Hypersurfaces of hyperbolic space with 1-type Gauss map

Uğur Dursun

Abstract. In this work we obtain the necessary and sufficient conditions on submanifolds of a pseudo-Euclidean space R_t^m to have 1-type Gauss map. Then we classify hypersurfaces of a hyperbolic space in Lorentz-Minkowski space R_1^m with at most two distinct principal curvatures and 1-type Gauss map.

M.S.C. 2010: 53C40, 53C50.
Key words: finite type map; Gauss map; hyperbolic space; principal curvature.

1 Introduction

In late 1970's B.Y. Chen introduced the notion of finite type submanifold of a Euclidean space. Since then the finite type submanifolds of Euclidean spaces or pseudo-Euclidean spaces have been studied by many geometers, and many interesting results have been obtained (see [2, 6, 7, 8, 14], etc.).

In [9], Chen and Piccinni extended the notion of finite type to differentiable maps, in particular, to Gauss map of submanifolds. A smooth map ϕ from a compact Riemannian manifold M into a Euclidean space $\mathbb{R}^m$ is said to be of *finite type* if ϕ can be expressed as a finite sum of $\mathbb{R}^m$-valued eigenfunctions of the Laplacian Δ of M, that is, $\phi = c + \sum_{i=1}^k \phi_i$, where c is a constant map, $\phi_1, \ldots, \phi_k$ are non-constant maps such that $\Delta\phi_i = \lambda_i\phi_i,\ \lambda_i \in \mathbb{R},\ i = 1, \ldots, k$. If $\lambda_1, \ldots, \lambda_k$ are different, then the map ϕ is said to be of k-type. If ϕ is an isometric immersion, then M is called a submanifold finite type (or of k-type) if ϕ does.

The notion of finite type Gauss map is especially a useful tool in the study of submanifolds, and many authors studied submanifolds with finite type Gauss map ([3, 4, 5, 9, 10], etc.). Chen and Piccinni characterized and classified compact hypersurfaces with 1-type Gauss map. Also, they proved that a compact surface M in a Euclidean space $\mathbb{R}^m$ has 1-type Gauss map if and only if M is one of the surfaces: a sphere $\mathbb{S}^2(r) \subset \mathbb{R}^3 \subset \mathbb{R}^m$ or the product of two plane circles $\mathbb{S}^1(a) \times \mathbb{S}^1(b) \subset \mathbb{R}^4 \subset \mathbb{R}^m$.

Later, in [13] Choi, Ki and Suh studied space-like surfaces in pseudo-Euclidean spaces $\mathbb{R}_p^{2+p}$. They proved that the only space-like surfaces in $\mathbb{R}_p^{2+p}$ with 1-type Gauss

BSG Proceedings 18. The Int. Conf. of Diff. Geom. and Dynamical Systems (DGDS-2010), October 8-11, 2010, Bucharest Romania, pp. 47-55.

map are locally the surfaces: the Euclidean space $\mathbb{R}^2$, the hyperbolic cylinder $\mathbb{H}^1 \times \mathbb{R}$ in $\mathbb{R}^3_1$ and the product $\mathbb{H}^1 \times \mathbb{H}^1$ of two hyperbolic curves in $\mathbb{R}^4_2 \subset \mathbb{R}^{2+p}_p$.

In this work we study hypersurfaces of the hyperbolic space $\mathbb{H}^{n+1}(-1) \subset \mathbb{R}^{n+2}_1$ with 1-type Gauss map. We mainly prove that if M is a hypersurface of $\mathbb{H}^{n+1}(-1) \subset \mathbb{R}^{n+2}_1$ with at most two distinct principal curvatures, then the Gauss map of M is of 1-type if and only if M is one of the hypersurfaces: $\mathbb{H}^n(-c) \subset \mathbb{H}^{n+1}(-1)$ with $0 < c \leq 1$, $\mathbb{S}^n(c) \subset \mathbb{H}^{n+1}(-1)$ with $c > 0$ or $\mathbb{H}^k(-b) \times \mathbb{S}^{n-k}(a) \subset \mathbb{H}^{n+1}(-1)$ with $1/a - 1/b = -1$.

2 Preliminaries

Let $\mathbb{R}^m_t$ be an m-dimensional pseudo-Euclidean space with an inner product of signature $(t, m-t)$ given by

$$\langle x, x\rangle = -\sum_{i=1}^{t} x_i^2 + \sum_{i=t+1}^{m} x_i^2,$$

where $(x_1, \ldots, x_m)$ is the natural coordinate of $\mathbb{R}^m_t$. We define pseudo-Riemannian manifolds $\mathbb{S}^n_t(c)$ and $\mathbb{H}^n_t(-c)$ as follows:

$$\mathbb{S}^n_t(c) = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}_t | -\sum_{i=1}^{t} x_i^2 + \sum_{i=t+1}^{n+1} x_i^2 = \frac{1}{c}\},\ (c > 0),$$

$$\mathbb{H}^n_t(-c) = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}_{t+1} | -\sum_{i=1}^{t+1} x_i^2 + \sum_{i=t+2}^{n+1} x_i^2 = -\frac{1}{c}\},\ (c > 0).$$

These spaces are complete and of constant curvature c and $-c$, respectively. $\mathbb{S}^n_t(c)$ and $\mathbb{H}^n_t(-c)$ are called a pseudo-sphere and a pseudo-hyperbolic space, respectively. Also, for $t = 0$ and $x_1 > 0$, $\mathbb{H}^n(-c) = \mathbb{H}^n_0(-c)$ is called a hyperbolic space of $\mathbb{R}^{n+1}_1$.

Let M_q be an n-dimensional pseudo-Riemannian submanifold with index q in an m-dimensional pseudo-Euclidean space $\mathbb{R}^m_t$. Let $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$ be an adapted local orthonormal frame on M_q in $\mathbb{R}^m_t$ such that $\langle e_A, e_B\rangle = \varepsilon_B \delta_{AB}$, ($\varepsilon_B = \langle e_B, e_B\rangle = \pm 1$), $e_1, \ldots, e_n$ are tangent to M_q, and $e_{n+1}, \ldots, e_m$ are normal to M_q. We use the following convention on the range of indices:

$$1 \leq A, B, C, \ldots \leq m, \quad 1 \leq i, j, k, \ldots \leq n, \quad n+1 \leq \beta, \gamma, \theta, \ldots \leq m.$$

Let $\widetilde{\nabla}$ be the Levi-Civita connection of $\mathbb{R}^m_t$ and ∇ the induced connection on M_q. Denote by $\{\omega^1, \ldots, \omega^m\}$ the dual frame and by $\{\omega_{AB}\}$ with $\omega_{AB} + \omega_{BA} = 0$ the connection forms associated to $\{e_1, \ldots, e_m\}$. Then we have the formulas of Gauss and Weingarten, respectively, as

$$\widetilde{\nabla}_{e_k} e_i = \sum_{j=1}^{n} \varepsilon_j \omega_{ij}(e_k) e_j + \sum_{\beta=n+1}^{m} \varepsilon_\beta \omega_{i\beta}(e_k) e_\beta$$

and

$$\widetilde{\nabla}_{e_k} e_\beta = -A_\beta(e_k) + \sum_{\nu=n+1}^{m} \varepsilon_\nu \omega_{\beta\nu}(e_k) e_\nu.$$

From the Cartan's Lemma we have $\omega_{i\beta} = \sum_{j=1}^{n} \varepsilon_j h^{\beta}_{ij}\omega_j$ with $h^{\beta}_{ij} = h^{\beta}_{ji}$, where h^{β}_{ij}'s are the coefficients of the second fundamental form h, and A_β the Weingarten map in direction e_β.

The mean curvature vector H and the squared length $\|h\|^2$ of the second fundamental form h are defined, respectively, by

$$H = \frac{1}{n}\sum_{\beta,i} \varepsilon_\beta \varepsilon_i h^{\beta}_{ii} e_\beta \quad \text{and} \quad \|h\|^2 = \sum_{\beta,i,j} \varepsilon_\beta \varepsilon_i \varepsilon_i h^{\beta}_{ij} h^{\beta}_{ji}.$$

The scalar curvature S of M_q is given by

$$S = n^2|H|^2 - \|h\|^2, \tag{2.1}$$

where $|H|^2$ is the squared length of the mean curvature vector H of M_q in $\mathbb{R}^m_t$.

The Codazzi equation of M_q in $\mathbb{R}^m_t$ is given by

$$\begin{aligned} &h^{\beta}_{ij,k} = h^{\beta}_{jk,i}, \\ &h^{\beta}_{jk,i} = e_i(h^{\beta}_{jk}) - \sum_{\ell=1}^{n} \varepsilon_\ell \left(h^{\beta}_{j\ell}\omega_{k\ell}(e_i) + h^{\beta}_{k\ell}\omega_{j\ell}(e_i)\right) + \sum_{\nu=n+1}^{m} \varepsilon_\nu h^{\nu}_{jk}\omega_{\nu\beta}(e_i). \end{aligned} \tag{2.2}$$

Also, from the Ricci equation of M_q in $\mathbb{R}^m_t$ we have

$$R^{\perp}(e_j, e_k; e_\theta, e_\beta) = \langle [A_{e_\theta}, A_{e_\beta}](e_j), e_k\rangle = \sum_{i=1}^{n} \varepsilon_i \left(h^{\theta}_{ki} h^{\beta}_{ij} - h^{\theta}_{ji} h^{\beta}_{ik}\right), \tag{2.3}$$

where $R^{\perp}$ is the normal curvature tensor.

A submanifold M_q of a Euclidean space $\mathbb{R}^m$ or a pseudo-Euclidean space $\mathbb{R}^m_t$ is said to be of *finite type* if its position vector x can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M_q, that is,

$$x = c + x_1 + \cdots + x_k,$$

where c is a constant map, $x_1, \ldots, x_k$ are non-constant maps satisfying $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, k$. If $\lambda_1, \ldots, \lambda_k$ are different, then M_q is said to be of k-type. It is known that if P is a polynomial defined by $P(t) = \prod_{i=1}^{k}(t-\lambda_k)$, then $P(\Delta)(x-c) = 0$. Conversely, if M is compact and if there exist a constant vector c and a non-trivial polynomial P such that $P(\Delta)(x - c) = 0$, then M is of finite type [6, pp.255-258].

If M is not compact, then the existence of a non-trivial polynomial P such that $P(\Delta)(x - c) = 0$ does not imply that M is of finite type in general. However, in [11] it was proved that the existence of such a polynomial P guaranties that M is of finite type when either M is of 1-dimensional or the polynomial P has exactly k distinct roots, where $k = \deg P$. In [12], Chen studied non-compact finite type pseudo-Riemannian submanifolds of pseudo-Euclidean spaces. He gave the following definition for the finite type submanifolds of the pseudo-Riemannian sphere $\mathbb{S}^{m-1}_t(r)$ or the pseudo-hyperbolic space $\mathbb{H}^{m-1}_{t-1}(-r)$.

Definition 2.1. An n-dimensional pseudo-Riemannian submanifold M_q of a pseudo-Riemannian sphere $\mathbb{S}^{m-1}_t(r)$ (resp., of a pseudo-hyperbolic space $\mathbb{H}^{m-1}_{t-1}(-r)$) is called of k-type in $\mathbb{S}^{m-1}_t(r)$ (resp., in $\mathbb{H}^{m-1}_{t-1}(-r)$) if the position vector x of M_q in $\mathbb{R}^m_t$ has the following form:

$$x = x_1 + \cdots + x_k, \quad \Delta x_i = \lambda_i x_i, \ \lambda_i \in \mathbb{R}, i = 1, \ldots, k$$

such that $\lambda_1, \ldots, \lambda_k$ are distinct.

Similarly we give

Definition 2.2. A smooth map $\phi : M_q \to \mathbb{S}^{m-1}_t(r) \subset \mathbb{R}^m_t$ (resp., $\phi : M_q \to \mathbb{H}^{m-1}_{t-1}(-r) \subset \mathbb{R}^m_t$) from a pseudo-Riemannian manifold M_q into a pseudo-Riemannian sphere $\mathbb{S}^{m-1}_t(r)$ (resp., into a pseudo-hyperbolic space $\mathbb{H}^{m-1}_{t-1}(-r)$) is called of k-type in $\mathbb{S}^{m-1}_t(r)$ (resp., in $\mathbb{H}^{m-1}_{t-1}(-r)$) if the map ϕ has the following form:

$$\phi = \phi_1 + \cdots + \phi_k, \quad \Delta \phi_i = \lambda_i \phi_i, \ \lambda_i \in \mathbb{R}, i = 1, \ldots, k$$

such that $\lambda_1, \ldots, \lambda_k$ are distinct.

The following proposition was given in [11].

Proposition 2.1. *Let $x : M \to \mathbb{R}^m$ be an immersion. If there exist a constant vector $c \in \mathbb{R}^m$ and a polynomial $P(t) = \prod_{i=1}^k (t - \lambda_k)$ with mutually distinct $\lambda_1, \ldots, \lambda_k$ such that $P(\Delta)(x - c) = 0$, then M is finite type.*

This proposition remains true if M is a pseudo-Riemannian submanifold of a pseudo-Euclidean space, [11]. For smooth maps we have the following result analogous to Proposition 2.1 whose proof is the same as that of Proposition 2.1.

Proposition 2.2. *Let $\phi : M_q \to \mathbb{R}^m_t$ be a smooth map from a pseudo-Riemannian manifold M_q into a pseudo-Euclidean space $\mathbb{R}^m_t$. If there exist a constant vector $c \in \mathbb{R}^m_t$ and a polynomial $P(t) = \prod_{i=1}^k (t - \lambda_k)$ with mutually distinct root $\lambda_1, \ldots, \lambda_k$ such that $P(\Delta)(\phi - c) = 0$, then the map ϕ is of finite type.*

By Definition 2.2 and Proposition 2.2 we can state

Corollary 2.3. *Let $\phi : M_q \to \mathbb{S}^{m-1}_t(r) \subset \mathbb{R}^m_t$ (resp., $\phi : M_q \to \mathbb{H}^{m-1}_{t-1}(-r) \subset \mathbb{R}^m_t$) be a smooth map from a pseudo-Riemannian manifold M_q into a pseudo-Riemannian sphere $\mathbb{S}^{m-1}_t(r)$ (resp., into a pseudo-hyperbolic space $\mathbb{H}^{m-1}_{t-1}(-r)$). Then the map ϕ is of finite type if there exists a polynomial $P(t) = \prod_{i=1}^k (t - \lambda_k)$ with mutually distinct roots $\lambda_1, \ldots, \lambda_k$ such that $P(\Delta)(\phi) = 0$. In particular, ϕ is of 1-type if and only if $\Delta\phi = \lambda\phi$ for some real number λ.*

3 The Gauss Map

The definition of Gauss map of a pseudo-Riemannian submanifold of pseudo-Euclidean space was given in [15] as follows. Let $G(n, m)$ be the Grassmannian manifold consisting of all oriented n-planes through the origin of $\mathbb{R}^m_t$ and $\bigwedge^n \mathbb{R}^m_t$ the vector space obtained by the exterior product of n vectors in $\mathbb{R}^m_t$. Let $f_{i_1} \wedge \cdots \wedge f_{i_n}$ and $g_{i_1} \wedge \cdots \wedge g_{i_n}$ be two vectors in $\bigwedge^n \mathbb{R}^m_t$, where $\{f_1, f_2, \ldots, f_m\}$ and $\{g_1, g_2, \ldots, g_m\}$ are two orthonormal bases of $\mathbb{R}^m_t$. Define an indefinite inner product $\langle , \rangle$ on $\bigwedge^n \mathbb{R}^m_t$ by

$$\langle f_{i_1} \wedge \cdots \wedge f_{i_n}, g_{i_1} \wedge \cdots \wedge g_{i_n} \rangle = \det(\langle f_{i_\ell}, g_{j_k} \rangle). \tag{3.1}$$

Therefore, for some positive integer s, we may identify $\bigwedge^n \mathbb{R}^m_t$ with some pseudo-Euclidean space $\mathbb{R}^N_s$, where $N = \binom{m}{n}$. Let $e_1, \dots, e_n, e_{n+1}, \dots, e_m$ be an oriented local orthonormal frame on M_q in $\mathbb{R}^m_t$ with $\varepsilon_B = \langle e_B, e_B \rangle = \pm 1$ such that $e_1, \dots, e_n$ are tangent to M_q and $e_{n+1}, \dots, e_m$ are normal to M_q. The map $\nu : M_q \to G(n,m) \subset \mathbb{R}^N_s$ from an oriented pseudo-Riemannian submanifold M_q into $G(n,m)$ defined by

$$\nu(p) = (e_1 \wedge e_2 \wedge \cdots \wedge e_n)(p)$$

is called the *Gauss map* of M_q that is a smooth map which assigns to a point p in M_q the oriented n-plane through the origin of $\mathbb{R}^m_t$ and parallel to the tangent space of M_q at p. We put $\varepsilon = \langle \nu, \nu \rangle = \varepsilon_1 \varepsilon_2 \cdots \varepsilon_n = \pm 1$ and

$$\widetilde{M}^{N-1}_s(\varepsilon) = \begin{cases} \mathbb{S}^{N-1}_s(1) & \text{in } \mathbb{R}^N_s \quad \text{if } \varepsilon = 1, \\ \mathbb{H}^{N-1}_{s-1}(-1) & \text{in } \mathbb{R}^N_s \quad \text{if } \varepsilon = -1. \end{cases}$$

Then the Gauss image $\nu(M_q)$ can be viewed as $\nu(M_q) \subset \widetilde{M}^{N-1}_s(\varepsilon)$.

Considering ν as an $\mathbb{R}^N_s$-valued function on M_q we have

$$e_i \nu = \sum_{j=1}^{n} e_1 \wedge \cdots \wedge \widetilde{\nabla}_{e_i} e_j \wedge \cdots \wedge e_n = \sum_{j,\beta} \varepsilon_\beta h^\beta_{ij}\, e_1 \wedge \cdots \wedge \overbrace{e_\beta}^{j-th} \wedge \cdots \wedge e_n. \tag{3.2}$$

Using $\nabla_{e_k} e_i = \sum_{j=1}^n \varepsilon_j \omega_{ij}(e_k) e_j$ we have

$$(\nabla_{e_i} e_i)\nu = \sum_{j,k,\beta} \varepsilon_k \varepsilon_\beta h^\beta_{kj} \omega_{ik}(e_i)\, e_1 \wedge \cdots \wedge \overbrace{e_\beta}^{j-th} \wedge \cdots \wedge e_n. \tag{3.3}$$

Since

$$\Delta \nu = \sum_{i=1}^{n} \varepsilon_i (\nabla_{e_i} e_i - e_i e_i)\nu,$$

considering the Codazzi equation (2.2) we obtain by a direct calculation that

$$\begin{aligned} \Delta \nu =& \|h\|^2 \nu - n \sum_{j=1}^{n} e_1 \wedge \cdots \wedge \overbrace{\nabla^\perp_{e_j} H}^{j-th} \wedge \cdots \wedge e_n \\ & - \sum_{i,j,\ell,\theta,\beta} \varepsilon_\beta \varepsilon_\theta \varepsilon_i h^\theta_{\ell i} h^\beta_{ij}\, e_1 \wedge \cdots \wedge \overbrace{e_\theta}^{\ell-th} \wedge \cdots \wedge \overbrace{e_\beta}^{j-th} \wedge \cdots \wedge e_n, \end{aligned} \tag{3.4}$$

where $\nabla^\perp$ is the normal connection induced on M_q. Hence, from (2.3) and (3.4) we have

Lemma 3.1. *Let M_q be an n-dimensional oriented pseudo-Riemannian submanifold of an m-dimensional pseudo-Euclidean space $\mathbb{R}^m_t$. Then the Laplacian of the Gauss* *map* $\nu : M_q \to G(n,m) \subset \widetilde{M}_s^{N-1}(\varepsilon) \subset \mathbb{R}_s^N$ *is given by*

$$\begin{aligned} \Delta\nu = \|h\|^2\nu - n\sum_{j=1}^{n} e_1 \wedge \cdots \wedge \overbrace{\nabla_{e_j}^{\perp} H}^{j-th} \wedge \cdots \wedge e_n \\ + \sum_{i,j;\theta<\beta} \varepsilon_\theta\varepsilon_\beta R^{\perp}(e_i, e_j; e_\theta, e_\beta) e_1 \wedge \cdots \wedge \overbrace{e_\theta}^{i-th} \wedge \cdots \wedge \overbrace{e_\beta}^{j-th} \wedge \cdots \wedge e_n. \end{aligned} \tag{3.5}$$

4 Hypersurfaces of Hyperbolic Space with 1-Type Gauss Map

We classify hypersurfaces of a hyperbolic space in the Lorentz-Minkowski space R_1^m with at most two distinct principal curvatures and 1-type Gauss map.

Theorem 4.1. *Let* M_q *be an n-dimensional oriented pseudo-Riemannian submanifold of an m-dimensional pseudo-Euclidean space* $\mathbb{R}_t^m$. *Then the Gauss map* $\nu : M_q \to \widetilde{M}_s^{N-1}(\varepsilon) \subset \mathbb{R}_s^N$, $N = \binom{m}{n}$, *is of 1-type if and only if* M_q *has constant scalar curvature, parallel mean curvature vector, and flat normal connection.*

Proof. Suppose that the Gauss map is of 1-type, that is, from Corollary 2.3 $\Delta\nu = \lambda\nu$ for some $\lambda \in \mathbb{R}$. It follows from (3.5) that $\|h\|^2 = \lambda$ is a constant, $R^{\perp}(e_i, e_j; e_\theta, e_\beta) = 0$ and $\nabla^{\perp} H = 0$. So, the normal connection is flat and the mean curvature vector is parallel which implies that the mean curvature is constant. Hence, from (2.1) the scalar curvature is constant.

The converse follows from (2.1) and (3.5). □

For example, all n-dimensional isoparametric hypersurfaces of $\mathbb{R}_t^{n+1}$ have 1-type Gauss map.

Corollary 4.2. *Let* M *be an oriented hypersurface of the hyperbolic space* $\mathbb{H}^{n+1}(-1) \subset \mathbb{R}_1^{n+2}$. *Then the Gauss map* ν *of* M *is of 1-type if and only if* M *has constant scalar curvature and constant mean curvature.*

Proof. Let $\{e_1, \ldots, e_{n+2}\}$ be an orthonormal local moving frame on M in $\mathbb{R}_1^{n+2}$ such that $e_1, \ldots, e_n$ are tangent to M, and $e_{n+1}, e_{n+2} = x$ are normal to M, where x is the position vector of M in $\mathbb{R}_1^{n+2}$. Since x is parallel in the normal bundle and the codimension of M is two, then e_{n+1} is also parallel, and thus the normal connection is flat. Therefore the proof follows from Theorem 4.1. □

For example, since isoparametric hypersurfaces of a hyperbolic space have constant mean curvature and constant scalar curvature, then all such hypersurfaces of a hyperbolic space have 1-type Gauss map by Corollary 4.2.

Now we consider the following hypersurfaces of $\mathbb{H}^{n+1}(-1)$:

$$\mathbb{H}^n(-c) = \{x \in \mathbb{H}^{n+1}(-1) \,|\, x_1 > 0,\ x_{n+2} = \sqrt{-1 + \frac{1}{c}}\ \},\ (0 < c \le 1),$$

$$\mathbb{S}^n(c) = \{x \in \mathbb{H}^{n+1}(-1) \,|\, x_1 = \sqrt{1 + \frac{1}{c}}\ \},\ (c > 0)$$

and

$$\mathbb{H}^k(-b)\times\mathbb{S}^{n-k}(a)=\{x\in\mathbb{H}^{n+1}(-1)\mid -x_1^2+\sum_{i=2}^{k+1}x_i^2=-\frac{1}{b},\ \sum_{i=k+2}^{n+2}x_i^2=\frac{1}{a},\quad x_1>0\},$$

where $1/a-1/b=-1$. Then we give the following classification theorem.

Theorem 4.3. *Let M be an oriented hypersurface of the hyperbolic space $\mathbb{H}^{n+1}(-1)\subset\mathbb{R}_1^{n+2}$ with at most two distinct principal curvatures. Then, the Gauss map $\nu: M\to\mathbb{S}_s^{N-1}(1)\subset\mathbb{R}_s^N$, $(N=(n+1)(n+2)/2,\ s=n(n+1)/2)$, is of 1-type if and only if M is congruent to one of the followings:*
1) $\mathbb{H}^n(-c)\subset\mathbb{H}^{n+1}(-1)$ with $0<c\le 1$,
2) $\mathbb{S}^n(c)\subset\mathbb{H}^{n+1}(-1)$ with $c>0$ or
3) $\mathbb{H}^k(-b)\times\mathbb{S}^{n-k}(a)\subset\mathbb{H}^{n+1}(-1)$ with $1/a-1/b=-1$.

Proof. Let M be one of $\mathbb{H}^n(-c)\subset\mathbb{H}^{n+1}(-1)$ with $0<c\le 1$, $\mathbb{S}^n(c)\subset\mathbb{H}^{n+1}(-1)$ with $c>0$ or $\mathbb{H}^{n-k}(-b)\times\mathbb{S}^k(a)\subset\mathbb{H}^{n+1}(-1)$ with $1/a-1/b=-1$. The principle curvatures in the unit normal direction of M in $\mathbb{H}^{n+1}(-1)$ are constant, and the number of distinct principle curvatures are at most two. Then, the Gauss map ν of M is of 1-type by Corollary 4.2.

Conversely, let M be a hypersurface in $\mathbb{H}^{n+1}(-1)\subset\mathbb{R}_1^{n+2}$ with 1-type Gauss map ν. Then M has constant mean curvature and constant scalar curvature by Corollary 4.2. Let $\{e_1,e_2,\dots,e_{n+2}\}$ be an orthonormal local moving frame on M in $\mathbb{R}_1^{n+2}$ such that $e_1,e_2,\dots,e_n$ are tangent to M, $e_{n+1},e_{n+2}=x$ are normal to M, and also $A_{e_{n+1}}(e_i)=\mu_i e_i,\ i=1,\dots,n$, where x denote the position vector of M in $\mathbb{R}_1^{n+2}$. Since M has at most two distinct principal curvature, we put $\mu_1=\cdots=\mu_k=\mu$ and $\mu_{k+1}=\cdots=\mu_n=\lambda$ for $0\le k<n$. Hence, the mean curvature vector H of M in $\mathbb{R}_1^{n+2}$ is $H=\alpha' e_{n+1}+x$, where $\alpha'=(k\mu+(n-k)\lambda)/n$ is the mean curvature of M in $\mathbb{H}^{n+1}(-1)$.

The mean curvature $|H|$ of M in $\mathbb{R}_1^{n+2}$ is constant if and only if $\alpha'=(k\mu+(n-k)\lambda)/n$ is constant. Also, the square of the length of the second fundamental form is $\|h\|^2=n+k\mu^2+(n-k)\lambda^2$ which is constant because of (2.1). Let X be a vector on M. From $X(|H|^2)=0$ and $X(\|h\|^2)=0$ we obtain

$$kX(\mu)+(n-k)X(\lambda)=0\quad\text{and}\quad k\mu X(\mu)+(n-k)\lambda X(\lambda)=0$$

which gives $(\mu-\lambda)X(\mu)=0$. So, $\mu-\lambda=0$ or $X(\mu)=0$ with $\mu-\lambda\neq 0$.

Case 1. $\mu=\lambda$. Clearly, μ and λ are constants as $\alpha'=(k\mu+(n-k)\lambda)/n$ is constant. The hypersurface M is totally umbilical in $\mathbb{H}^{n+1}(-1)$, i.e., $A_{e_{n+1}}=\mu I$, where I denotes the identity transformation. From the Gauss equation it is easily seen that M is of constant curvature $-1+\mu^2$. By Theorem 5.1 in [1], M is locally congruent to $\mathbb{H}^n(-c)\subset\mathbb{H}^{n+1}(-1)$ with the curvature $-c$, $0<c\le 1$ or $\mathbb{S}^n(c)\subset\mathbb{H}^{n+1}(-1)$ with the curvature $c>0$.

Case 2. $X(\mu)=0$ with $\mu-\lambda\neq 0$. Hence, μ and λ are constants. Again, by Theorem 5.1 in [1], M is locally congruent to $\mathbb{H}^k(-b)\times\mathbb{S}^{n-k}(a)\subset\mathbb{H}^{n+1}(-1)$ with $1/a-1/b=-1$, where $-b=-1+\mu^2$, $(0<b\le 1)$, $a=-1+\lambda^2$, $(a>0)$ and $\mu\lambda=1$. $\square$

We conclude that

Corollary 4.4. *The totally geodesic hypersurface* $\mathbb{H}^n(-1)$ *of* $\mathbb{H}^{n+1}(-1) \subset \mathbb{R}^{n+2}_1$ *is the only minimal hypersurface of* $\mathbb{H}^{n+1}(-1)$ *with at most two distinct principal curvatures and 1-type Gauss map.*

From Corollary 4.4, the totally geodesic surface $\mathbb{H}^2(-1)$ of $\mathbb{H}^3(-1) \subset \mathbb{R}^4_1$ is the only minimal surface of $\mathbb{H}^3(-1)$ with 1-type Gauss map.

Corollary 4.5. *An oriented surface* M *of the hyperbolic space* $\mathbb{H}^3(-1) \subset \mathbb{R}^4_1$ *has 1-type Gauss map* $\nu : M \to \mathbb{S}^5_3(1) \subset \mathbb{R}^6_3$ *if and only if* M *is congruent to one of the followings:* $\mathbb{H}^2(-c) \subset \mathbb{H}^3(-1)$ *with* $0 < c \leq 1$, $\mathbb{S}^2(c) \subset \mathbb{H}^3(-1)$ *with* $c > 0$ *or* $\mathbb{H}^1(-b) \times \mathbb{S}^1(a) \subset \mathbb{H}^3(-1)$ *with* $1/a - 1/b = -1$.

References

[1] N. Abe, N. Koike and S. Yamaguchi, *Congruence theorems for proper semi-Riemannian hypersurfaces in a real space forms*, Yokohama Math. J., 35 (1987), 123-136.

[2] A. Arvanitoyeorgos, G. Kaimakamis and M. Magid, *Lorentz hypersurfaces in* E^4_1 *satisfying* $\Delta\vec{H} = \alpha\vec{H}$, Illinois J. Math., 53 (2009), 581-590.

[3] C. Baikoussis and D.E. Blair, *On the Gauss map of ruled surfaces*, Glasgow Math.J., 34 (1992), 355–359.

[4] C. Baikoussis, B.Y. Chen and L. Verstraelen, *Ruled surfaces and tubes with finite type Gauss map*, Tokyo J. Math., 16 (1993), 341–348.

[5] C. Baikoussis, *Ruled submanifolds with finite type Gauss map*, J. Geom., 49 (1994), 42–45.

[6] B.Y. Chen, *Total Mean Curvature and Submanifolds of Finite Type*, World Scientific, Singapore-New Jersey-London, 1984.

[7] B.Y. Chen, *Finite type submanifolds in pseudo-Euclidean spaces and its applications*, Kodai Math. J., 8 (1985), 358-374.

[8] B.Y. Chen, *A report on submanifolds of finite type*, Soochow J. Math., 22 (1996), 117-337.

[9] B.Y. Chen and P. Piccinni, *Submanifolds with finite type Gauss map*, Bull. Austral. Math. Soc., 35 (1987), 161–186.

[10] B.Y. Chen and S.J. Li, *Spherical hypersurfaces with 2-type Gauss map* Beiträge zur Algebra und Geometrie/ Contributions to Algebra and Geometry, 39 (1998), 169-179.

[11] B.Y. Chen and M. Petrovic, *On spectral decomposition of immersions of finite type*, Bull. Austral. Math. Soc., 44 (1991), 117-129.

[12] B.Y. Chen, *Finite-type pseudo-Riemannian submanifolds*, Tamkang J. of Math., 17 (1986), 137-151.

[13] S.M. Choi, U-H. Ki and Y.J. Suh, *Space-like surfaces with 1-type Gauss map*, J. Korean Math. Soc., 35 (1998), 315-330.

[14] U. Dursun, *Null 2-type space-like submanifolds of* E^5_t *with normalized parallel mean curvature vector*, Balkan J. Geom. Appl., 11 (2006), 61–72.

[15] Y.H. Kim and D.W. Yoon, *Classification of rotation surfaces in pseudo-Euclidean space*, J. Korean Math., 41 (2004), 379-396.

[16] S. Shu, *Complete hypersurfaces in a hyperbolic space*, Differ. Geom. Dyn. Syst., 11 (2009), 166-174.

[17] B. Wu, *On complete hypersurfaces with two distinct principal curvatures in a hyperbolic space*, Balkan J. Geom. Appl., 15 (2010), 126-137.

Author's address:

Uğur Dursun
Istanbul Technical University, Faculty of Science and Letters
Department of Mathematics, 34469 Maslak, Istanbul, Turkey.
E-mail: udursun@itu.edu.tr