Geometrical structures associated to rheonomic first order dynamical systems

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Abstract. It was shown that to any scleronomic first order dynamical system we can associate, canonically, a geometrical structure built by a generalized d-metric and a nonlinear d-connection. Thus, we can define a covariant differential operator, a parallel transport, autoparallel vector fields, d-1-forms and in general d-tensors and autoparallel curves. The existence of the curvature tensor and of other geometrical properties follows. In this paper we extend these considerations to the case of rheonomic first order ordinary dynamical systems.

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1 Introduction

Let $M = M_m$ be a differentiable manifold of dimension m and of local coordinates (x^i). Let J^1M be the first order jet space. We consider a change of local chart on M given by $\bar{x}^i = \bar{x}^i(x^h)$ and on J^1M by:

$$\bar{t} = t,\ \bar{x}^i = \bar{x}^i(x^h),\ \dot{\bar{x}}^i = \frac{\partial \bar{x}^i}{\partial x^h}\dot{x}^h. \tag{1.1}$$

By a geometrical structure we understand a d-1-form, a "d-metric" and a nonlinear d-connection.

A d-1-form is locally written $\omega = a_i\left(t, x^h, \dot{x}^h\right) d\dot{x}^i$ with the property that, on a change of local chart, its coefficients change by the rules:

$$\bar{a}_h = \frac{\partial x^i}{\partial \bar{x}^h} a_i. \tag{1.2}$$

We call "d-metric" a second order distinguished tensor, two times covariant, nondegenerated and parameterized by t. Thus, a "d-metric" is defined by a set of functions

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$a_{ij}^1 = a_{ij}^1\left(t, x^h, \dot{x}^h\right)$ with $\det\left(a_{ij}^1\right) \neq 0$ and which change, on a change of local chart, by the rules:

$$\bar{a}_{hk}^1 = \frac{\partial x^i}{\partial \bar{x}^h}\frac{\partial x^j}{\partial \bar{x}^k}a_{ij}^1. \tag{1.3}$$

The matrix (a_{ij}^1) being nondegenerated, the existence of the contravariant components a_1^{ij} $(a_1^{ih}a_{hj}^1 = \delta_j^i)$ follows. In another local chart, the contravariant components are given by: $\bar{a}_1^{hk} = \dfrac{\partial \bar{x}^h}{\partial x^i}\dfrac{\partial \bar{x}^k}{\partial x^j}a_1^{ij}$.

A d-connection is defined by a set of functions $a_{ij}^0 = a_{ij}^0\left(t, x^h, \dot{x}^h\right)$, which change, on a change of local chart, by the rules:

$$\bar{a}_{hk}^0 = \frac{\partial x^i}{\partial \bar{x}^h}\left(\frac{\partial x^j}{\partial \bar{x}^k}a_{ij}^0 + \frac{\partial \dot{x}^j}{\partial \bar{x}^k}a_{ij}^1\right). \tag{1.4}$$

They are called Christoffel symbols of the first type of the d-connection.

We can define the functions: $M_j^i = a_1^{ih}a_{hj}^0$, called the Christoffel symbols of the second type of the nonlinear connection ([6], [8]). They change, on a change of chart, by the rules:

$$\overline{M}_j^i = \frac{\partial \bar{x}^i}{\partial x^h}\left(\frac{\partial x^k}{\partial \bar{x}^j}M_k^h + \frac{\partial \dot{x}^k}{\partial \bar{x}^j}\delta_k^h\right) \tag{1.5}$$

In [9], [10], [11], [12], [13], we associated to a dynamical system whose evolution is described by ordinary or partial differential equations, geometrical objects as metrical tensors, connections, covariant differential operators, we defined a parallel transport of vector fields and autoparallel curves. The geometrical structure built in this way is perfectly suitable to holonomic scleronomic dynamical systems. For them, the following property holds: the trajectories of the given system are autoparallel curves with respect to this geometrical structure and conversely.

These results can be generalized to systems of rheonomous implicit first order differential equations.

2 Extended geometrical structures

We extend the above geometrical structure to the product manifold $\mathbf{R} \times M$ ([3], [4]). We consider the time t as an $(m+1)$-coordinate. The atlas of $\mathbf{R} \times M$ is the product of the canonical atlas of $\mathbf{R}$, given by one chart $i_{\mathbf{R}}$, and the atlas of M.

We denote by τ the parameter of evolution, $t = \varphi(\tau)$. For simplicity, we define τ by the relation $t = \tau$, reason to understand by the derivatives with respect to t the same as by the derivatives with respect to τ.

2.1 Extended metric

Given a set of functions $a_{ij}^1(t, x, \dot{x})$ satisfying (1.3), we extend the d-metric to $\mathbf{R} \times M$ by the nondegenerated matrix: $\begin{pmatrix} a_{ij}^1 & 0 \\ 0 & 1 \end{pmatrix}$.

Its reciprocal matrix is $\begin{pmatrix} a_1^{ij} & 0 \\ 0 & 1 \end{pmatrix}$, where a_1^{ij} are the components of the reciprocal matrix of $\left(a_{ij}^1\right)$. Indeed, we have: $\begin{pmatrix} a_1^{ih} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_{hj}^1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \delta_j^i & 0 \\ 0 & 1 \end{pmatrix} = I_{m+1}$.

2.2 Extended connection

We build on $\mathbf{R} \times M$ the Christoffel symbols of the first type of a nonlinear connection ([1], [2]), by the matrix: $\begin{pmatrix} a_{ij}^0 & a_i \\ 0 & 0 \end{pmatrix}$, where $a_{ij}^0\,(t,x,\dot{x})$ are the Christoffel symbols of the first type of a nonlinear d-connection on M and a_i are the components of a d-1-form.

By contracting these coefficients with the given metric, we obtain the Christoffel symbols of the second type:

$$\begin{pmatrix} M_j^i & M^i \\ M_j & M_0 \end{pmatrix} = \begin{pmatrix} a_1^{ih} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_{hj}^0 & a_h \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_1^{ih}a_{hj}^0 & a_1^{ih}a_h \\ 0 & 0 \end{pmatrix}, \tag{2.1}$$

where $M_j^i = a_1^{ih}a_{hj}^0$ are the previous coefficients, $M^i = a_1^{ih}a_h$ are new coefficients and $M_j = M_0 = 0$. On a change of local chart, they change by the rules:

$$\overline{M}_j^i = \frac{\partial \bar{x}^i}{\partial x^h}\left(\frac{\partial x^k}{\partial \bar{x}^j}M_k^h + \frac{\partial \dot{x}^k}{\partial \bar{x}^j}\delta_k^h\right), \quad \overline{M}^i = \frac{\partial \bar{x}^i}{\partial x^h}M^h. \tag{2.2}$$

2.3 Covariant differential operator associated to an extended nonlinear connection

A differential operator $D : \Lambda^o(J^1M) \to \Lambda^o(J^2M)$, is defined, along a curve $x^i = c^i(t)$, by $D : f \to Df = \dfrac{df}{dt} = \dfrac{\partial f}{\partial t} + \dot{x}^i\dfrac{\partial f}{\partial x^i} + \ddot{x}^i\dfrac{\partial f}{\partial \dot{x}^i}$.

Let us consider the spaces (rings) $\Lambda^o(J^1(\mathbf{R} \times M))$, $\Lambda^o(J^2(\mathbf{R} \times M))$, the module $\mathfrak{X}_1^d(\mathbf{R} \times M)$, where $X \in \mathfrak{X}_1^d(\mathbf{R} \times M)$ if $X = X^i(t,x,\dot{x})\dfrac{\partial}{\partial x^i} + Y(t,x,\dot{x})\dfrac{\partial}{\partial t}$ is distinguished, and let N be a nonlinear connection of local components M_i^j, M^j. To the connection N it is associated a linear differential operator D_N, defined on the set $\mathfrak{X}_1^d(\mathbf{R} \times M)$, of the d-vector fields and with values in $\mathfrak{X}_2^d(\mathbf{R} \times M)$, given by its values on the basis vectors $\left(\dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial t}\right)$:

$$\begin{aligned} D_N\left(\frac{\partial}{\partial x^i}\right) &= M_i^j\frac{\partial}{\partial x^j} + M_i\frac{\partial}{\partial t} = M_i^j\frac{\partial}{\partial x^j}, \\ D_N\left(\frac{\partial}{\partial t}\right) &= M^j\frac{\partial}{\partial x^j} + M_0\frac{\partial}{\partial t} = M^j\frac{\partial}{\partial x^j} \end{aligned} \tag{2.3}$$

and satisfying the relation $D_N(fX) = fD_N(X) + D(f)X$, for any $f \in \Lambda^o(J^1(\mathbf{R}\times M))$, $X \in \mathfrak{X}_1^d(\mathbf{R} \times M))$. The local expression of the operator D_N on an arbitrary d-vector field X, is given by:

$$D_N(X) = \left(\frac{dX^i}{dt} + M_j^iX^j + M^iY\right)\frac{\partial}{\partial x^i} + \frac{dY}{dt}\frac{\partial}{\partial t}$$

This new vector field is called the *N-covariant derivative* of X.

Definition 2.1. The expressions $\frac{\delta X^i}{\delta t} = \frac{dX^i}{dt} + M^i_j X^j + M^i Y$ and $\frac{\delta Y}{\delta t} = \frac{dY}{dt}$ are called *extended covariant derivatives* of the components of the d-vector field X with respect to the nonlinear connection N.

On a change of local chart, the covariant derivatives change by the rules:

$$\frac{\delta \overline{X}^i}{\delta t} = \frac{\partial \bar{x}^i}{\partial x^h}\frac{\delta X^h}{\delta t}, \qquad \frac{\delta \overline{Y}}{\delta t} = \frac{\delta Y}{\delta t}.$$

Proposition 2.1. *To any connection N it corresponds a covariant differential operator.*

Conversely, we have:

Proposition 2.2. *To any differential operator D it can be associated a d-connection.*

Definition 2.2. We say that a d-vector field $X = X^i \frac{\partial}{\partial x^i} + Y\frac{\partial}{\partial t}$ is *transported by parallelism*, along a curve $x^i = c^i(t)$, if, along this curve, its covariant derivative vanish:

$$\frac{dX^i}{dt} + M^i_j X^j + M^i Y = 0, \qquad \frac{dY}{dt} = 0. \tag{2.4}$$

A curve $x^i = c^i(t)$, is called *autoparallel* if

$$\frac{d^2c^i}{dt^2} + M^i_j \frac{dc^j}{dt} + M^i = 0, \tag{2.5}$$

that means its tangent vector $X^i = \frac{dc^i}{dt}$, $Y = 1$ is transported by parallelism.

Let $\alpha \in \Lambda^1_d(\mathbf{R} \times M)$ be a d-1-form on $\mathbf{R} \times M$, locally written $\alpha = Y_i dx^i + Y dt$. We define the differential operator on d-1-forms by its values on (dx^i, dt):

$$\begin{aligned} D_N(dx^i) &= \overset{*}{M}{}^i_j dx^j + \overset{*}{M}{}^i dt, \\ D_N(dt) &= \overset{*}{M}_i dx^i + \overset{*}{M}_0 dt. \end{aligned} \tag{2.6}$$

On an arbitrary d-1-form α, the local expression of the operator D_N is given by:

$$D_N(\alpha) = \left(\frac{dY_i}{dt} + Y_j \overset{*}{M}{}^j_i + Y \overset{*}{M}_i\right) dx^i + \left(\frac{dY}{dt} + Y_i \overset{*}{M}{}^i + Y \overset{*}{M}_0\right) dt.$$

By the definition of D_N on d-vector fields and by the compatibility conditions, it follows that: $\overset{*}{M}{}^i_j = -M^i_j$, $\overset{*}{M}{}^i = -M^i$, $\overset{*}{M}_i = 0$, $\overset{*}{M}_0 = 0$. The relations (2.6) become:

$$D_N(dx^i) = -M^i_j dx^j - M^i dt, \quad D_N(dt) = 0.$$

and we have: $D_N(\alpha) = \left(\frac{dY_i}{dt} - Y_j M^j_i\right) dx^i + \left(\frac{dY}{dt} - Y_i M^i\right) dt.$

2.4 Adapted frames

Given a nonlinear d-connection, in order to define an adapted frame ([4], [5]), we build at first an adapted dual frame by:

$$\begin{aligned}
&\delta x^i = dx^i,\\
&\delta t = dt,\\
&\delta \dot{x}^i = d\dot{x}^i + M^i_j dx^j + M^i dt,\\
&\delta \dot{t} = d\dot{t} + M_j dx^j + M_0 dt.
\end{aligned} \tag{2.7}$$

By $d\dot{t} = 0$ and the expressions (2.1) of the coefficients of the connection, the last relation becomes: $\delta\dot{t} = 0$.

The adapted frame is defined by the duality conditions as follows:

$$\begin{aligned}
&\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - M^h_i \frac{\partial}{\partial \dot{x}^h},\\
&\frac{\delta}{\delta t} = \frac{\partial}{\partial t} - M^h \frac{\partial}{\partial \dot{x}^h},\\
&\frac{\delta}{\delta \dot{x}^i} = \frac{\partial}{\partial \dot{x}^i}.
\end{aligned} \tag{2.8}$$

On a change of local chart, we have respectively:

$$\begin{aligned}
&\delta \bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^h}\delta x^h, & & \frac{\delta}{\delta \bar{x}^i} = \frac{\partial x^h}{\partial \bar{x}^i}\frac{\delta}{\delta x^h},\\
&\delta \bar{t} = \delta t, & \text{and}\quad & \frac{\delta}{\delta \bar{t}} = \frac{\delta}{\delta t},\\
&\delta \dot{\bar{x}}^i = \frac{\partial \bar{x}^i}{\partial x^h}\delta \dot{x}^h & & \frac{\delta}{\delta \dot{\bar{x}}^i} = \frac{\partial x^h}{\partial \bar{x}^i}\frac{\delta}{\delta \dot{x}^h}.
\end{aligned}$$

The Lie brackets of the vector fields $\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right]$ and $\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta t}\right]$ are expressed by the formulas:

$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = R^h_{ij}\frac{\partial}{\partial \dot{x}^h}, \qquad \left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta t}\right] = R^h_i \frac{\partial}{\partial \dot{x}^h}, \tag{2.9}$$

where: $R^h_{ij} = \frac{\delta M^h_i}{\delta x^j} - \frac{\delta M^h_j}{\delta x^i}$, $R^h_i = \frac{\delta M^h_i}{\delta t} - \frac{\delta M^h}{\delta x^i}$. The geometrical object build by these components is called *tensor of curvature.* On a change of local chart, its components change by the rules:

$$\overline{R}^q_{hk} = \frac{\partial \bar{x}^q}{\partial x^p}\frac{\partial x^i}{\partial \bar{x}^h}\frac{\partial x^j}{\partial \bar{x}^k}R^p_{ij}, \quad \overline{R}^q_h = \frac{\partial \bar{x}^q}{\partial x^p}\frac{\partial x^i}{\partial \bar{x}^h}R^p_i.$$

2.5 Vertical and horizontal operators

Let us consider on $T(\mathbf{R} \times M)$, which is equivalent to $\mathbf{R} \times TM$, a vector field $\widetilde{X}$ locally written in a natural chart $\widetilde{X} = X^i \frac{\partial}{\partial x^i} + Y \frac{\partial}{\partial t} + Z^i \frac{\partial}{\partial \dot{x}^i}$.

In the adapted frame (2.8), $\widetilde{X}$ is written as:

$$\widetilde{X} = X^i \frac{\delta}{\delta x^i} + Y\frac{\delta}{\delta t} + (M^i_j X^j + M^i Y + Z^i)\frac{\delta}{\delta \dot{x}^i}, \tag{2.10}$$

and we obtain a vertical component $v\widetilde{X} = (M^i_j X^j + M^i Y + Z^i)\dfrac{\delta}{\delta \dot{x}^i}$ and a horizontal component $h\widetilde{X} = X^i \dfrac{\delta}{\delta x^i} + Y\dfrac{\delta}{\delta t}$ such that $\widetilde{X} = v\widetilde{X} + h\widetilde{X}$.

In the natural frame, these components become:

$$\begin{aligned} v\widetilde{X} &= (M^i_j X^j + M^i Y + Z^i)\frac{\partial}{\partial \dot{x}^i}, \\ h\widetilde{X} &= X^i \left(\frac{\partial}{\partial x^i} - M^j_i \frac{\partial}{\partial \dot{x}^j}\right) + Y\left(\frac{\partial}{\partial t} - M^j \frac{\partial}{\partial \dot{x}^j}\right). \end{aligned} \tag{2.11}$$

Proposition 2.3. *The following properties hold:*

$$vv\widetilde{X} = v\widetilde{X}, \quad hh\widetilde{X} = h\widetilde{X}, \quad vh\widetilde{X} = 0, \quad hv\widetilde{X} = 0.$$

3 First order dynamical systems

Given an implicit dynamical system, locally written by the equations:

$$F_i(t, x, \dot{x}) = 0, \quad i = \overline{1, m}, \tag{3.1}$$

with $det\left(\dfrac{\partial F_i}{\partial \dot{x}^j}\right) \neq 0$ and where the functions F_i change, on a change of local chart, by the rules: $\overline{F}_i = \dfrac{\partial x^h}{\partial \bar{x}^i} F_h$, we denote:

$$\frac{\partial F_i}{\partial \dot{x}^j} = \overset{1}{a}_{ij}, \quad \frac{\partial F_i}{\partial x^j} = \overset{0}{a}_{ij}, \quad \frac{\partial F_i}{\partial t} = a_i. \tag{3.2}$$

On the solutions ($F_i = 0$), these functions change by the rules (1.3), (1.4) and (1.2) respectively.

Thus, to the system (3.1) we can associate a "d-metric" of local components $\overset{1}{a}_{ij}$, a "nonlinear d-connection" of components $\overset{0}{a}_{ij}$ and a "d-1-form" of components a_i.

It follows that the dynamical system (3.1) defines on M a geometrical structure (a d-metric and a d-connection) ([5], [7]).

The functions a_i are the components of a d-1-form $\omega = a_i(t, x, \dot{x})d\dot{x}^i$. If the given system does not depend explicitly on t, we have $a_h = 0$.

These objects are defined on the configuration manifold M. This geometrical model corresponds to holonomic scleronomic dynamical systems. There exists a parallel transport of the vector fields along a curve, autoparallel curves such that they are solutions of the given system and conversely. If the system is rheonomic, the above defined geometrical structure does not correspond. The property that the solutions of the system are autoparallel curves holds no longer.

That is why we extend the geometrical objects (3.2) to $\mathbf{R} \times M$. Now, the number of equations does not correspond. It is necessary to add another equation to the system (3.1). We consider the equation: $F_0 = \dot{t} - 1 = 0$ and we have a new system:

$$\begin{aligned} &F_i(t, x, \dot{x}) = 0, \quad i = \overline{1, m}, \\ &F_0(t, x, \dot{x}) = \dot{t} - 1 = 0. \end{aligned} \tag{3.3}$$

3.1 Extended metric defined by the system

The extended matrix: $\begin{pmatrix} \dfrac{\partial F_i}{\partial \dot{x}^j} & \dfrac{\partial F_i}{\partial \dot{t}} \\ \dfrac{\partial F_0}{\partial \dot{x}^j} & \dfrac{\partial F_0}{\partial \dot{t}} \end{pmatrix} = \begin{pmatrix} a^1_{ij} & 0 \\ 0 & 1 \end{pmatrix}$ is nondegenerated. Thus, we can consider it as a metric on $\mathbf{R} \times M$.

3.2 Extended nonlinear connection defined by the system

The local coefficients of an extended d-connection associated to the system (3.3), called Christoffel symbols of the first type, are: $\begin{pmatrix} \dfrac{\partial F_i}{\partial x^j} & \dfrac{\partial F_i}{\partial t} \\ \dfrac{\partial F_0}{\partial x^j} & \dfrac{\partial F_0}{\partial t} \end{pmatrix} = \begin{pmatrix} a^0_{ij} & a_i \\ 0 & 0 \end{pmatrix}$.

The Christoffel symbols of the second type are obtained by:

$$\begin{pmatrix} M^i_j & M^i \\ M_j & M_0 \end{pmatrix} = \begin{pmatrix} a^{ih}_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^0_{hj} & a_h \\ 0 & 0 \end{pmatrix},$$

where $M^i_j = a^{ih}_1 a^0_{hj}$, $M^i = a^{ih}_1 a_h$ and $M_j = M_0 = 0$.

On a change of local chart, they change on the solutions ($F_i = 0$) by the rules (2.2).

Thus, we can build a covariant differential operator, we can define a parallel transport of vector fields and autoparallel curves.

We have:

Theorem 3.1. *Any solution of the dynamical system* (3.1) *is an autoparallel curve with respect to the associated geometrical structure and conversely.*

Proof. Let $x^i = c^i(t)$ be a solution of the system, that means $F_p(t, c(t), \dot{c}(t)) = 0$. The total derivative $\dfrac{dF_p}{dt} = 0$ is written as: $\dfrac{\partial F_p}{\partial \dot{x}^j}\dfrac{d\dot{x}^j}{dt} + \dfrac{\partial F_p}{\partial x^j}\dfrac{dx^j}{dt} + \dfrac{\partial F_p}{\partial t} = 0$ or, equivalent, $a^1_{pj}\dfrac{d^2x^j}{dt^2} + a^0_{pj}\dfrac{dx^j}{dt} + a_p = 0$. By multiplying and contracting with a^{ip}_1, it follows (2.5).

Conversely, by (2.5), we obtain $\dfrac{dF_p}{dt} = 0$, which lead us to the relations $F_p = C_p$ (const.), $\forall p = \overline{1, m}$. These equations represent a holonomic manifold and for the corresponding initial conditions ($C_p = 0$), it gives us the solutions of the system. □

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