential equation:

$$1+2bp_2+2g\frac{\partial p_2}{\partial z}-2dg\frac{\partial p_2}{\partial y}-2by\frac{\partial p_2}{\partial y}-2cz\frac{\partial p_2}{\partial y}-2y\frac{\partial p_2}{\partial x}=0. \tag{5.9}$$

Then, (5.9) and (5.5) has the following solutions:

$$p_1(x,y,z)=-\text{ProductLog}\left(\frac{(-cg+bdg+b^2y+bcz)\text{Exp}\left(\frac{-cg+bdg+b^2y+bcz}{cg}\right)}{cg}\right.$$

$$\left.\varphi\left[\frac{(-cg+bbdg+b^2y+bcz)\text{Exp}\left(\frac{bz}{g}\right)}{b^2}\right]\frac{dg+cz}{g}+\frac{dg+by+cz}{2bg},\right.$$

$$p_2(x,y,z)=\text{ProductLog}\left(\frac{(-cg+bdg+b^2y+bcz)\text{Exp}\left(\frac{-cg+bdg+b^2y+bcz}{cg}\right)}{cg}\right.$$

$$\left.\varphi\left[\frac{(-cg+bbdg+b^2y+bcz)\text{Exp}\left(\frac{bz}{g}\right)}{b^2}\right]+\frac{1}{2b},\right.$$

$$p_3(x,y,z)=b\text{ProductLog}\left(\frac{(-cg+bdg+b^2y+bcz)\text{Exp}\left(\frac{-cg+bdg+b^2y+bcz}{cg}\right)}{cg}\right.$$

$$\left.\varphi\left[\frac{(-cg+bbdg+b^2y+bcz)\text{Exp}\left(\frac{bz}{g}\right)}{b^2}\right],\right.$$

where $\varphi\in C^1(\mathbb{R})$ is an arbitrary real function. As a consequence, we obtain

Proposition 5.3. *The system* (5.2) *has the Hamilton-Poisson realization:*

$$(\mathbb{R}^3,\Pi:=\left[\Pi^{ij}\right],H),$$

where

$$\Pi=\begin{pmatrix} 0 & \frac{dg+by+cz}{2bg}-\frac{dg+cz}{g}P & \frac{1}{2b}+P \\ -\frac{dg+by+cz}{2bg}+\frac{dg+cz}{g}P & 0 & bP \\ -\frac{1}{2b}-P & -bP & 0 \end{pmatrix},$$

$$P=\text{ProductLog}\left(\frac{(-cg+bdg+b^2y+bcz)\text{Exp}\left(\frac{-cg+bdg+b^2y+bcz}{cg}\right)}{cg}\right)\cdot$$

$\varphi\left[\frac{(-cg+bbdg+b^2y+bcz)\text{Exp}\left(\frac{bz}{g}\right)}{b^2}\right]$, $\varphi\in C^1(\mathbb{R})$ *, is an arbitrary real function,*

and $H(x,y,z)=-2bgx+cz^2+2g(y+dz),\ b,d,g\in\mathbb{R}.$

6 Results and further work

The aim of this paper is to study the dynamics of the Li-ion battery separator equations. It outlines the difference between trajectories given by the system (2.2) integration using Runge-Kutta inregrator and the obtained Kahan integrator (3.1) for the analyzed system as well as the cases in which it has equilibrium states (Theorem 4.1). Four cases in which the system concedes prime integrals are figured out in this paper (Proposition 5.1). Finally, the Hamilton-Poisson realization was found for two cases but it is still an open problem to find the Casimir for each of them. Further work consists in discussig the results according to the values of the control considered, finding a Hamilton-Poisson realization for the two electrodes [6] eventually using a control [9] or associate the fractional equations [1] and implementation of the state-of-charge monitoring system suitable for integration on WSN nodes.

Acknowledgements. The first and third authors' work was supported by the project "Development and support of multidisciplinary postdoctoral programmes in major technical areas of national strategy of Research - Development - Innovation" 4D-POSTDOC, contract no. POSDRU/89/1.5/S/52603, project co-funded by the European Social Fund through Sectoral Operational Programme Human Resources Development 2007-2013. The second author's work was partially supported by the strategic grant POSDRU/88/1.5/S/50783, Project ID50783 (2009), co-financed by the European Social Fund "Investing in People, within the Sectoral Operational Programme Human Resources Development 2007 - 2013".

References

[1] M. Ivan, Gh. Ivan, D Opriş, *The Maxwell-Bloch equations on fractional Leibniz algebroids*, Balkan J. Geom. Appl. 13, 2(2008) 50-58.

[2] A.M. Lyapunov, *The general problem of the stability of motion [1893]*, Taylor and Francis, 1992.

[3] J. Manwell, J. McGowan, *Lead acid battery storage model for hybrid energy systems.* Solar Energy, 50 (1993), 399-405.

[4] D, Panigrahi, S. Dey, R. Rao, K. Lahiri, C. Chiasserini, A.Raghunathan, *Battery life estimation of mobile embedded systems*, The 14th Int. Conf. on VLSI Design (VLSID-01) 2001, 57.

[5] V. Pop, H.J. Bergveld, D. Danilov, P.P.L. Regtien, P.H.L. Notten, *Battery Management Systems. Accurate State-of-Charge Indication for Battery-Powered Applications*, Philips Research Book Series 9, Springer-Verlag, 2008.

[6] L. Popescu, *A note on Poisson-Lie algebroids (I)*, Balkan J. Geom. Appl. 14, 2 (2009), 79-89.

[7] D. Rakhmatov, S. Vrudhula, *An analytical high-level battery model for use in energy management of portable electronic systems*, Proc. of Int. Conf. on Computer Aided Design (IC-CAD01),(2001), 488-493.

[8] V.R. Subramanian, V. Boovaragavan, V. Ramadesigan, M. Arabandi, *Mathematical model reformulation for lithium-ion battery simulations: galvanostatic boundary conditions*, Jour. Electrochem. Soc. 156 (4), (2009), 260-271.

[9] C. Udriste, *Equivalence of multitime optimal control problems*, Balkan J. Geom. Appl. 15, 1(2010) 155-162.

Authors' addresses:

A. Aron, C. Pop
Department of Mathematics,"Politehnica" University of Timisoara,
Pta. Victoriei 2, Timisoara, 300006, Romania.
E-mail: anania.girban@gmail.com , cariesanu@yahoo.com

G. Gîrban
Faculty of Automation and Computers, "Politehnica" University of Timisoara,
Bvd. Vasile Parvan, Nr. 2, Timisoara, 300223, Romania.
E-mail: girban@cs.upt.ro