# On a theorem of Walkup-Wets

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**Abstract.** The Walkup-Wets theorem involves the continuity of polarity and the bicontinuity of the Legendre-Fenchel transform by reports to the Attouch-Wets topology. Some similar results for bornological convergence are proposed. When $\mathfrak{B}$ is a bornology on $X$, a bornology $\mathfrak{B}^\star$ on $X'$ is associated such that if $(A)_{n\in\mathbb{N}}$ is a sequence of closed convex cones $\mathfrak{B}-$convergent at $A$ then $(A^\circ)_{n\in\mathbb{N}}$ is $\mathfrak{B}^\star-$convergent at $A^\circ$.

**M.S.C. 2010**: 54B20, 54E35.
**Key words**: convex cones, Walkup-Wets theorem, bornologies.

## 1 Notations and preliminary results

Throughout this paper $X$ is a normed linear space with its topological dual $X'$. The closed unit ball of $X$ (resp. $X'$) is denoted by $U$ (resp. $U^\star$). $C(X)$ denotes the family of all nonempty convex closed subsets of $X$. If $A$ and $C$ are nonempty subsets of $X$, we define the excess of $A$ over $C$ by:

$$e(A, C) = \sup\{d(x, C), x \in A\},$$

and

$$haus_r(A, C) = \max\{e(A \cap rU, C), e(C \cap rU, A)\}, \quad r > 0.$$

It is well-known that the family of pseudodistances $(haus_r(., .))_{r>0}$ defines a topology on $C(X)$ called the Attouch-Wets topology called elsewhere the bounded-Hausdorff topology (see [1, 2]). The polar $A^\circ$ of a nonempty subset $A$ of $X$, is the subset of $X'$ defined by $A^\circ = \{y \in X', \langle y, x\rangle \leq 1, \forall x \in A\}$. Similarly, if $E$ is a nonempty subset of $X'$ then its polar is the subset of $X$ defined by $E^\circ = \{x \in X, \langle y, x\rangle \leq 1, \forall y \in E\}$. For $x \in X$, we let $T_x = \{\lambda x, \lambda \geq 0\}$ and we call this set the ray emanating from 0 passing through $x$. For $y \in X'$, the half-space in $X$ is denoted by $E_y$ and defined by $E_y = \{x \in X, \langle y, x\rangle \leq 0\}$. Similarly, for $x \in X$, we denote by $E_x$ the half-space in $X'$ limited by the continuous linear functional $x$. Then, when we take polars, the ray becomes a half-space, and the half-space becomes ray ( $(T_x)^\circ = E_x$ et $(E_y)^\circ = T_y$).

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BSG Proceedings 18. The Int. Conf. of Diff. Geom. and Dynamical Systems (DGDS-2010), October 8-11, 2010, Bucharest Romania, pp. 56-59.

The Legendre-Fenchel transform of an extended real valued function $f$ on $X$ is the function $f^\star$ defined on $X'$ with values in $[-\infty, +\infty]$ as

$$y \in X' \longmapsto f^\star(y) = \sup_{x \in X}\{\langle y, x\rangle - f(x)\}.$$

The new function $f^\star$ is automatically convex lower semi-continuous on $X'$.

Recall the well-known Ascoli's formula: Let $H = y^{-1}\{\alpha\}$ be a closed hyperplane in $X$, where $y$ is a nonzero element of $X'$ and $\alpha \in \mathbb{R}$. Then for each $x \in X$, we have

$$d(x, H) = \frac{|\langle y, x\rangle - \alpha|}{\|y\|}.$$

Let $\mathcal{S}$ be a family of nonempty subsets of $X$. A net of subsets $(A_\lambda)_\lambda$ of $X$ is said to be $\mathcal{S}$-convergent to a subset $A$ of $X$ if for every $S \in \mathcal{S}$ and every $\epsilon > 0$, we have $A \cap S \subset A_\lambda^\epsilon$ and $A_\lambda \cap S \subset A^\epsilon$.

No finer convergence results if we replace $\mathcal{S}$ by the family of all subsets of finite unions of members of $\mathcal{S}$. Therefore, we will only consider ideals, i.e., families of nonempty subsets of $X$ that is hereditary and is closed under finite unions. When an ideal $\mathcal{S}$ is also a cover then it is called a bornology. *In the sequel we will consider bornologies* $\mathfrak{B}$ *of type* $\mathcal{H}$ *such that* $\forall B \in \mathfrak{B}$ *and* $\forall \lambda > 0$ *we have* $\lambda B \in \mathfrak{B}$. For example: The d-bounded subsets, the d-totally bounded sets, the finite subsets, the subsets of a normed linear space with weakly compact closure, the weak* bounded subsets of a dual space $X', \ldots$ are bornologies of type $\mathcal{H}$.

Bornologies have appear to deal with large phenomena:

1) Geometry of Banach spaces [5] focusing on the bornology of metrically bounded sets, the bornology of finite sets, and the bornology of subsets of weakly compact sets.
2) Beginning with the seminal paper of Lechicki, Levi and Spakowski [6], several authors have studied a generalization of Attouch-Wets convergence - a central object in convex analysis [1, 2]

Let us recall the Walkup-Wets theorem [2] which is in the origin of this paper.

**Theorem 1.1.** *(Walkup-Wets ) Let* $X$ *be a normed linear space and let* $A$ *and* $C$ *be two closed convex cones in* $X$*. Then* $haus_1(A, C) = haus_1(A^\circ, C^\circ)$.

This theorem implies that if $(A_\lambda)_{\lambda>0}, A$ is a net of convex cones then $A = aw - \lim A_\lambda$ if and only if $A^\circ = aw - \lim A_\lambda{}^\circ$. The Attouch-Wets-topology in $X$ or in $X'$ is associated with the bornology formed by the bounded subsets of $X$ or $X'$.

## 2 Main result

For $A \subset X$, put $A^\star = \{y \in X', \exists x \in A, \|y\| = \|x\|\}$.

**Proposition 2.1.** *If* $\mathfrak{B}$ *is a bornology of* $X$ *then* $\mathfrak{S}^\star = \{S^\star, S \in \mathfrak{B}\}$ *is a base for a bornology* $\mathfrak{B}^\star$ *in* $X'$*. Moreover if* $\mathfrak{B}$ *is stable under small enlargements or if* $\mathfrak{B}$ *is of type* $\mathcal{H}$ *, so is* $\mathfrak{B}^\star$.

*Proof.* Let us prove that $\mathfrak{B}^\star = \{B \subseteq X', B \neq \emptyset, \exists S^\star \in \mathfrak{S}^\star, B \subset S^\star\}$ is a bornology in $X'$ with $\mathfrak{S}^\star$ as a base. It is clear that $\mathfrak{B}^\star$ is hereditary. Let $B_1$ and $B_2$ two elements of $\mathfrak{B}^\star$ and $S_1^\star$ and $S_2^\star$ two elements of $\mathfrak{S}^\star$ such that $B_1 \subset S_1^\star$ and $B_2 \subset S_2^\star$. Then $B_1 \cup B_2 \subset S_1^\star \cup S_2^\star = (S_1 \cup S_2)^\star \in \mathfrak{S}^\star$. Hence $B_1 \cup B_2 \in \mathfrak{B}^\star$. It remains to prove that $\mathfrak{B}^\star$ is a cover of $X'$. For every $y \in X'$ put $x = \|y\|u$ where $u$ is an element of norm one. Then $y \in (\{x\})^\star$. Since $\{x\} \in \mathfrak{B}$ we have $(\{x\})^\star \in \mathfrak{S}^\star$. Then $\mathfrak{B}^\star$ is a cover of $X'$.

Suppose that $\mathfrak{B}$ is stable under small enlargement. Let $B \in \mathfrak{B}^\star$ and $S^\star \in \mathfrak{S}^\star$ such that $B \subset S^\star$. $S \in \mathfrak{B}$ implies that there exists $\epsilon > 0$ such that $S^\epsilon \in \mathfrak{B}$. We are going to show that $(S^\star)^\epsilon \subset (S^\epsilon)^\star$. Let $y \in (S^\star)^\epsilon$, then there exists $z \in S^\star$ such that $\|y - z\| < \epsilon$. Choose $a \in S$ such that $\|z\| = \|a\|$. If $a = 0$ then $z = 0$ and therefore $\|y\| < \epsilon$. Pick $x \in X$ such that $\|x\| = \|y\|$ then we have $x \in S^\epsilon$. Hence $y \in (S^\epsilon)^\star$. If $a \neq 0$ put $x = \frac{\|y\|}{\|a\|}a$ then $\|x - a\| = |\|y\| - \|z\|| \leq \|y - z\| < \epsilon$ and then $x \in S^\epsilon$. We have $\|y\| = \|x\|$ then $y \in (S^\epsilon)^\star$. Therefore $(S^\star)^\epsilon \subset (S^\epsilon)^\star$. Consequently, $(S^\star)^\epsilon \in \mathfrak{B}^\star$ and $\mathfrak{B}^\star$ is stable under small enlargement.

Suppose that $\mathfrak{B}$ is of type $\mathcal{H}$ and let $B \in \mathfrak{B}^\star$ and $\lambda > 0$. Then there exists $S \in \mathfrak{B}$ such that $B \subset S^\star$. Hence $\lambda B \subset \lambda S^\star = (\lambda S)^\star$ . $\lambda S \in \mathfrak{B}$ implies that $\lambda B \in \mathfrak{B}^\star$ which proves that $\mathfrak{B}^\star$ is of type $\mathcal{H}$. ☐

Our main result relates a bornological version of the Walkup-Wets theorem.

**Theorem 2.1.** *Let $X$ be a normed linear space, $\mathfrak{B}$ a bornology of type $\mathcal{H}$ in $X$ and $\mathfrak{B}^\star$ its dual bornology. If $A$ and $C$ are two closed convex cones of $X$, then the following conditions are equivalent:*

*i)* $A \cap S \subset C^\epsilon, C \cap S \subset A^\epsilon, \forall S \in \mathfrak{B}$ *and* $\forall \epsilon > 0$.

*ii)* $A^\circ \cap T \subset (C^\circ)^\epsilon, C^\circ \cap T \subset (A^\circ)^\epsilon, \forall T \in \mathfrak{B}^\star$ *and* $\forall \epsilon > 0$.

*Proof.* *i)* $\Longrightarrow$ *ii)* Let $T \in \mathfrak{B}^\star$ and $\epsilon > 0$. We are going to show that $A^\circ \cap T \subset (C^\circ)^\epsilon$. If $y \in A^\circ \cap T$ we will show that $d(y, C^\circ) < \epsilon$. If $y \in C^\circ$, there is nothing to prove. If $y \notin C^\circ$, we denote $r = d(y, C^\circ)$ and $F = y + rU^\star$. $F$ is a nonempty convex $\sigma(X', X)-$compact of $X'$ disjoint from the nonempty $\sigma(X', X)$-closed convex subset $C^\circ$. The Hahn Banach separation theorem implies that there exists $x \in X$ of norm 1 and $\alpha \in \mathbb{R}$ such that $C^\circ \subset \{z \in X', \langle z, x\rangle \leq \alpha\}$ and $F \subset \{z \in X', \langle z, x\rangle > \alpha\}$. Since $C^\circ$ is a cone, we can take $\alpha = 0$. $C^\circ$ is a cone limited by the half-space $T_x^\circ$ and $y \neq 0$ implies that $d(y, C^\circ) = d(y, T_x^\circ)$. From the Ascoli's formula we get $d(y, T_x^\circ) = |\langle y, x\rangle| = \frac{|\langle y, \|y\|x\rangle|}{\|y\|} = d(\|y\|x, E_y) \leq d(\|y\|x, A)$ because $A \subset E_y$ . On the other hand, we have $x \in (C^\circ)^\circ = C$ then $\|y\|x \in C$. Put $S = \{\|y\|x\} \in \mathfrak{B}$ then we have $\|y\|x \in C \cap S$. By *i)* we deduce that $d(\|y\|x, A) < \epsilon$ so $d(y, C^\circ) < \epsilon$ and consequently $A^\circ \cap T \subset (C^\circ)^\epsilon$. The inclusion $C^\circ \cap T \subset (A^\circ)^\epsilon$ is deduced by symmetry and this finished *i)* $\Longrightarrow$ *ii)*.

*ii)* $\Longrightarrow$ *i)* Let $S \in \mathfrak{B}$ and $\epsilon > 0$. Let $x \in A \cap S$ we show that $d(x, C) < \epsilon$. If $x \in C$ there is nothing to prove. Otherwise, we denote $r = d(x, C) > 0$ and $G = x + r(intU)$, where $intU$ is the interior of $U$. Then the nonempty open convex subset $G$ of $X$ does not intersects the nonempty convex subset $C$. The Hahn Banach separation theorem implies that there exists $y \in X'$ and a real $\alpha$ such that $C \subset \{u \in X, \langle y, u\rangle \leq \alpha\}$ and $G \subset \{u \in X, \langle y, u\rangle \geq \alpha\}$ . Since $C$ is a cone in $X$ we can take $\alpha = 0$ , hence $C \subset E_y$ , $\{u \in X, \langle y, u\rangle = 0\}$ is a support hyperplane to $C$ and $r = d(x, E_y)$. The

Ascoli's formula implies that $d(x, E_y) = \frac{|\langle y,x\rangle|}{\|y\|} = d(y, T_x^{\circ})$. On the other hand we have $x \in A$ then $T_x \subset A$ since $A$ is a cone and then $A^{\circ} \subset T_x^{\circ}$. This implies that $d(y, A^{\circ}) \geq d(y, T_x^{\circ}) = r$. We have $\|\frac{y}{\beta}\| = \|x\|$ where $\beta = \frac{\|y\|}{\|x\|}$ and $x \in S$ hence $\frac{y}{\beta} \in C^{\circ} \cap S^{\star}$ then $y \in C^{\circ} \cap (\beta S)^{\star}$ . Applying $ii)$ with $T = (\beta S)^{\star} \in \mathfrak{B}^{\star}$ ( $\mathfrak{B}$ is of type $\mathcal{H}$ ) and $\epsilon$ we deduced that $d(y, A^{\circ}) < \epsilon$ then $r < \epsilon$ and consequently $d(x, C) < \epsilon$. And then $A \cap S \subset C^{\epsilon}$. By symmetry $C \cap S \subset A^{\epsilon}$ finishes $ii) \Longrightarrow i)$. □

The Walkup-Wets theorem ensures that the "distance" between two closed convex cones is the same with the "distance" between their polars. This result is in the base of the important theorem of the bicontinuity theorem of the Legendre-Fenchel transform for the Attouch-Wets topology. This bicontinuity theorem has many applications in mechanics problems, optimization and optimal control problems (see [7, 8]). In the present paper we presented a generalization of this result to bornologies. It is an open problem to prove the same result for the bicontinuity theorem of the Legendre-Fenchel transform.

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