Vector and affine bundles related to Riemannian foliations

Paul Popescu and Marcela Popescu

Abstract. The purpose of this paper is to give some new criteria by which a foliation is Riemannian. The following conditions are equivalent for a foliation to be Riemannian: the foliation allows a positively admissible transverse Lagrangian (in particular a transverse Finslerian); the foliation that allows a positively admissible transverse Hamiltonian (in particular a transverse Cartan form); the lifted foliation F^r (on the bundle of r-jets of sections of the normal bundle νF) is Riemannian; the lifted foliation F_0^r on the slashed bundle is Riemannian and vertically exact ; there is a positively admissible transverse Lagrangian on $J^r E$, for a foliated vector bundle $p : E \to M$, all for some $r \geq 1$.

M.S.C. 2010: 53C12, 53C60.
Key words: transverse vector bundle, Riemannian foliation, admissible Lagrangian, admissible Hamiltonian, transverse jet bundle.

The Ghys problem in foliation theory was raised by E. Ghys' Appendix E of [5] and recently reformulated in [3, Theorem 3.2] by Miernowski and Mozgawa: is any Finslerian foliation (see [2, 3, 6]) a Riemannian foliation? A partial result of the problem is given in [2], for a Finslerian foliation on a compact manifold. Using a different method, a more general form of the result is proved by the authors in a Lagrangian setting: *a foliation that allows a positively admissible and transverse Lagrangian (in particular a transverse Finslerian) is a Riemannian foliation* [6, Theorem 4]. The main idea in the proof is averaging the transverse vertical Hessian of the Lagrangian, using a measure that in the Finslerian case is the Bussemann-Hausdorff measure (see [10, Section 5.1]). A similar idea is used also in the present paper, first in Proposition 0.4, then applied to transverse Lagrangians on transverse vector bundles in Corollary 0.1. Moreover, we announce some new results concerning new criteria for a foliation to be a Riemannian one, as follows. First, we have that *the existence of a positively admissible and transverse Lagrangian implies that the foliation is Riemanian* (Corollary 0.2). Then, the dual Hamiltonian result: *the existence of a positively admissible and transverse Hamiltonian implies that the foliation is Riemanian* (Corollary 0.3). Taking into account the fact that the Legendre duality does not assure the allowance

BSG Proceedings 18. The Int. Conf. of Diff. Geom. and Dynamical Systems (DGDS-2010), October 8-11, 2010, Bucharest Romania, pp. 82-86.

of the dual Hamiltonian in the general case (it works only in the Finsler-Cartan case), we are compelled to make a direct proof, using similar techniques.

The following result improves the main result of Tarquini [11, Theorem 1.2.]: *The lifted foliation F^r is Riemannian for some $r \geq 1$ iff F is Riemannian* (Theorem 0.1). But Theorem 0.1 can not give an answer to the following question: *when is $\mathcal{F}$ Riemannian if $\mathcal{F}_0^r$ is Riemannian for some $r \geq 1$?* Relating to this question, we have the following result: *Let F be a foliation on a manifold M and F_0^r be the lifted foliation on the slashed bundle of r-jets of sections of the normal bundle νF. Then F_0^r is Riemannian and vertically exact for some $r \geq 1$ iff F is Riemannian* (Theorem 0.2). An other result concerning foliated vector bundles is the following one: *Let $p : E \to M$ be a foliated vector bundle over a foliated manifold (M, F). There is a positively admissible and transverse Lagrangian on J^rE for some $r \geq 1$ iff the foliation F is Riemannian* (Theorem 0.3).

All the objects considered are of class C^∞. We use notations and general references on vector bundles and Lagrangians from [4]. Let $E \xrightarrow{p} M$ be a vector bundle. A *positively admissible Lagrangian* on E is a differentiable map $L : E_* = E\backslash\{0\} \to I\!R$, where $\{0\}$ is the image of the null section, such that the following conditions hold: 1) L is positively defined (i.e. its basic Hessian is positively defined) and $L(x, y) \geq 0 = L(x, 0)$, $(\forall)x \in M$ and $y \in E_x = p^{-1}(x)$; 2) the Lagrangian L can be projected locally on transverse Lagrangians; 3) there is a smooth function $\varphi : M \to (0, \infty)$, such that for every $x \in M$ there is $y \in E_x$ such that $L(x, y) = \varphi(x)$. If a positively Lagrangian F is 2–homogeneous (i.e. $F(x, \lambda y) = \lambda^2 F(x, y)$, $(\forall)\lambda > 0$), one say that F is a *Finslerian*; it is also a positively admissible Lagrangian, since one can take $\varphi \equiv 1$, or any positive constant.

Proposition 0.1. *There is an $\mathcal{F}(M)$–linear integration operator $\Phi_L : \mathcal{F}(E_*) \to \mathcal{F}(M)$.*

The next step is to consider the foliate case. This case worth to be studied as a special case, when the Lagrangian is not regular. Let $\mathcal{F}$ be a given foliation on the base M of the vector bundle $E \xrightarrow{p} M$. One say that the vector bundle is *foliated* if there is a vectorial atlas of local trivialisations on E such that all the components of the structural matrices are basic functions. Notice that a canonical foliation $\mathcal{F}_E$ on E is induced. Let us consider now a local base (s_a) of the module of sections on the restriction of an open $U \subset U_0 \subset M$, where U_0 is an open domain of a vectorial chart of the atlas. A section s on U is called *foliated* if $s = f^a s_a$ and f^a are basic functions. For example every section s_a is foliated. It is easy to see that a section s on $U \cup U'$, foliated on U and U' is also foliated on $U \cap U'$, thus the definition of a foliated section can be extended to every section on an open $U \subset M$, particularly on M. Examples of foliated vector bundles are the transverse bundle of the foliation itself, denoted by $\nu\mathcal{F}$, and the various tensor bundles constructed using $\nu\mathcal{F}$. Analogously, given a foliated vector bundle, its tensor bundles give rise to foliated vector bundles. For example, one can consider the transverse vector bundle of bilinear forms on the fibers of E; a *transverse bilinear form* b on E is a section in this vector bundle; a bilinear forms b on the fibers of E is transverse iff for any two transverse (local) sections s_1, s_2, then $b(s_1, s_2)$ is a (local) basic function. A transverse bilinear form b gives rise to a (canonical) transverse Lagrangian on E, given by the quadratic form defined by b. A special case can be considered when the foliated vector bundle is transversely parallelizable; if $\nu\mathcal{F}$ is parallelizable, then $\mathcal{F}$ is a Riemannian foliation. A transverse Lagrangian on the foliate vector bundle E is a Lagrangian $L : E_* \to I\!R$, such that for every foliated section $s : U \to E$, the function $x \to L(x, s(x))$ is basic on U. The definition of a *positively admissible transverse Lagrangian* is analogous to the definition of a positively admissible Lagrangian, asking in the second condition that $\varphi \in \mathcal{F}(M)$ be a basic function. The integration operator can be adapted to the transverse structure, as follows.

Proposition 0.2. *Let $E \stackrel{p}{\to} M$ be a transverse vector bundle and $L : E_* \to I\!R$ be a positively admissible and transverse Lagrangian. Then there is an$\mathcal{F}(M)$–linear integration operator $\bar{\Phi}_L : \mathcal{F}(E_*) \to \mathcal{F}(M)$ that sends basic functions to basic functions.*

Nevertheless, we use in the sequel only basic functions.

Theorem 0.1. *The lifted foliation $\mathcal{F}^r$ is Riemannian for some $r \geq 1$ iff $\mathcal{F}$ is Riemannian.*

According to [11, Definition 1.1], one say that the foliation $\mathcal{F}$ is of *finite type* if there exists $r \geq 1$ such that $\mathcal{F}^r$ is transversely parallelizable. If moreover all the leaves of $\mathcal{F}^r$ are relatively compact, one say that $\mathcal{F}$ is a *compact finite type foliation.* In [11, Theorem 1.2.] it is proved that *any compact finite type foliation is Riemannian.* But a foliation that is transversely parallelizable is also Riemannian. Thus the Tarquini's result is improved by the above Theorem. But considering the induced foliation $\mathcal{F}_0^r$ on the slashed vector bundle $\mathcal{J}_*^r = \mathcal{J}^r \backslash \{\bar{0}\}$, then Theorem 0.1 can not give an answer to the following question: *when is $\mathcal{F}$ Riemannian if $\mathcal{F}_0^r$ is Riemannian for some* $r \geq 1$?

If a positively Lagrangian F is 2–homogeneous (i.e. $F(x, \lambda y) = \lambda^2 F(x, y)$, $(\forall)\lambda > 0$), one say that F is a *Finslerian*; it is also a positively admissible Lagrangian, since one can take $\varphi \equiv 1$, or any positive constant. We can regard the vertical bundle $VTE = \ker p_* \to E$ as a vector subbundle of $\nu F_E \to E$ by mean of the canonical projection $TE \to \nu F_E$, since VTE is transverse to τF_E. We say that an invariant Riemannian metric G' on νF_E is *vertically exact* if its restriction to the vertical foliated sections is the transverse vertical hessian of a positively admissible Lagrangian $L : E \to I\!R$; in this case, we say that the foliation $\mathcal{F}_E$ is *vertically exact.* Notice that if $p : E \to M$ is an affine bundle, then the vertical hessian Hess L of a Lagrangian $L : E \to I\!R$ is a symmetric bilinear form on the fibers of the vertical bundle VTE, given by the second order derivatives of L, using the fiber coordinates (see [6, 10] for more details using coordinates).

Theorem 0.2. *Let $\mathcal{F}$ be a foliation on a manifold M and $\mathcal{F}_0^r$ be the lifted foliation on the slashed bundle of r-jets of sections of the normal bundle $\nu\mathcal{F}$. Then $\mathcal{F}_0^r$ is Riemannian and vertically exact for some $r \geq 1$ iff $\mathcal{F}$ is Riemannian.*

The sufficiency is implied by the following result.

Proposition 0.3. *Any invariant metric g on νF gives a canonical vertically exact invariant Riemannian metric on νF^r, for any $r \geq 1$.*

In particular, it follows that any invariant metric g on νF gives rise to a canonical Lagrangian on $\mathcal{J}^r$, coming from the vertical part of the vertically exact invariant Riemannian metric on νF^r. So, one can ask for the converse: does the existence of a Lagrangian on $\mathcal{J}^r$ guaranties that $\mathcal{F}$ is Riemannian?

Theorem 0.3. *Let $p : E \to M$ be a foliated vector bundle over a foliated manifold $(M, \mathcal{F})$. There is a positively admissible and transverse Lagrangian on $\mathcal{J}^r E$ for some $r \geq 1$ iff the foliation $\mathcal{F}$ is Riemannian.*

The main technical tool to prove the necessity of the above theorem has independent interest, as follows.

Proposition 0.4. *Let $p_1 : E_1 \to M$ and $p_2 : E_2 \to M$ be foliated vector bundles over a foliated manifold $(M, \mathcal{F})$ and $q_2 : E_{2*} \to M$ be the slashed bundle. If there are a positively admissible Lagrangian $L : E_2 \to I\!R$ and a metric b on the pull back bundle $q_2^* E_1 \to E_{2*}$, foliated with respect to $\mathcal{F}_{E_{2*}}$, then there is a foliated metric on E_1, with respect to $\mathcal{F}$.*

We state as a corollary the case when $E_1 = E_2 = E$ and b is $HessL$ seen as a metric on $p^* E_* \to E$ for some foliated bundle $p : E \to M$.

Corollary 0.1. *Let $p : E \to M$ be a foliated vector bundle over a foliated manifold $(M, \mathcal{F})$. If $L : E \to I\!R$ is a positively admissible and transverse Lagrangian, then there is a foliated metric on E.*

Specializing further to the case $E = \nu \mathcal{F}$ and L is a foliated Finsler metric we get back the following result.

Corollary 0.2. *Any foliation having an invariant, positively admissible and transverse Lagrangian (in particular a transverse Finsler structure) is a Riemannian foliation.*

The problem for Finsler foliations is raised in [3], as a special case of a problem of E. Ghys in Appendix E of P. Molino's book [5]; see [2, 3, 6]. Another interesting special case is when $E = \nu^* F$, specially concerning the duality Lagrangian-Hamiltonian.

Corollary 0.3. *Any foliation having an invariant, positively admissible and transverse Hamiltonian (in particular a Cartan structure) is a Riemannian foliation.*

The following question arises finally: *can we drop in Theorem 0.2 the condition that $\mathcal{F}_0^r$ be vertically exact?*

Acknowledgments. The second author was partially supported by a CNCSIS Grant, code 536/2008 (695/2009).

References

[1] Anastasiei, M., *Metrizable linear connections in a Lie algebroid*, J. Adv. Math. Studies, 3, 1 (2010), 09-18.

[2] Józefowicz M., Wolak R., *Finsler foliations of compact manifolds are Riemannian*, Diff. Geom. Appl., 26, 2 (2008), 224-226.

[3] Miernowski A., Mozgawa W., *Lift of the Finsler foliation to its normal bundle*, Diff. Geom. Appl., 24 (2006), 209-214.

[4] Miron R., Anastasiei, M., *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Academic Publishers, 1994.

[5] Molino P., *Riemannian foliations*, Progress in Mathematics 73, Birhäuser, Boston, 1988.

[6] Popescu M., Popescu P., *Lagrangians adapted to submersions and foliations*, Diff. Geom. Appl., 27, 2 (2009), 171-178.

[7] Popescu M., Popescu P., *Lagrangians and higher order tangent spaces*, Balkan J. Geom. Appl. 15, 1 (2010), 142-148.

[8] Popescu M., Popescu P., *A new setting for higher order Lagrangians in the time dependent case*, J. Adv. Math. Studies, 3, 1 (2010), 83-92.

[9] Saunders D.J., *Horizontal forms on jet bundles*, Balkan J. Geom. Appl. 15, 1 (2010), 149-154.

[10] Shen Z., *Differential Geometry of Spray and Finsler Spaces*, Kluwer Academic Publishers, 2001.

[11] Tarquini C., *Feuilletages de type fini compact*, C. R. Acad. Sci. Paris, Ser. I, 339 (2004), 209-214.

Authors' address:

Marcela Popescu and Paul Popescu
University of Craiova, Department of Applied Mathematics,
13 Al.I.Cuza st., 200585 Craiova, Romania.
E-mail: marcelacpopescu@yahoo.com ; paul_p_popescu@yahoo.com