Standard controllable realizations for nD separable transfer matrices

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Abstract. A class of nD discrete-time systems is considered and its state-space and frequency-domain representations are provided. By using the multidimensional (nD) $\mathcal{Z}$-transformation, it is shown that the transfer matrices for these systems are proper rational functions in n indeterminates with separable denominators. A realization problem for these systems is stated and an algorithm is proposed to construct standard controllable realizations for transfer matrices with separable denominators. A Matlab program is given for the provided algorithm and two examples illustrate the advantages of the proposed method.

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1 Introduction

In the past three decades a lot of published paper and books have been designed to the theory of multidimensional (nD) systems, which become a distinct and important branch of the systems theory. This increasing interest in the nD systems domain is determined on one side by the richness in potential application fields and on the other side by the richness and significance of the theoretical approaches. The main application fields are represented by control and signal processing, image processing, computer tomography, circuits, gravity and magnetic field mapping, geophysics, seismology, control of multipass processes, etc.

From the theoretical point of view, the domain of nD systems develops a specific approach, since many aspects of the 1D systems do not generalize to nD and there are many nD systems phenomena which have no 1D systems counterparts.

One of the most important problems in this approach is that of the state-space realization of nD transfer matrices. Various state-space 2D discrete-time models have been proposed in literature by Roesser [14], Fornasini-Marchesini [6], Attasi [1],

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Eising [5] and others, as well as some multidimensional and hybrid systems and their applications([8], [9], [11], [2], [4]]). For more informations concerning realizations in linear systems theory see [3], [7], [10] and [15].

In this paper the class of Attasi type nD discrete-time systems is considered. In Section 2 the state-space representation of this class and its input-output map are provided. By using the multidimensional (nD) $\mathcal{Z}$-transformation, in Section 3 it is shown that the transfer matrices for these systems are proper rational matrix functions in n indeterminates with separable denominators. A realization problem for these systems is stated in Section 4. Section 5 presents a realization algorithm which constructs standard controllable realizations for transfer matrices with separable denominators. A Matlab program is given for the provided algorithm in Section 6 as well as two examples which illustrate the advantages of the proposed method.

2 Multidimensional linear discrete-time systems

We shall use the following notations: if $n \in \mathbb{N}^*$, $\langle n \rangle$ is the set $\{1, 2, \ldots, n\}$; for a subset of $\langle n \rangle$, $\delta = \{i_1, \ldots, i_l\}$, $|\delta| := l$ and $\hat{\delta} := \langle n \rangle \setminus \delta$; for $i \in \langle n \rangle$, $\hat{i} := \langle n \rangle \setminus \{i\}$ and $\hat{\hat{i}} = \{i+1, \ldots, n\}$. The relation $\delta \subset \langle n \rangle$ means that δ is a subset of $\langle n \rangle$ or $\delta = \emptyset$, and $\delta \neq \langle n \rangle$. For a function $x : \mathbb{Z}^n \to \mathbb{R}^N$, $n, N \in \mathbb{N}^*$, $n > 2$, $x(t)$ denotes the value $x(t_1, t_2, \ldots, t_n)$ where $t = (t_1, t_2, \ldots, t_n) \in \mathbb{Z}^n$ and $x(t_\delta, 0_{\hat{\delta}})$ denotes $x(0, \ldots, 0, t_{i_1}, 0, \ldots, 0, t_{i_2}, 0, \ldots, t_{i_l}, 0, \ldots, 0)$. The operator σ_δ is defined by $\sigma_\delta x(t) = x(t + e_\delta)$ where $e_\delta = e_{i_1} + e_{i_2} + \cdots + e_{i_l}$, $e_j = (\underbrace{0, \ldots, 0}_{j-1}, 1, 0, \ldots, 0) \in \mathbb{R}^n$; we denote by σ the operator σ_δ for $\delta = \langle n \rangle$, i.e. $\sigma x(t_1, t_2, \ldots, t_n) = x(t_1 + 1, t_2 + 1, \ldots, t_n + 1)$.

By O_r and I_r (or I) are denoted respectively the null matrix and the unit matrix of order $r \in \mathbb{N}^*$. If $\{A_i\}_{i \in I}$ is a family of matrices, we consider $\prod\limits_{i \in \emptyset} A_i = I$.

The time set of the Attasi-type nD discrete-time system is $T = \mathbb{Z}_+^n$.

Definition 2.1. An nD linear time-invariant discrete-time system is a set $\Sigma = (\{A_i, i \in \langle n \rangle\}; B; C; D)$ where A_i are commuting $N \times N$ real matrices, with the state equation

$$\sigma x(t) = \sum_{\delta \subset \langle n \rangle} (-1)^{n-|\delta|-1} \left(\prod_{i \in \hat{\delta}} A_i \right) \sigma_\delta x(t) + Bu(t) \tag{2.1}$$

and the output equation

$$y(t) = Cx(t) + Du(t); \tag{2.2}$$

$x(t) \in \mathbb{R}^N$ is the *state* at the moment $t = (t_1, t_2, \ldots, t_n) \in \mathbb{Z}_+^n$, $u(t) \in \mathbb{R}^m$ is the *input* (*control*) and $y(t) \in \mathbb{R}^p$ is the *output* of the system Σ. The number N is called the *dimension* of the system Σ and it is denoted dim Σ

Definition 2.2. The vector $x_0 \in \mathbb{R}^N$ is called an *initial state* of the system Σ if, for any $\delta \subset \langle n \rangle$, the following initial conditions of Σ hold

$$x(t_\delta, 0_{\hat{\delta}}) = \prod_{i \in \delta} A_i^{t_i} x_0. \tag{2.3}$$

3 Multidimensional transfer matrices

Assume that the states $x(\cdot)$ and the inputs $u(\cdot)$ are original functions [13, Def. 2.1].
Definition 3.1. The $\mathcal{Z}_n$-*transform* of the original f is the function

$$F^*(z) = F^*(z_1, \dots, z_n) = \mathcal{Z}_n[f(t_1, \dots, t_n)] = \sum_{t_1=0}^{\infty} \dots \sum_{t_n=0}^{\infty} f(t_1, \dots, t_n) z_1^{-t_1} \cdots z_n^{-t_n}. \tag{3.1}$$

For $\alpha = \{j_1, \dots, j_k\} \subset \langle n \rangle$ the α-partial $\mathcal{Z}_n$-transform of the original f is the function

$$\mathcal{Z}_n^{\alpha}[f(t_1, \dots, t_n)] = \sum_{t_{j_1}=0}^{\infty} \dots \sum_{t_{j_k}=0}^{\infty} f(t_1, \dots, t_{j_1}, \dots, t_{j_k}, \dots, t_n) z_1^{-t_{j_1}} \cdots z_k^{t_{j_k}}. \tag{3.2}$$

By applying [13, Theorem 2.15] we get
Proposition 3.2. *The following equalities hold:*

$$\mathcal{Z}_n[\sigma x(t_1, \dots, t_n)] = \left(\prod_{i=1}^{n} z_i \right) (F^*(z_1, \dots, z_n) + \sum_{\alpha \subset \langle n \rangle} (-1)^{|\alpha|} \mathcal{Z}_n^{\hat{\alpha}}[x(t_{\hat{\alpha}}, 0_\alpha)]) \tag{3.3}$$

$$\mathcal{Z}_n[\sigma_\delta x(t_1, \dots, t_n)] = \left(\prod_{i \in \delta} z_i \right) (F^*(z_1, \dots, z_n) + \sum_{\alpha \subset \delta} (-1)^{|\alpha|} \mathcal{Z}_n^{\hat{\alpha}}[x(t_{\hat{\alpha}}, 0_\alpha)]) \tag{3.4}$$

for $\delta \subset \langle n \rangle$.

Now, we shall apply the operator $\mathcal{Z}_n$ to equations (2.1) and (2.2), taking into account the linearity of $\mathcal{Z}_n$. We obtain by (3.3) and (3.4) the equations

$$\begin{aligned} &\left(\prod_{i=1}^{n} z_i \right) (X^*(z_1, \dots, z_n) + \sum_{\alpha \subset \langle n \rangle} (-1)^{|\alpha|} \mathcal{Z}_n^{\hat{\alpha}}[x(t_{\hat{\alpha}}, 0_\alpha)]) = \sum_{\delta \subset \langle n \rangle} (-1)^{n-|\delta|-1} \cdot \\ &\cdot \left(\prod_{i \in \hat{\delta}} A_i \right) (X^*(z_1, \dots, z_n) + \sum_{\alpha \subset \delta} (-1)^{|\alpha|} \mathcal{Z}_n^{\hat{\alpha}}[x(t_{\hat{\alpha}}, 0_\alpha)]) + BU^*(z_1, \dots, z_n), \end{aligned} \tag{3.5}$$

$$Y^*(z_1, \dots, z_n) = CX^*(z_1, \dots, z_n) + DU^*(z_1, \dots, z_n). \tag{3.6}$$

Using the equality (where $I = I_N$)

$$\prod_{i=1}^{n} z_i I - \sum_{\delta \subset \langle n \rangle} (-1)^{n-|\delta|-1} \left(\prod_{i \in \delta} z_i \right) \left(\prod_{i \in \hat{\delta}} A_i \right) = \prod_{i=1}^{n} (z_i I - A_i),$$

we solve (3.5) with respect to $X^*(z_1, \dots, z_n)$, for $z_i \in \mathbb{C} \setminus \sigma(A_i)$, $i \in \langle n \rangle$ and we obtain

$$\begin{aligned} X^*(z_1, \dots, z_n) = \left(\prod_{i=1}^{n} (z_i I - A_i)^{-1} \right) BU^*(z_1, \dots, z_n) + \\ + \left(\prod_{i=1}^{n} (z_i I - A_i)^{-1} \right) \left(\sum_{\delta \subseteq \langle n \rangle} (-1)^{n-|\delta|-1} \left(\prod_{i \in \hat{\delta}} A_i \right) \sum_{\alpha \subset \delta} (-1)^{|\alpha|} \mathcal{Z}_n^{\hat{\alpha}}[x(t_{\hat{\alpha}}, 0_\alpha)] \right). \end{aligned} \tag{3.7}$$

We replace X^* given by (3.7) in (3.6) and we obtain

Theorem 3.3. *The input-output map $U^* \to Y^*$ of the system Σ in the frequency domain is given by*

(3.8)
$$Y^*(z_1,\ldots,z_n) = \left[C\left(\prod_{i=1}^{n}(z_iI - A_i)^{-1}\right)B + D\right]U^*(z_1,\ldots,z_n)+$$
$$+C\left(\prod_{i=1}^{n}(z_iI - A_i)^{-1}\right)\left(\sum_{\delta\subseteq\langle n\rangle}(-1)^{n-|\delta|-1}\left(\prod_{i\in\hat{\delta}}A_i\right)\sum_{\alpha\subset\delta}(-1)^{|\alpha|}\mathcal{Z}_n^{\hat{\alpha}}[x(t_{\hat{\alpha}},0_\alpha)]\right).$$

Definition 3.4. The matrix

$$H_\Sigma(z_1,\ldots,z_n) = C\left(\prod_{i=1}^{n}(z_iI - A_i)^{-1}\right)B + D \tag{3.9}$$

is called the *transfer matrix* of the system Σ.

For the initial state $x_0 = 0$ the initial conditions (2.3) become $x(t_\delta, 0_{\hat{\delta}}) = 0$, $\forall\delta \subset \langle n\rangle$ and we get

Corollary 3.5. *For $x_0 = 0$, the input output map of Σ in the frequency domain is given by $Y^*(z_1,\ldots,z_n) = H_\Sigma(z_1,\ldots,z_n)U^*(z_1,\ldots,z_n)$.*

We can see from (3.9) that the matrix $C\left(\prod_{i=1}^{n}(z_iI - A_i)^{-1}\right)B$ is strictly proper in each variable z_i, $i \in \langle n\rangle$ and it has a separable denominator $\prod_{i=1}^{n}\pi(z_i)$, where $\pi_i(z_i) = \det(z_iI - A_i)$ is the characteristic polynomial of the matrix A_i. It follows that the transfer matrix $H_\Sigma(z_1,\ldots,z_n)$ is a proper $p\times m$ matrix and we call it a *separable* matrix.

4 Realization problem

Let $H(z_1, z_2,\ldots,z_n)$ be a proper separable matrix (i.e. with separable denominator).

Definition 4.1. A system $\Sigma = (\{A_i, i\in\langle n\rangle\}; B; C; D)$ is a *realization* of the matrix $H(z_1, z_2,\ldots,z_n)$ if $H = H_\Sigma$ i.e.

$$H(z_1, z_2,\ldots,z_n) = C\left(\prod_{i=1}^{n}(z_iI - A_i)^{-1}\right)B + D. \tag{4.1}$$

The system Σ is said to be a *minimal realization* of H if $\dim\Sigma \le \dim\tilde{\Sigma}$ for any realization $\tilde{\Sigma}$ of the matrix H.

Since the matrix $C\left(\prod_{i=1}^{n}(z_iI - A_i)^{-1}\right)B$ in (4.1) is strictly proper, by taking the limit in (4.1) as $z_i\to\infty$, $\forall i\in\langle n\rangle$, the problem of determining the matrix D in a realization has the solution

$$D = \lim_{z_1\to\infty}\lim_{z_2\to\infty}\ldots\lim_{z_n\to\infty}H(z_1, z_2,\ldots,z_n). \tag{4.2}$$

and the matrix $\hat{H}(z_1, z_2, \ldots, z_n) = H(z_1, z_2, \ldots, z_n) - D$ is strictly proper in each variable z_i, $i \in \langle n \rangle$. Therefore it remains to solve the following

Realization Problem. Given a strictly proper separable matrix $H(z_1, z_2, \ldots, z_n)$, determine the system $\Sigma = (\{A_i, i \in \langle n \rangle\}; B; C)$ such that Σ is a realization of H, i.e.

$$H(z_1, z_2, \ldots, z_n) = C \left(\prod_{i=1}^{n} (z_i I - A_i)^{-1} \right) B. \tag{4.3}$$

5 Standard controllable realizations

Let $H(z_1, z_2, \ldots, z_n)$ be a $p \times m$ strictly proper separable matrix.

Algorithm 5.1. (of standard controllable realization).

Stage I. Determine the l.c.d. (least common denominator) of the entries of matrix H. Let it be $\pi(z_1, z_2, \ldots, z_n) = \prod_{i=1}^{n} \pi_i(z_i)$, where

$$\pi_i(z_i) = z_i^{r_i} + \sum_{k=0}^{r_i - 1} a_{ik} z_i^k \tag{5.1}$$

Stage II. Determine the matrices $H_{j_1, j_2, \ldots, j_n} \in \mathbb{R}^{p \times m}$, $j_i \in \langle r_i - 1 \rangle$, $i \in \langle n \rangle$ such that

(5.2)

$$\pi(z_1, z_2, \ldots, z_n) H(z_1, z_2, \ldots, z_n) = \sum_{j_1=0}^{r_1-1} \sum_{j_2=0}^{r_2-1} \cdots \sum_{j_n=0}^{r_n-1} H(j_1, j_2, \ldots, j_n) z_1^{j_1} z_2^{j_2} \cdots z_n^{j_n}.$$

Stage III. Find the numbers $\bar{r}_1 = m$, $\bar{r}_2 = r_1 m, \ldots, \bar{r}_i = r_1 \cdots r_{i-1} m, \ldots, \bar{r}_{n+1} = r_1 \cdots r_n m$ and $q_i = r_{i+1} r_{i+2} \cdots r_n$, $i \in \langle n - 1 \rangle$.

Stage IV. Determine the matrices

$$L_i = \begin{bmatrix} O_{\bar{r}_i} & I_{\bar{r}_i} & O_{\bar{r}_i} & \ldots & O_{\bar{r}_i} & O_{\bar{r}_i} \\ O_{\bar{r}_i} & O_{\bar{r}_i} & I_{\bar{r}_i} & \ldots & O_{\bar{r}_i} & O_{\bar{r}_i} \\ . & . & . & \ldots & . & . \\ O_{\bar{r}_i} & O_{\bar{r}_i} & O_{\bar{r}_i} & \ldots & O_{\bar{r}_i} & I_{\bar{r}_i} \\ -a_{i0} I_{\bar{r}_i} & -a_{i1} I_{\bar{r}_i} & -a_{i2} I_{\bar{r}_i} & \ldots & -a_{i,r_i-2} I_{\bar{r}_i} & -a_{i,r_i-1} I_{\bar{r}_i} \end{bmatrix}, i \in \langle n \rangle. \tag{5.3}$$

Stage V. Write the completely controllable realization $\Sigma = (\{A_i, i \in \langle n \rangle\}; B; C)$ of the matrix $H(z_1, z_2, \ldots, z_n)$ where

$$A_i = \operatorname{diag}_{q_i} \{L_i\}, \ i \in \langle n - 1 \rangle, \ A_n = L_n, \tag{5.4}$$

$$B = [\underbrace{O_m \, O_m \, \ldots \, O_m \, I_m}_{r_1 r_2 \cdots r_n}]^t, \tag{5.5}$$

$$C = [C_1\ C_2\ \dots\ C_k\ \dots\ C_{r_1 r_2 \cdots r_n}]. \tag{5.6}$$

Proof. Let us denote by K_i the companion cell corresponding to the polynomial $\pi_i(z_i)$, i.e.

$$K_i = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_{i0} & -a_{i1} & -a_{i2} & \dots & -a_{i,r_i-2} & -a_{i,r_i-1} \end{bmatrix}, i \in \langle n \rangle. \tag{5.7}$$

and by g_i and f_i the r_i vectors

$$g_i = [0\ 0\ \dots\ 0\ 1]^t, \quad f_i = [1\ z_i\ z_i^2\ \dots\ z_i^{r_i-1}]^t. \tag{5.8}$$

We obtain from (5.1)-(5.8) the following equalities, for $i \in \langle n \rangle$:

$$(z_i I_{r_i} - K_i) f_i = \pi_i(z_i) g_i; \tag{5.9}$$

$$L_i = K_i \otimes I_{r_{i-1}} \otimes \cdots I_{r_1} \otimes I_m \tag{5.10}$$

since $L_i = K_i \otimes I_{\bar{r}_i}$ and $I_{\bar{r}_i} = I_{r_1 \cdots r_{i-1} m} = I_{r_i - 1} \otimes \cdots \otimes I_{r_1} \otimes I_m$, where $\otimes$ denotes the Kronecker product of matrices;

$$A_i = I_{r_n} \otimes \cdots \otimes I_{r_{i+1}} \otimes K_i \otimes I_{r_{i-1}} \otimes \cdots \otimes I_{r_1} \otimes I_m \tag{5.11}$$

$$B = g_n \otimes g_{n-1} \otimes \cdots \otimes g_1 \otimes I_m \tag{5.12}$$

Using the property

$$(M_1 \otimes M_2)(M_3 \otimes M_4) = M_1 M_3 \otimes M_2 M_4 \tag{5.13}$$

of the Kronecker product, where M_1, M_2, M_3, M_4 are matrices of appropriate dimensions, we get by (5.11), for any $i, j \in \langle n \rangle$, $i < j$:

$$A_i A_j = (I_{r_n} \otimes \cdots \otimes I_{r_{j+1}} \otimes I_{r_j} \otimes I_{r_{j-1}} \otimes \cdots \otimes I_{r_{i+1}} \otimes K_i \otimes I_{r_{i-1}} \otimes \cdots \otimes I_{r_1} \otimes I_m) \cdot$$
$$\cdot (I_{r_n} \otimes \cdots \otimes I_{r_{j+1}} \otimes K_j \otimes I_{r_{j-1}} \otimes \cdots \otimes I_{r_{i+1}} \otimes I_{r_i} \otimes I_{r_{i-1}} \otimes \cdots \otimes I_{r_1} \otimes I_m) =$$
$$= I_{r_n} \otimes \cdots \otimes I_{r_{j+1}} \otimes K_j \otimes I_{r_{j-1}} \otimes \cdots \otimes I_{r_{i+1}} \otimes K_i \otimes I_{r_{i-1}} \otimes \cdots \otimes I_{r_1} \otimes I_m) = A_j A_i,$$

hence the matrices $A_1, A_2, \dots, A_n$ commute.

Obviously, the dimension of the system Σ (5.4)-(5.6) is $N = \dim \Sigma = \bar{r}_{n+1} = r_1 \cdots r_n m$. Since the Kronecker product is distributive with respect to the addition, we have

$$\begin{aligned} z_i I_N - A_i &= z_i I_{r_n} \otimes \cdots \otimes I_{r_{i+1}} \otimes I_{r_i} \otimes I_{r_{i-1}} \otimes \cdots \otimes I_{r_1} \otimes I_m - \\ &- I_{r_n} \otimes \cdots I_{r_{i+1}} \otimes K_i \otimes I_{r_{i-1}} \otimes \cdots \otimes I_{r_1} \otimes I_m, \text{ hence} \\ z_i I_N - A_i &= I_{r_n} \otimes \cdots \otimes I_{r_{i+1}} \otimes (z_i I_{r_i} - K_i) \otimes I_{r_{i-1}} \otimes \cdots \otimes I_{r_1} \otimes I_m. \end{aligned} \tag{5.14}$$

Now, let us consider the matrix

$$P = f_n \otimes f_{n-1} \otimes \cdots \otimes f_1 \otimes I_m. \tag{5.15}$$

Obviously

$$P = [I_m \; z_1 I_m \; \dots \; z_1^{r_1-1} I_m \vdots z_2 I_m \; z_1 z_2 I_m \; \dots \; z_1^{r_1-1} z_2 I_m \vdots \dots \; z_1^{j_1} z_2^{j_2} \cdots z_n^{j_n} I_m \; \dots \; z_1^{r_1-1} z_2^{r_2-1} \cdots z_n^{r_n-1} I_m]^t. \tag{5.16}$$

where the block $z_1^{j_1} z_2^{j_2} \cdots z_n^{j_n} I_m$ is on the position $k = \sum_{l=2}^{n} j_l \left(\prod_{\alpha=1}^{l-1} r_\alpha \right) + j_1 + 1$, $k \in \langle r_1 r_2 \dots r_n \rangle$. By (5.2), (5.6) and (5.16) we get

$$CP = \sum_{j_1=0}^{r_1-1} \sum_{j_2=0}^{r_2-1} \dots \sum_{j_n=0}^{r_n-1} H_{j_1, j_2, \dots, j_n} z_1^{j_1} z_2^{j_2} \cdots z_n^{j_n} = \left(\prod_{i=1}^{n} \pi_i(z_i) \right) H(z_1, z_2, \dots, z_n). \tag{5.17}$$

On the other hand, by (5.9), (5.12), (5.13), (5.14) and (5.15) it follows:

$$\left(\prod_{i=1}^{n} (z_i I_N - A_i) \right) P = (z_n I_{r_n} - K_n) f_n \otimes \cdots \otimes (z_1 I_{r_1} - K_1) f_1 \otimes I_m =$$

$$= \left(\prod_{i=1}^{n} \pi_i(z_i) \right) g_n \otimes \cdots \otimes g_1 \otimes I_m = \prod_{i=1}^{n} \pi_i(z_i) B$$

hence

$$\left(\prod_{i=1}^{n} \pi_i(z_i) \right)^{-1} P = \left(\prod_{i=1}^{n} (z_i I_N - A_i)^{-1} \right) B. \tag{5.18}$$

By (5.17) and (5.18) we get:

$$C \left(\prod_{i=1}^{n} (z_i I_N - A_i)^{-1} \right) B = \left(\prod_{i=1}^{n} \pi_i(z_i)^{-1} \right) CP = H(z_1, z_2, \dots, z_n),$$

and it follows from (4.3) that the system Σ (5.4)-(5.6) is a realization of the strictly proper matrix $H(z_1, z_2, \dots, z_n)$.

Now we consider the controllability matrix of the system Σ

$$\mathcal{C}_\Sigma = [B \; A_1 B \; \dots \; A_1^{N-1} B \vdots A_2 B \; A_1 A_2 B \; \dots \; A_1^{N-1} A_2 B \vdots \dots \vdots$$

$$\vdots A_2^{N-1} \cdots A_n^{N-1} B \; A_1 A_2^{N-1} \cdots A_n^{N-1} B \; \dots \; A_1^{N-1} A_2^{N-1} \cdots A_n^{N-1} B].$$

For the sake of simplicity we will consider the case $n = 2$. Hence $\Sigma = (A_1, A_2, B, C)$ where

$$A_1 = I_{r_2} \otimes K_1 \otimes I_m, \; A_2 = K_2 \otimes I_{r_1} \otimes I_m, \; B = g_2 \otimes g_1 \otimes I_m,$$

with K_i (5.7) and g_i (5.8), $i = 1, 2$, and let us the matrices

$$\mathcal{C}_1 = \left[B\ A_1B\ \ \dots\ A_1^{r_1-1}B\right] \text{ and } \mathcal{C}_2 = \left[\mathcal{C}_1\ A_2\mathcal{C}_1\ \ \dots\ A_2^{r_2-1}\mathcal{C}_1\right].$$

Obviously, $\mathcal{C}_2$ is a submatrix of the controllability matrix

$$\mathcal{C}_\Sigma = [B\ A_1B\ \dots\ A_1^{N-1}B \vdots A_2B\ A_1A_2B\ \dots\ A_1^{N-1}A_2B \vdots \dots \vdots$$

$$\vdots A_2^{N-1}B\ A_1A_2^{N-1}B\ \dots\ A_1^{N-1}A_2^{N-1}B].$$

and by (5.13) we have $A_1^{j_1} = (I_{r_2} \otimes K_1 \otimes I_m)^{j_1} = I_{r_2} \otimes K_1^{j_1} \otimes I_m$ and similarly $A_2^{j_2} = K_2^{j_2} \otimes I_{r_1} \otimes I_m$. Then

$$\mathcal{C}_1 = [g_2 \otimes g_1 \otimes I_m\ g_2 \otimes K_1g_1 \otimes I_m\ \dots\ g_2 \otimes K_1^{r_1-1}g_1 \otimes I_m] = g_2 \otimes R_1 \otimes I_m$$

where R_1 is the $r_1 \times r_1$ matrix

$$R_1 = \left[g_1\ K_1g_1\ \dots\ K_1^{r_1-1}g_1\right] = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & \times \\ . & . & \dots & . & . \\ 0 & 1 & \dots & \times & \times \\ 1 & \times & \dots & \times & \times \end{bmatrix} \tag{5.19}$$

where $\times$ denotes some real numbers. Then again by (5.13), $A_2^{j_2} = K_2^{j_2}g_2 \otimes R_1 \otimes I_m$ and by defining the matrix $R_2 = \left[g_2\ K_2g_2\ \dots\ K_2^{r_2-1}g_2\right]$ we obtain $\mathcal{C}_2 = R_2 \otimes R_1 \otimes I_m$. Obviously R_2 and $R_2 \otimes R_1$ are respectively $r_2 \times r_2$ and $r_1r_2 \times r_1r_2$ matrices of the form (5.19).

We can prove by induction from (5.19) that $\det R_1 = (-1)^{\left[\frac{r_1}{2}\right]}$, hence $\det(R_1 \otimes R_2) = (-1)^{\left[\frac{r_1r_2}{2}\right]}$. Since for a $p \times p$ matrix M_1 and a $q \times q$ matrix M_2 $\det(M_1 \otimes M_2) = (\det(M_1)^q(\det(M_2)^p$, we obtain

$$\det\mathcal{C}_2 = (\det(R_1 \otimes R_2))^m(\det I_m)^{r_1r_2} = (-1)^{m\left[\frac{r_1r_2}{2}\right]} \neq 0.$$

It follows that the controllability matrix $\mathcal{C}_\Sigma$ has a nonzero minor $\det\mathcal{C}_2$ of order r_1r_2m, hence $\operatorname{rank}\mathcal{C}_\Sigma = r_1r_2m = \dim\Sigma$, and the system Σ is completely controllable by [12].

6 Matlab program illustrating the algorithm

We shall consider the case $n = 2$. Let $H(z_1, z_2)$ be a $p \times m$ strictly proper matrix with separable denominators.

From Section 5 we have:

Algorithm (of standard controllable realization)

Stage I. Determine the l.c.d. of the entries of matrix H:

$$\pi(z_1, z_2) = \pi_1(z_1)\pi_2(z_2) \quad \text{where} \quad \pi_i(z_i) = z_i^{r_i} + \sum_{k=0}^{r_i-1} a_{ik} z_i^k, \quad i = 1, 2.$$

Stage II. Determine the matrices $H_{j_1,j_2} \in \mathbb{R}^{p\times m}$ such that

$$\pi(z_1, z_2)H(z_1, z_2) = \sum_{j_1=0}^{r_1-1} \sum_{j_2=0}^{r_2-1} H_{j_1,j_2} z_1^{j_1} z_2^{j_2}.$$

Stage III. Determine the completely controllable realization $\Sigma = (A_1, A_2, B, C)$ of the matrix H:

$$A_1 = \text{diag}_{r_2}\{L_1\}$$

where

$$L_1 = \begin{bmatrix} O_m & I_m & O_m & \dots & O_m & O_m \\ O_m & O_m & I_m & \dots & O_m & O_m \\ . & . & . & \dots & . & . \\ O_m & O_m & O_m & \dots & O_m & I_m \\ -a_{10}I_m & -a_{11}I_m & -a_{12}I_m & \dots & -a_{1,r_1-2}I_m & -a_{1,r_1-2}I_m \end{bmatrix}$$

$$A_2 = \begin{bmatrix} O_{mr_1} & I_{mr_1} & O_{mr_1} & \dots & O_{mr_1} & O_{mr_1} \\ O_{mr_1} & O_{mr_1} & I_{mr_1} & \dots & O_{mr_1} & O_{mr_1} \\ . & . & . & \dots & . & . \\ O_{mr_1} & O_{mr_1} & O_{mr_1} & \dots & O_{mr_1} & I_{mr_1} \\ -a_{20}I_{mr_1} & -a_{21}I_{mr_1} & -a_{22}I_{mr_1} & \dots & -a_{1,r_2-2}I_{mr_1} & -a_{2,r_2-1}I_{mr_1} \end{bmatrix}$$

$$B = \underbrace{[O_m\ O_m \dots\ O_m\ I_m]}_{r_1 r_2}^t$$

$$C = [H_{00}\ H_{10}\ \dots\ H_{r_1-1,0} \vdots H_{01}\ H_{11}\ \dots\ H_{r_1-1,1} \vdots\ \dots\ \vdots H_{0,r_2-1}\ H_{1,r_2-1}\ \dots\ H_{r_1-1,r_2-1}]$$

(It follows that $H(z_1, z_2) = C(z_1 I_N - A_1)^{-1}(z_2 I_N - A_2)^{-1}B = H(z_1, z_2)$ where $N = \text{dim}\Sigma = r_1 r_2 m$).

The *Matlab* program presented below illustrates the above detailed bi-dimensional case, but it can be easily seen that it can be rewritten with no major difficulty for any other dimension. The program consists of 2 M-scripts and an M-function. The first M-script, *Transfmat.m* just clears all variables and defines a bi-dimensional strictly proper transfer matrix *H*, displaying this matrix in *pretty* form:

```
clear all,   echo off,
% Define a transfer matrix of two variables
syms z1 z2; H=[(z1-z2)/((z1^2-1)*(z2^2+1)) (z1+2)/((z1^2-1)*(z2+1));...
    2/((z1+1)*(z2+1)) , 3/((z1-1)*(z2-1))]; pretty(H)
```

The M-function *least_com_mul.m* provides the least common multiple of the 2 input polynomials:

```
function m = least_com_mul(p1, p2), echo off , syms t d x m;
%The output M will be the least common multiple of the input poly-
% nomials p1 and p2 as functions of x; Euclid's algorithm is used
t=p1; d=p2; [q,r]=quorem(t,d, x); if q== 0, t=p2;  d=p1;
[q,r]=quorem(t,d, x); end, t=d; while r ~= 0, d=r;
[q,r]=quorem(t,d, x); t=d; end, [q,r]=quorem(p1, d, x); m=q*p2;
```

The main script, *Find_contreal.m* applies the algorithm presented in the previous sections, finds a completely controllable realization of the given matrix H and completes also the verification:

```
% Find_contreal. This script is finding a standard controllable
% realization for the bidimensional strictly proper matrix H
syms z1 z2 x; [nrows,ncols]=size(H);
% Get numerators and denominators
for i=1:nrows, for j=1:ncols, temp=H(i,j);  [n,d]=numden(temp);
N(i,j)=n; D(i,j)=d;    end, end
% Split denominators and then find the least common denominator(s)
for i=1:nrows, for j=1:ncols,  f_temp=inline(D(i,j), 'z1', 'z2');
p_temp=sym(f_temp(z1,0)); cz1 = sym2poly(p_temp);
if cz1(1)==0, Dz1(i,j)=sym(1); else, div=sym(cz1(1));
p1_temp=p_temp/div;Dz1(i,j)=p1_temp; end
p_temp=sym(f_temp(0,z2)); cz2 = sym2poly(p_temp);
if cz2(1)==0, Dz2(i,j)=sym(1); else, div=sym(cz2(1));
p1_temp=p_temp/div; Dz2(i,j)=p1_temp; end,
end, end
% least common denominator z1
p1 = subs(Dz1(1,1), z1, x);
for i=1:nrows, for j=1:ncols, p2=subs(Dz1(i,j), z1, x);
m=least_com_mul(p1, p2); p1 = expand(m); end, end
cmz1 = sym2poly(m);m=m/cmz1(1);r1=length(cmz1)-1;mz1=subs(m, x, z1);
% least common denominator z2
p1 = subs(Dz2(1,1), z2, x); for i=1:nrows, for j=1:ncols
p2=subs(Dz2(i,j), z2, x); m=least_com_mul(p1, p2); p1 = m; end, end
cmz2 = sym2poly(m); r2=length(cmz2)-1; m=m/cmz2(1);
mz2=subs(m, x, z2);
% Find numerators after bringing to the lcd
for i=1:nrows, for j=1:ncols, [ampz1, r]= quorem(mz1,Dz1(i,j));
[ampz2, r]= quorem(mz2,Dz2(i,j)); T(i,j)= N(i,j)*ampz1*ampz2;
T(i,j)=expand(T(i,j));end, end
% Find the standard controllable realization
Im=eye(ncols); cL1=fliplr(flipud(compan(cmz1))); L1=kron(cL1, Im);
A1=kron(eye(r2),L1); Imr1=eye(ncols*r1);
cL2=fliplr(flipud(compan(cmz2)));
A2=kron(cL2, Imr1); Bprep=zeros(r1*r2,1); Bprep(r1*r2,1)=1;
```

```
B=kron(Bprep,Im); C=zeros(nrows,ncols*r1*r2);
for k=0:r2-1, for i=1:nrows, for j=1:ncols
T_temp=inline(T(i,j), 'z1', 'z2'); Tz1_temp=(T_temp(z1,0));
if Tz1_temp ~=0, c_temp=fliplr(sym2poly(sym(T_temp(z1,0))));
for l=1:length(c_temp), C(i,j+k*r1*ncols+ncols*(l-1))=c_temp(l); end
T(i,j)=T(i,j)-T_temp(z1,0); end, T(i,j)=simple(T(i,j)/z2);
end, end, end
The_standard_controllable_realization_for 'H=', pretty(H)
is = '(A1, A2, B, C) where',  A1, A2, B_transp=B', C
```

For example, given the matrix H from the script *Transfmat.m* above and denoting by $e1, e2, ..., e16$ the unit column vectors in $\mathbb{R}^{16}$, the answer provided by the *Matlab* program is:

```
A1 = [e3 e4 e1 e2 e7 e8 e5 e6 e11 e12 e9 e10 e15 e16 e13 e14]
A2 = [e13 e14 e15 e16 e1 e2 e3 e4 e5 e6 e7 e8 e9 e10 e11 e12]
B = [e15 e16]
C =  [0, -2, -1, -1, 1, 2, 0, 1, 0, -2, 1, -1, -1, 2, 0, 1; 2, 3, -2,
3, -2, 3, 2, 3, 2, 3, -2, 3, -2, 3, 2,  3]
```

Conclusion

A class of nD discrete-time systems is considered and its state-space and frequency-domain representations are provided. By using the multidimensional (nD) $\mathcal{Z}$-transformation, it is shown that the transfer matrices for these systems are proper rational functions in n indeterminates with separable denominators. A realization problem for these systems is stated and an algorithm is proposed to construct standard controllable realizations for transfer matrices with separable denominators. A Matlab program is given for the provided algorithm and two examples illustrate the advantages of the proposed method.

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