Subharmonic morphisms

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Abstract. In this paper we will introduce the notions of subharmonic morphisms and subharmonic maps on Riemannian manifolds in a natural way, based on notions of harmonic morphisms and harmonic maps. Our original results include comparisons between definitions and properties of harmonic maps, harmonic morphisms, subharmonic maps, subharmonic morphisms and semi-conformal maps.

Section 1 recalls the definition of harmonic morphisms. Section 2 introduces the tension field and studies the harmonic maps. Section 3 gives properties of subharmonic maps. Section 4 studies similar properties for semi-conformal maps. Section 5 includes original results regarding the theory of subharmonic morphisms.

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1 Harmonic morphisms

There are many equivalent definitions of harmonic morphisms on Riemannian manifolds (see [1]-[23]).

Let (M, g) and (N, h) be two Riemannian manifolds.

Definition 1.1. (with subharmonic functions) A smooth differentiable map $\varphi : (M, g) \to (N, h)$ is called a *harmonic morphism* if it pulls back local subharmonic functions to local subharmonic functions.

More exactly, if $\varphi : M \to N$ is a smooth map between two Riemannian manifolds, φ is called *harmonic morphism*, if, for any subharmonic function $f : V \to \mathbb{R}$, where V is an open subset of N with $\varphi^{-1}(V) \neq \varnothing$, the composition $f \circ \varphi$ is a subharmonic function on $\varphi^{-1}(V)$, where

$$M \supset \varphi^{-1}(\mathrm{V}) \xrightarrow{\varphi} \mathrm{V}(\subset \mathrm{N}) \xrightarrow{\mathrm{f}} \mathbb{R},$$

or it *pulls back germs of subharmonic functions in germs of subharmonic functions.*

We shall prove the following

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Theorem 1.1. *Let $\varphi : (M, g) \to (N, h)$ be a smooth map. Then the following statements are equivalent: (i) the map φ is a harmonic morphism; (ii) φ pulls back local harmonic functions to local harmonic functions (or pulls back germs of harmonic functions to germs of harmonic functions); (iii) φ is a semi-conformal harmonic map; (iv) φ is a semi-conformal map whose local components in harmonic local coordinates on the second manifold are harmonic functions.*

Remark 1.1. The definition (iii) seems to be the most important for developing the theory of harmonic morphisms. It links the definition (ii) of Constantinescu and Cornea ([2], [3]) to that of Fuglede ([10]-[12]) and of Ishihara ([14]). The definition (iv) is based on the idea of local components in harmonic local coordinates.

2 Tension field and harmonic maps

We start with notations and some introductory notions. We use two smooth Riemannian manifolds (M, g) and (N, h) (connected, compact, orientable, without bord, if not explicitly consider otherwise). In local coordinates, we consider (U, ϕ) a local system of coordinates around the point $p \in U$, where U is an open set in the manifold M and $\phi : U \to \mathbb{R}^m$ is smooth, $\phi = (x^1, ..., x^m), x^i : U \to \mathbb{R}, i = 1, ..., m$. Let $\varphi : M \to N$ be a smooth map, between two smooth differentiable manifolds, and (V, ψ) a local coordinates system around the point $q = \varphi(p) \in \varphi(U), \varphi(U) \subseteq V$, where V is an open set in N and $\psi : V \to \mathbb{R}^n$ is smooth, $\psi = (y^1, ..., y^n), y^\alpha : V \to \mathbb{R}, \alpha = 1, .., n$. We define the local components φ^γ of φ, by

$$\varphi^\gamma \overset{def}{=} y^\gamma \circ \varphi|_U : U \to \mathbb{R}, (\varphi^1, ..., \varphi^n) = (\varphi^\gamma) = \psi \circ \varphi|_U : U \to \mathbb{R}^n, \gamma = 1, .., n.$$

The local representation of $d\varphi$ in the neighborhood of $p \in M$, is $\varphi_i^\alpha = \frac{\partial \varphi^\alpha}{\partial x^i}$, i=1,..,m, so that $d\varphi_p \left(\frac{\partial}{\partial x^i}\Big|_p \right) = \varphi_i^\alpha(p) \left.\frac{\partial}{\partial y^\alpha}\right|_{\phi(p)}$. Since $\varphi_i^\alpha = \frac{\partial \varphi^\alpha}{\partial x^i}$: U$\to \mathbb{R}$, we find d$\varphi = \frac{\partial\ (\mathrm{y}^\alpha \mathrm{o} \varphi)}{\partial x^i} \frac{\partial}{\partial y^\alpha}$ $= \varphi_i^\alpha \frac{\partial}{\partial y^\alpha}$. The matrix (φ_i^α) is the Jacobian of φ relative to the given coordinate systems.

For a smooth map $\varphi : (M, g) \to (N, g)$, between two Riemannian manifolds, the bundle $E = T^*M \otimes \varphi^{-1}TN \to M$ has a connection ${}^E\nabla$, induces by the Levi-Civita connection ∇^M on M and the pull-back connection ∇^φ of the bundle $\varphi^{-1}TN \to M$, in turn induced by the Levi-Civita connection on N.

Definition 2.1. (second fundamental form of a map) The covariant differential

$$\beta(\varphi) \overset{not}{=} {}^E\nabla(\mathrm{d}\varphi) \in C^\infty(\odot^2 T^*M \otimes \varphi^{-1}TN)$$

is called the *second fundamental form* of the application φ.

The second fundamental form of an application φ is a 2-covariant tensor field ([6]),

$$\beta : C^\infty(M, N) \to C^\infty(\odot^2 T^*M \otimes \varphi^{-1}TN) \subset C^\infty(T^*M \otimes T^*M \otimes \varphi^{-1}TN)$$

$$\beta(\varphi) : C^\infty(T^*M) \otimes C^\infty(T^*M) \to C^\infty(\varphi^{-1}TN).$$

If $X, Y \in C^\infty(TM)$, then

$$(2.1) \qquad {}^E\nabla \mathrm{d}\varphi(X, Y) = ({}^E\nabla X \mathrm{d}\varphi)Y = \nabla_X^\varphi(d\varphi(Y)) - \mathrm{d}\varphi(\nabla_X^M Y).$$

In local coordinates $(U,(x^1,..,x^m))$ at $p\in M$ and $(V,(y^1,\dots,y^n))$ at $q=\varphi(p)\in\varphi(U)\subseteq V$ on N, we have

$$({}^E\nabla d\varphi)_{ij} = {}^E\nabla d\varphi\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right) = \varphi^\gamma_{;ij}\frac{\partial}{\partial y^\gamma},$$

where the symbol ; means the partial covariant derivative of second order, given by

$$\varphi^\gamma_{;ij} = ({}^E\nabla(\mathrm{d}\phi))^\gamma_{\mathrm{ij}} = ({}^E\nabla\mathrm{d}\varphi^\gamma)_{ij} + {}^h\Gamma^\gamma_{\alpha\beta}\varphi^\alpha_i\varphi^\beta_j.$$

Consequently

$$({}^E\nabla\mathrm{d}\varphi)_{ij} = \sum_{\gamma=1}^n ({}^E\nabla(\mathrm{d}\varphi))^\gamma_{\mathrm{ij}} = \sum_{\gamma=1}^n\left(\frac{\partial^2\varphi^\gamma}{\partial x^i\partial x^j} - {}^g\Gamma^k_{ij}\frac{\partial\varphi^\gamma}{\partial x^k} + {}^h\Gamma^\gamma_{\alpha\beta}\frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial x^j}\right)\frac{\partial}{\partial y^\gamma}. \tag{2.2}$$

Here ${}^g\Gamma^k_{ij}$ are the Christoffel symbols of (M,g), respectively ${}^h\Gamma^\gamma_{\alpha\beta}$ of (N,h), relative to the chosen local coordinates. We observe that this formula show that ${}^E\nabla\mathrm{d}\varphi$ is symmetrical.

If we choose local normal coordinates, the Christoffel symbols ${}^g\Gamma^k_{ij}$ and ${}^h\Gamma^\gamma_{\alpha\beta}$ vanish at the centers of local coordinates and the last formula becomes

$$({}^E\nabla\mathrm{d}\varphi)_{ij} = \sum_{\gamma=1}^n ({}^E\nabla(\mathrm{d}\varphi))^\gamma_{\mathrm{ij}} = \sum_{\gamma=1}^n \frac{\partial^2\varphi^\gamma}{\partial x^i\partial x^j}\frac{\partial}{\partial y^\gamma}. \tag{2.3}$$

Let $\{e_i\}$be a local orthonormal frame for the tangent bundle TM of M.

Definition 2.2. (tension field) Given a smooth application $\varphi:(M,g)\to(N,h)$ between two smooth Riemannian manifolds, the trace, relative to the metric g, of the second fundamental form, $\mathrm{Trace}^E_g\nabla\mathrm{d}\varphi = \sum_{i=1}^m {}^E\nabla d\varphi(e_i,e_i) \overset{not}{=} \tau(\varphi)$ is called the *tension field* of φ.

Remark 2.1. For an 1-form θ, we define $\mathrm{div}\theta = g^{ij}\left(\frac{\partial\theta_i}{\partial x^j} - \Gamma^k_{ij}\theta_k\right)$. If we put $\theta = \mathrm{d}\varphi$, then $\tau(\varphi) = \mathrm{divd}\varphi = -\mathrm{d}^*\mathrm{d}\varphi \in C^\infty(\varphi^{-1}TN)$ is a section of the pull-back bundle. This section defines the *Euler-Lagrange operator* $\tau : C^\infty(M,N)\to C^\infty(\varphi^{-1}TN)$, which is an elliptic linear self-adjoint operator of second order.

From (2.1) we obtain the formula

$$\tau(\varphi) = \sum_{i=1}^m(\nabla^\varphi_{e_i}(\mathrm{d}\varphi(e_i) - \mathrm{d}\varphi(\nabla^M_{e_i}e_i)). \tag{2.4}$$

This formula simplifies at $x\in M$, if we choose an orthonormal frame $\{e_i\}$ on the bundle TM, with $\nabla^M_{e_i}e_i = 0$ at x. Such frame is achieved by parallel translation at

x of an orthonormal frame along geodesics starting from x and is called *normal or adapted frame.* ([1], p.71).

For a smooth differentiable function $f : M \to \mathbb{R}$ and an orthonormal frame $\{e_i\}$ on the bundle TM, we obtain

$$\tau(f) = \mathrm{Trace}^E \nabla \mathrm{d}f = \sum_{i=1}^{m} {}^E\nabla \mathrm{d}f(e_i, e_i) = \sum_{i=1}^{m} [\nabla^f_{e_i}(\mathrm{d}f(e_i) - \mathrm{d}f(\nabla^M_{e_i} e_i)] =$$

$$= \sum_{i=1}^{m} [(e_i(e_i(f)) - (\nabla^M_{e_i} e_i) f] = \sum_{i=1}^{m} g(\nabla^M_{e_i} \mathrm{grad} f, e_i) =$$

$$= \mathrm{div}(\mathrm{grad} f) = \frac{\partial^2 f}{\partial x_1^2} + \ldots + \frac{\partial^2 f}{\partial x_m^2} = \Delta_g f.$$

Since $\tau(\varphi) = \mathrm{trace_g}\left(\sum_{i,j} g^{ij} ({}^E\nabla(\mathrm{d}\varphi))_{ij}\right) = g^{ij}({}^E\nabla(\mathrm{d}\varphi))_{ij}$, the contravariant components of the tension field in local coordinates at $\varphi(x)$, on N, are

$$\tau(\varphi)^\gamma = \mathrm{trace_g}\left(\sum_{i,j} g^{ij} ({}^E\nabla(d\varphi))^\gamma_{ij}\right) = g^{ij}\varphi^\gamma_{ij} + g^{ij} \frac{\partial \varphi^\alpha}{\partial x^i} \frac{\partial \varphi^\beta}{\partial x^j} \; {}^h\Gamma^\gamma_{\alpha\beta} =$$

$$= \Delta_g \varphi^\gamma + g(\mathrm{grad}\varphi^\alpha, \mathrm{grad}\varphi^\beta)^h\Gamma^\gamma_{\alpha\beta},$$

i.e.,

$$\tau(\varphi)^\gamma = \Delta_g \varphi^\gamma + g(\mathrm{grad}\varphi^\alpha, \mathrm{grad}\varphi^\beta)({}^h\Gamma^\gamma_{\alpha\beta} \circ \varphi), \tag{2.5}$$

where

$$\Delta_g \varphi^\gamma = g^{ij}\varphi^\gamma_{ij} = g^{ij}\left(\frac{\partial^2 \varphi^\gamma}{\partial x^i \partial x^j} - {}^g\Gamma^k_{ij} \frac{\partial \varphi^\gamma}{\partial x^k}\right) = g^{ij} \frac{\partial^2 \varphi^\gamma}{\partial x^i \partial x^j} - g^{ij}\, {}^g\Gamma^k_{ij} \frac{\partial \varphi^\gamma}{\partial x^k}. \tag{2.6}$$

It follows

$$\tau(\varphi)^\gamma(p) = \Delta_g \varphi^\gamma(p) + g^{ij}_p \frac{\partial \varphi^\alpha}{\partial x^i}(p) \frac{\partial \varphi^\beta}{\partial x^j}(p)^h\Gamma^\gamma_{\alpha\beta}(\varphi(p)), \;\; 1 \le \gamma \le n \tag{2.7}$$

$$\tau(\varphi)(x) \in T_{\varphi(x)}N, \quad \tau(\varphi) = \tau(\varphi)^\gamma \frac{\partial}{\partial y^\gamma}. \tag{2.8}$$

We choose local normal coordinates centered at $p \in M$ and, respectively at $q = \varphi(p)$. We have $g_{ij}(p) = \delta^{ij}$, ${}^g\Gamma^k_{ij}(p) = 0$, ${}^h\Gamma^\gamma_{\alpha\beta}(\varphi(p)) = 0$. Therefore, we obtain

$$\tau(\varphi)^\gamma(p) = \sum_{i=1}^{m} \frac{\partial^2 \varphi^\gamma}{(\partial x^i)^2}(\mathrm{p}) \; = \Delta_g \varphi^\gamma(\mathrm{p}) \; = \tau(\varphi^\gamma)(p), \quad 1 \le \gamma \le n \tag{2.9}$$

So, only at the centers p and $q = \varphi(p)$, of the two local normal coordinates systems, we get the formula

$$\tau(\varphi)(p) = \tau(\varphi)^{\gamma}(p)\frac{\partial}{\partial y^{\gamma}} = (\Delta_g \varphi^{\gamma})(p)\frac{\partial}{\partial y^{\gamma}}. \tag{2.10}$$

In general, $\tau^{\gamma}(\varphi) = \Delta_g \varphi^{\gamma} + g(\mathrm{grad}\varphi^{\alpha}, \mathrm{grad}\varphi^{\beta})\Gamma^{\gamma}_{\alpha\beta}$, while $\tau(\varphi^{\gamma}) = \Delta_g \varphi^{\gamma}$, since φ^{γ} is a function, and

$$\Delta_g \varphi^{\gamma} = g^{ij}\varphi^{\gamma}_{ij} = g^{ij}\frac{\partial^2 \varphi^{\gamma}}{\partial x^i \partial x^j} - g^{ij}\ {}^{g}\Gamma^{k}_{ij}\frac{\partial \varphi^{\gamma}}{\partial x^k},$$

cf. (2.6).

Definition 2.3. (differential-geometric definition of harmonic map, [1]-[23]) A smooth map $\varphi : (M, g) \to (N, h)$ is called *harmonic* if its tension field vanishes on M, i.e.,

$$\tau(\varphi) = 0. \tag{2.11}$$

The equation (2.11) is called *the tension field equation* or *the harmonic equation.* In local coordinates, from (2.5), the harmonic equation is a system of semilinear second order partial differential equations of elliptic type,

$$\tau^{\gamma}(\varphi) = \Delta_g \varphi^{\gamma} + g^{ij}\frac{\partial \varphi^{\alpha}}{\partial x^i}\frac{\partial \varphi^{\beta}}{\partial x^j}\ {}^{h}\Gamma^{\gamma}_{\alpha\beta} = 0, \quad \gamma = 1, \ldots, n. \tag{2.12}$$

In other words, *a smooth map between Riemannian manifolds is a harmonic map if the local components of the tension field vanishes simultaneously.*

Note that this definition extends to non-degenerate semi-Riemannian manifolds.

Remark 2.2. In normal coordinates, at $p \in M$ and $q = \varphi(p) \in N$, from (2.9), the function φ is a harmonic map, if for any $(\forall)(p, q) \in M \times N$, $q = \varphi(p)$,

$$\sum_{i=1}^{m} \frac{\partial^2 \varphi^{\gamma}}{(\partial x^i)^2}(\mathrm{p}) \ = \Delta_g \varphi^{\gamma}(\mathrm{p}) \ = 0, \quad 1 \leq \gamma \leq n \tag{2.13}$$

i.e. φ satisfies the Euclidean-Laplace equations at p ([6]).

3 Subharmonic Maps

We give two equivalent definitions of a subharmonic map that is

1. A differential-geometric definition, as a differentiable map, with positive values of the local components of its tension field, at the centers of the local coordinates;
2. The extended definition of Ishihara ([14]), as a differentiable map which pulls-back germ of certain convex functions to germs of subharmonic functions.

Definition 3.1. (differential geometric definition of subharmonic map) A differential smooth map between Riemannian manifolds is called a *subharmonic map* if the local components of the tension field are positive, i.e.,

$$(\forall)p \in M,\ \tau^{\gamma}(\varphi)(p) \geq 0, \quad (\forall)\gamma = 1, \ldots, n. \tag{3.1}$$

Similarly are defined the *superharmonic map*, the *strict subharmonic map* and the *strict superharmonic map*.

In normal coordinates, at $p \in M$, this means $(\Delta_g \varphi^\gamma)(p) \geq 0$, $\gamma = 1, \ldots, n$, i.e.

$$\sum_{i=1}^{m} \frac{\partial^2 \varphi^\gamma}{(\partial x^i)^2}(\mathrm{p}) \geq 0 \tag{3.2}$$

and it is a superharmonic map, if, for any $p \in M$, in normal coordinates centered at p,

$$(\Delta_g \varphi^\gamma)(p) \leq 0, \quad \gamma = 1, \ldots, n.$$

If the inequalities are strict, we say that φ is a *strict subharmonic map*, respectively a *strict superharmonic map*.

Obviously we have the following equivalent definition of the harmonic map:

Definition 3.2. (definition of harmonic map) A differential smooth map between Riemannian manifolds is a *harmonic map* if it is simultaneously a subharmonic map and a superharmonic map.

We can also give another equivalent definition of subharmonic map,

Definition 3.3. (first extended Ishihara definition of subharmonic maps) A differential smooth map $\varphi : (M, g) \to (N, h)$ is a *subharmonic map* if and only if, for any $p \in M$, (U, φ) local coordinates centered at p and (V, ψ) local coordinates centered at $q = \varphi(p)$, $\varphi(U) \subseteq V$, it pulls back local convex functions $f : V \to \mathbb{R}$, such that $\dfrac{\partial f}{\partial y^\gamma}(\varphi(p)) \geq 0$, $(\forall)\gamma = 1, \ldots, n$, to local subharmonic functions.

Definition 3.4. (second extended Ishihara definition of subharmonic maps) A differential smooth map $\varphi : (M, g) \to (N, h)$ is a *subharmonic map*, if and only if, for any $p \in M$, (U, φ) local coordinates centered at p and (V, ψ) local coordinates centered at $q = \varphi(p)$, $\varphi(U) \subseteq V$, pulls back local concave functions $f : V \to \mathbb{R}$, such that $\dfrac{\partial f}{\partial y^\gamma}(\varphi(p)) \leq 0$, $(\forall)\gamma = 1, \ldots, n$, to local superharmonic functions.

4 Semi-conformal maps

Definition 4.1. (semi-conformal map) A differential smooth map $\varphi : (M, g) \to (N, h)$ is called *semi-conformal* at $x \in M$, if one of the following equivalent statements is true:

1. $\mathrm{d}\varphi_x = 0$
2. $\mathrm{d}\varphi_x$ maps the orizontal subspace $H_x = (\mathrm{Ker}\mathrm{d}\varphi_x)^\perp$ of $T_x M$ conformally onto $T_{\varphi(x)}N$, i.e., $\mathrm{d}\varphi_x$ is surjective and $(\exists)\ \Lambda(x) \in (0, +\infty) \subseteq \mathbb{R}$, such that

$$h_{\varphi(x)}(\mathrm{d}\varphi_x(X), \mathrm{d}\varphi_x(Y)) = \Lambda(x) g_x(X, Y) \quad (\forall)X, Y \in H_x. \tag{4.1}$$

The map φ is semi-conformal on M if it is conformal at every point of M.

Now, we give some caracterizations and properties of semi-conformal maps that we need further.

We recall that, for frames $\{X_i\}_i$ at $x \in M$ and $\{Y_\alpha\}_\alpha$ at $\varphi(x) \in N$, we have $g_{ij} = g(X_i, X_j)$, $h_{\alpha\beta} = h(Y_\alpha, Y_\beta)$ and $\mathrm{d}\varphi(X_i) = \varphi_i^\alpha Y_\alpha$. The differential $\mathrm{d}\varphi_x^* : T_{\varphi(x)}N \to T_xM$ is caracterized by the formula

$$g(X, \mathrm{d}\varphi_x^*(Y)) = h(\mathrm{d}\varphi_x(X), Y) \quad (X \in T_xM,\ Y \in T_{\varphi(x)}N). \tag{4.2}$$

Proposition 4.1. (semi-conformallity) *For a smooth map $\varphi : (M, g) \to (N, h)$, between two Riemannian manifolds, and $x \in M$ the following statements are equivalent*

(i) φ is semi-conformal at x, with the dilation $\lambda(x)$ and square dilation $\Lambda(x) = \lambda^2(x)$ i.e. $d\varphi_x = 0$ or $(\exists)$ $\Lambda(x) \in (0, +\infty) \subseteq \mathbb{R}$, such that

$$h_{\varphi(x)}(\mathrm{d}\varphi_x(X), \mathrm{d}\varphi_x(Y)) = \Lambda(x) g_x(X, Y) \quad (\forall) X, Y \in H_x; \tag{4.3}$$

(ii) $(\exists)$ $\Lambda(x) \in \mathbb{R}_+$, such that, $(\forall) f_1, f_2 : V \to \mathbb{R}$, C^1-functions, $\varnothing \neq V \subseteq N$ open, we have:

$$g_x(\mathrm{grad}(f_1 \circ \varphi), \mathrm{grad}(f_2 \circ \varphi)) = \Lambda(x)[h_{\varphi(x)}(\mathrm{grad} f_1, \mathrm{grad} f_2) \circ \varphi];$$

(iii) $(\exists)\Lambda(x) \in \mathbb{R}_+$, such that, $(\forall)\{Y_\alpha\}_\alpha$ frame at $\varphi(x) \in N$,

$$g(\mathrm{d}\varphi_x^*(Y_\alpha), \mathrm{d}\varphi_x^*(Y_\beta)) = \Lambda(x) h_{\alpha\beta}, \quad (\forall)\alpha, \beta \in \{1, 2, \ldots, n\}; \tag{4.4}$$

(iv) $(\exists)\Lambda(x) \in \mathbb{R}_+$, such that, $(\forall)\{X_i\}_i$ the frame at $x \in M$ and $(\forall)\{Y_\alpha\}_\alpha$ the frame in $\varphi(x) \in N$,

$$g^{ij} \varphi_i^\alpha \varphi_j^\beta = \Lambda(x) h^{\alpha\beta}, \quad (\forall)\alpha, \beta \in \{1, 2, \ldots, n\} \tag{4.5}$$

(v) $(\exists)\Lambda(x) \in \mathbb{R}_+$, such that, the cometrics g_x^* on T_x^*M and $h_{\varphi(x)}^*$ on $T_{\varphi(x)}^*N$ are related by $\varphi_*(g_x^*) = \Lambda(x) h_{\varphi(x)}^*$,

(vi) $(\exists)\Lambda(x) \in \mathbb{R}_+$, such that, the adjoint $d\varphi_x^*$ of $\mathrm{d}\varphi_x$ satisfies the relation

$$\mathrm{d}\varphi_x \circ \mathrm{d}\varphi_x^* = \Lambda(x) \mathrm{Id}_{T_{\varphi(x)}N}; \tag{4.6}$$

(vii) $(\exists)\Lambda(x) \in \mathbb{R}_+$, such that, $(\forall)\{Y_\alpha\}_\alpha$ orthonormal frame at $\varphi(x) \in N$, the vectors $\mathrm{d}\varphi_x^*(Y_\alpha)$, are orthogonal and of the same norm, $\lambda(x)$;

(viii) $(\exists)\Lambda(x) \in \mathbb{R}_+$, $\Lambda(x) = \lambda^2(x)$, such that, either $\mathrm{d}\varphi_x = 0$, or, $(\forall)\{Y_\alpha\}_\alpha$ orthonormal frame at $\varphi(x) \in N$, $(\exists)\{X_i\}_i$ an orthonormal frame at $x \in M$, such that

$$\mathrm{d}\varphi_x(X_i) = \lambda(x) Y_i, \quad (i = 1, \ldots, n) \quad \text{or} \quad 0,\ \text{if} i > n; \tag{4.7}$$

(ix) $(\exists)\Lambda(x) \in \mathbb{R}_+$, such that, either $\mathrm{d}\varphi_x = 0$ or $\mathrm{d}\varphi_x$ is surjective and the pull-back of h satisfies

$$\varphi^* h\,|_{H_x \times H_x} = \Lambda(x) g\,|_{H_x \times H_x}\ ;$$

(x) $(\exists)\Lambda(x) \in \mathbb{R}_+$, such that, $(\forall)(y^\gamma)$ local coordinates at $\varphi(x) \in N$,

$$g(\mathrm{grad}\varphi^\alpha, \mathrm{grad}\varphi^\beta) = \Lambda[h^{\alpha\beta} \circ \varphi], \quad (\forall)\alpha, \beta \in \{1, 2, \ldots, n\}; \tag{4.8}$$

(xi) $(\exists)\Lambda(x) \in \mathbb{R}_+$, such that,

$$g(\mathrm{d}\varphi_x^*(Y), \mathrm{d}\varphi_x^*(Y')) = \Lambda(x) h(Y, Y'), \quad (\forall) Y, Y' \in T_{\varphi(x)}N. \tag{4.9}$$

Proof. We notice immediately that they are the same things expressed in different ways (see also [1], [10]-[12]). □

Lemma 4.2. ([10]-[12]) *Let $\varphi : M \to N$ a C^2 semi-conformal map, between even non-degenerate semi-Riemannian manifolds with the square dilation Λ. Then, the tension field of φ in local coordinates (y^γ) on N is*

$$\tau^\gamma(\varphi) = \Delta_g \varphi^\gamma - \Lambda[(\Delta_h y^\gamma) \circ \varphi]. \tag{4.10}$$

Remark 4.1. The tension field can be written $\tau^\gamma(\varphi) = \Delta_g \varphi^\gamma + g^{ij} \frac{\partial \varphi^\alpha}{\partial x^i} \frac{\partial \varphi^\beta}{\partial x^j} \, {}^h\Gamma^\gamma_{\alpha\beta}$. In particular, $\tau^\gamma(\varphi) = \Delta_g \varphi^\gamma$, if the local coordinates on N are chosen as harmonic functions. But

$$\Delta_g \varphi^\gamma = g^{ij} \varphi^\gamma_{ij} = g^{ij} \left(\frac{\partial^2 \varphi^\gamma}{\partial x^i \partial x^j} - {}^g\Gamma^k_{ij} \frac{\partial \varphi^\gamma}{\partial x^k} \right) = g^{ij} \frac{\partial^2 \varphi^\gamma}{\partial x^i \partial x^j} - g^{ij} \, {}^g\Gamma^k_{ij} \frac{\partial \varphi^\gamma}{\partial x^k}.$$

So, if we take local harmonic coordinates on M too, $\Delta_g \varphi^\gamma = g^{ij} \frac{\partial^2 \varphi^\gamma}{\partial x^i \partial x^j}$. In particular, if $g^{ij} = \delta^{ij}$, we have $\Delta_g \varphi^\gamma = \sum_{i=1}^{m} \frac{\partial^2 \varphi^\gamma}{(\partial x^i)^2}$.

Lemma 4.3. ([10]-[12]) *A C^2-map between two Riemannian manifolds is harmonic and semi-conformal, if and only if, there exist a scalar Λ on M, such that*

$$\Delta_g (f \circ \varphi) = \Lambda[\Delta_h f) \circ \varphi] \tag{4.11}$$

for any C^2-function on N. The function Λ is called the dilation of φ.

5 Subharmonic Morphisms

Lemma 5.1. ([10]-[12]) *For a C^2-map $\varphi : (M, g) \to (N, h)$ between two non-degenerate semi-Riemannian manifolds, the following statements are equivalent:*

(i) There is a scalar $\lambda \geq 0$ on M such that, for any C^2-function on N, we have

$$\Delta_M (f \circ \varphi) = \lambda[(\Delta_N f) \circ \varphi].$$

(ii) For any point $p \in M$, $q = \varphi(p)$, and any C^2-function on N, we have

$$\Delta_N f(q) \geq 0 \Rightarrow \Delta_M (f \circ \varphi)(p) \geq 0.$$

(iii) For any C^2-function defined on an open subset $V \subseteq N$, with $\varphi^{-1}(V) \neq \varnothing$ we have

$$\nabla_N f \geq 0 \text{ in } V \Rightarrow \nabla_M (f \circ \varphi)(p) \geq 0 \text{ in } \varphi^{-1}(V).$$

Lemma 5.2. ([10]-[12]) *For a C^2-map $\varphi : (M, g) \to (N, h)$ between two non-degenerate semi-Riemannian manifolds, the following statements are equivalent:*

(i) $(\exists)\lambda \in \mathbb{R}$, a scalar on M, such that for any C^2-function $f : N \to \mathbb{R}$,

$$\Delta_g (f \circ \varphi) = \lambda \cdot [(\Delta_h f) \circ \varphi]$$

(ii) $(\forall) p \in M$, $q = \varphi(p)$ *and* $(\forall) f : N \to \mathbb{R}$, C^2*-function on* N,

$$\Delta_h f(q) = 0 \Rightarrow \Delta_g (f \circ \varphi)(p) = 0.$$

(iii) $(\forall) f : V \to \mathbb{R}$, *a harmonic function definite on an open subset* $V \subseteq N$, $f \circ \varphi$ *is a harmonic function on* $\varphi^{-1}(V) \neq \varnothing$.

Lemma 5.3. *Let* (M, g), (N, h) *be Riemannian smooth manifolds. Then* $\varphi : M \to N$ *a semi-conformal harmonic map pulls back germs of harmonic functions to germs of harmonic functions.*

Proof. Let (V, ψ) be a local smooth coordinate system on N and $f : V \to \mathbb{R}$ be a smooth function. Then the tension field of the composition of two maps, $\tau(\psi \circ \varphi) = \mathrm{d}\psi(\tau(\varphi)) + \mathrm{Trace}\Delta\mathrm{d}\psi(\mathrm{d}\varphi, \mathrm{d}\varphi)$ can be written as

$$\Delta(f \circ \varphi) = \mathrm{d}f(\tau(\varphi)) + \nabla \mathrm{d}f \left(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta} \right) g(\mathrm{grad}\varphi^\alpha, \mathrm{grad}\varphi^\beta), \tag{5.1}$$

If φ is a harmonic map the first term from the right member canceled, since $\mathrm{d}f(\tau(\varphi)) = \sum_{\gamma=1}^{n} \frac{\partial f}{\partial y^\gamma} \cdot \tau(\varphi)^\gamma = 0$. If, φ is a semi-conformal map, from Proposition 4.1, it follows that $g(\mathrm{grad}\varphi^\alpha, \mathrm{grad}\varphi^\beta) = \Lambda h^{\alpha\beta}$, and we obtain:

$$\Delta(f \circ \varphi) = \Lambda h^{\alpha\beta} \nabla \mathrm{d}f \left(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta} \right) = \Lambda \Delta f. \tag{5.2}$$

Consequently, if f is a harmonic function, then $f \circ \varphi$ is too. So φ pulls back local harmonic functions to local harmonic functions ([1]) □

Remark 5.1. Moreover, if φ is a semi-conformal harmonic map and f is a subharmonic function, then $f \circ \varphi$ is a subharmonic function.

If φ is a semi-conformal harmonic map and f is a superharmonic function, then $f \circ \varphi$ is a superharmonic function.

If φ is a semi-conformal subharmonic map and f is a subharmonic function with $\frac{\partial f}{\partial y^\gamma}(\varphi(x)) \geq 0$ $(\forall)\gamma = 1, \ldots, n$, then $f \circ \varphi$ is a subharmonic function.

If φ is a semi-conformal superharmonic map and f is a superharmonic function with $\frac{\partial f}{\partial y^\gamma}(\varphi(x)) \leq 0$ $(\forall)\gamma = 1, \ldots, n$, then $f \circ \varphi$ is a superharmonic function.

Theorem 5.4. ([10]-[12], [14]) *A smooth map between Riemannian manifolds pulls back germs of harmonic functions in germs of harmonic functions, if and only if, it is a semi-conformal harmonic map.*

Lemma 5.5. *The following conditions are equivalent:*

(i) The smooth map $\varphi : (M, g) \to (N, h)$ *between Riemannian manifolds is a harmonic morphism;*

(ii) $(\exists)\Lambda : M \to [0, \infty)$ *a smooth function such that, for any smooth function* $f : V \to \mathbb{R}$, *defined on an open set* $V \subseteq N$ *with* $\varphi^{-1}(V) \neq \varnothing$,

$$\Delta(f \circ \varphi)(x) = \Lambda(x) \Delta f(x), \quad (\forall) x \in \varphi^{-1}(V);$$

(iii) $(\exists)\Lambda : M \to [0, \infty)$ *a smooth function such that,* $(\forall)\psi : V \to P$ *a smooth map defined on an open set* $V \subseteq N$ *with* $\varphi^{-1}(V) \neq \varnothing$ *and* P *a Riemannian manifold, we have*

$$\tau(\psi \circ \varphi) = \Lambda \tau(\psi) \tag{5.3}$$

(iv) $(\forall)\psi : V \to P$ *a smooth map defined on an open set* $V \subseteq N$ *with* $\varphi^{-1}(V) \neq \varnothing$ *and* P *a Riemannian manifold, the composition* $\psi \circ \varphi$ *is a harmonic map.*

Moreover, if (ii) or (iii) is true, then Λ *is the square of the dilation of* φ*.*

Proof. (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) immediately; (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) immediately; it remains to prove (i) $\Rightarrow$ (iii). If (i) is true, then from Theorem 1.1, the map φ is harmonic and semi-conformal.

We prove now (iii). Indeed, let $(\forall)\psi : V \to P$ be a smooth map defined on an open set $V \subseteq N$ with $\varphi^{-1}(V) \neq \varnothing$ and P a Riemannian manifold. In the formula of the tension field of the composition of two maps, $\tau(\psi \circ \varphi) = \mathrm{d}\psi(\tau(\varphi)) + \mathrm{Trace}\nabla \mathrm{d}\psi(\mathrm{d}\varphi, \mathrm{d}\varphi)$, φ is a harmonic map, so $\tau(\varphi) = 0$ and that implies $\mathrm{d}\psi(\tau(\varphi)) = 0$. Moreover, φ is a semi-conformal map, so, from Proposition 1, $g(\mathrm{grad}\varphi^\alpha, \mathrm{grad}\varphi^\beta) = \Lambda h^{\alpha\beta}$, and we obtain:

$$\tau(\psi \circ \varphi) = \mathrm{Trace}\nabla \mathrm{d}\psi(\mathrm{d}\varphi, \mathrm{d}\varphi) = \nabla \mathrm{d}\psi\left(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta}\right) g(\mathrm{grad}\varphi^\alpha, \mathrm{grad}\varphi^\beta) =$$

$$= \Lambda h^{\alpha\beta} \nabla \mathrm{d}\psi\left(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta}\right) = \Lambda h^{\alpha\beta} (\nabla \mathrm{d}\psi)_{\alpha\beta} = \Lambda \tau(\psi)$$

i.e. (iii) (see also ([1]). □

Theorem 5.6. ([10]-[12]) (equivalence between the first and the second definition of a harmonic morphism) *Let be* $\varphi : (M, g) \to (N, h)$ *a smooth map between Riemannian manifolds. Then* φ *pulls back germs of subharmonic functions to germs of subharmonic functions, if and only if,* φ *pulls back germs of harmonic functions to germs of harmonic functions.*

Proof. If φ pulls back germs of subharmonic functions to germs of subharmonic functions, then φ pulls back germs of superharmonic functions to germs of superharmonic functions and so φ pulls back germs of harmonic functions in germs of harmonic functions.

If φ pulls back germs of harmonic functions to germs of harmonic functions, then, from Lemma 5.5 (ii), $\Delta(f \circ \varphi) = \Lambda \Delta f$ and so $\Delta f \geq 0$, implies $\Delta(f \circ \varphi) \geq 0$, i.e. φ pulls back germs of subharmonic functions to germs of subharmonic functions. □

Remark 5.2. Moreover, when φ is a harmonic morphism, if f is a strict subharmonic function, then $f \circ \varphi$ is a strict subharmonic function.

Proposition 5.7. (equivalence between the second and the fourth definition of a harmonic morphism) *A smooth map between non-degenerate semi-Riemannian manifolds pulls back germs of harmonic functions to germs of harmonic functions, if and only if,* φ *is a semi-conformal map whose local components* $\varphi^\gamma = y^\gamma \circ \varphi$*, in local harmonic coordinates on* N*, are harmonic functions.*

Proof. "$\Rightarrow$" If φ pulls back germs of harmonic functions in germs of harmonic functions, (i) from Lemma 5.2 is true for $f = y^\gamma$ too and so $\Delta_g(\varphi^\gamma) = \lambda \cdot [(\Delta_h y^\gamma) \circ \varphi] = 0$.

"$\Leftarrow$" If φ is a semi-conformal map those local components $\varphi^\gamma = y^\gamma \circ \varphi$, in local harmonic coordinates on N, are harmonic functions, for any $\gamma = 1, \ldots, n$ and let be $f : V \to \mathbb{R}$, a harmonic function definite on an open set $V \subseteq N$, $\Delta_h f = 0$. Then, from Lemma 4.2, $\tau^\gamma(\varphi) = \Delta_g \varphi^\gamma - \Lambda[(\Delta_h y^\gamma) \circ \varphi]$, cf. (4.10).

$\Delta_h y^\gamma = 0$ implies $\tau^\gamma(\varphi) = \Delta_g \varphi^\gamma = 0$ i.e φ is a harmonic map. Since φ is a semi-conformal harmonic map, from Lemma 4.3, there exists a scalar Λ on M, so that $\Delta_g(f \circ \varphi) = \Lambda[\Delta_h f) \circ \varphi] = 0$ and therefore $f \circ \varphi$ is a harmonic function on $\varphi^{-1}(V) \neq \varnothing$. □

Definition 5.1. (subharmonic morphism) Let $\varphi : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds. If φ is a semi-conformal subharmonic map, we say that it is a *subharmonic morphism.*

Analogously, if φ is a semi-conformal strict subharmonic map, we say that it is a *strict subharmonic morphism.* Respectively, if φ is a semi-conformal superharmonic map, we say that it is a *superharmonic morphism* and if φ is a semi-conformal strict superharmonic map, we say that it is a *strict superharmonic morphism.*

Definition 5.2. (5th definition of a harmonic morphism) A smooth map between Riemannian manifolds is a *harmonic morphism*, if and only if, it is a subharmonic morphism and a superharmonic morphism, at the same time.

Form the Lemma 4.2, we have the following characterisation of a subharmonic morphism:

Proposition 5.8. *Let $\varphi : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds. If φ is a semi-conformal map whose local components are subharmonic functions with respect to superharmonic coordinates of the manifold N, then it is a subharmonic morphism.*

Proof. From Lemma 4.4, since the tension field of φ is given in local coordinates (y^γ) on N by (4.11), we infer

$$\tau^\gamma(\varphi) = \Delta_g \varphi^\gamma - \Lambda[(\Delta_h y^\gamma) \circ \varphi], \quad \Delta_h y^\gamma \leq 0, \quad \Delta_g \varphi^\gamma \geq 0,$$

and it follows $\tau^\gamma(\varphi) \geq 0$. □

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