Some results on complex Douglas spaces

Nicoleta Aldea

Abstract. In this paper we give several descriptions of the complex Douglas spaces. We study the conformal changes of some complex Finsler spaces, such as complex Berwald, generalized Berwald or more general, complex Douglas spaces. Finally, the necessary and sufficient conditions under which a two - dimensional complex Finsler space is a complex Douglas space, are pointed out.

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1 Introduction

Many great contributions to the geometry of the real Douglas and Berwald spaces, strongly connected to the equation of geodesics, are due to S. Bácsó, M. Matsumoto [7, 8]. The topic continues to be of interest for the conformal transformations of such real Finsler spaces, [12, 13, 23]. The conformal complex Berwald spaces are studied by T. Aikou in [2].

In the previous papers [3], we introduced and studied the complex Douglas spaces. This study is made from the viewpoint of the equations of a complex geodesic curve and by means of the projectively related complex Finsler metrics, [4]. Two complex Finsler metrics F and $\tilde{F}$, on a common underlying manifold M, are called projectively related if any complex geodesic curve, in [1]' s sense, of the first is also complex geodesic curve for the second and the other way around. This means that between the spray coefficients G^i and $\tilde{G}^i$ there is a so-called *projective change* $\tilde{G}^i = G^i + B^i + P\eta^i$, where P is a smooth function on $T'M$ with complex values and $B^i := \frac{1}{2}(\tilde{\theta}^{*i} - \theta^{*i})$. The projective change gives rise to projective curvature invariants: three of Douglas type and two of Weyl type. The vanishing of the projective curvature invariants of Douglas defines the complex Douglas spaces. A projective curvature invariant of Weyl type characterizes the complex Berwald spaces. They must be either purely Hermitian of constant holomorphic curvature or non purely Hermitian of vanish holomorphic curvature, (for more see [3]).

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In the present paper, the outcomes on complex Douglas spaces are enriched with the new results about conformal complex Douglas spaces. Two complex Finsler manifolds (M,F) and $(M,\tilde{F})$ are called conformal if $\tilde{F}^2 = e^{\rho(z)}F^2$, where ρ is a real valued function which depends only on the position z. Our aim is to give an answer how can we check the conformality of a complex Finsler manifold with a complex Douglas manifold and with a generalized Berwald manifold, respectively. We show that the generalized Berwald property is preserved to a conformal change, (Theorem 4.1). A weakly Kähler space remains weakly Kähler under any conformal change if and only if the function ρ is homothetic, (Theorem 4.2). A necessary and sufficient condition for that the complex Douglas property to persist under a conformal change of metrics is proven in Theorem 4.3. Also, the C - conformal changes are discussed.

The general theory on complex Douglas spaces is applied to the class of 2 - dimensional complex Finsler spaces. Theorem 5.2 reports on the necessary and sufficient conditions for a 2 - dimensional complex Finsler space to be a complex Douglas space.

The organization of the paper is as follows. In §2, we recall some preliminary properties of the n - dimensional complex Finsler spaces. Section §3 contains an overview of the complex Douglas spaces and in the sections §4 and §5, we prove aforementioned outcomes.

2 Preliminaries

Let M be an n - dimensional complex manifold and $z = (z^k)_{k=\overline{1,n}}$ be the complex coordinates in a local chart. The complexified $T_C M$ of the real tangent bundle $T_R M$, splits into the sum of the holomorphic tangent bundle $T'M$ and its conjugate $T''M$. The bundle $T'M$ is itself a complex manifold and the local coordinates in a local chart will be denoted by $u = (z^k, \eta^k)_{k=\overline{1,n}}$. These are changed into $(z'^k, \eta'^k)_{k=\overline{1,n}}$ by the rules $z'^k = z'^k(z)$ and $\eta'^k = \frac{\partial z'^k}{\partial z^l}\eta^l$.

A *complex Finsler space* is a pair (M,F), where $F : T'M \to \mathbb{R}^+$ is a continuous function satisfying the following conditions:

i) $L := F^2$ is smooth on $\widetilde{T'M} := T'M \backslash \{0\}$;

ii) $F(z,\eta) \geq 0$, the equality holds if and only if $\eta = 0$;

iii) $F(z,\lambda\eta) = |\lambda| F(z,\eta)$ for $\forall \lambda \in \mathbb{C}$;

iv) the Hermitian matrix $\left(g_{i\bar{j}}(z,\eta)\right)$ is positive definite, where $g_{i\bar{j}} := \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$ is the fundamental metric tensor. Equivalently, this means that the indicatrix is strongly pseudo-convex.

Consequently, from *iii*) we have $\frac{\partial L}{\partial \eta^k}\eta^k = \frac{\partial L}{\partial \bar{\eta}^k}\bar{\eta}^k = L$, $\frac{\partial g_{i\bar{j}}}{\partial \eta^k}\eta^k = \frac{\partial g_{i\bar{j}}}{\partial \bar{\eta}^k}\bar{\eta}^k = 0$ and $L = g_{i\bar{j}}\eta^i\bar{\eta}^j$.

Consider the sections of the complexified tangent bundle of $T'M$. $VT'M \subset T'(T'M)$ is the vertical bundle, locally spanned by $\{\frac{\partial}{\partial \eta^k}\}$, and $VT''M$ is its conjugate. A complex nonlinear connection, briefly (*c.n.c.*), is a supplementary complex subbundle to $VT'M$ in $T'(T'M)$, i.e. $T'(T'M) = HT'M \oplus VT'M$. The horizontal distribution $H_u T'M$ is locally spanned by $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\}$, where $N_k^j(z,\eta)$ are the coefficients of the (*c.n.c.*). The pair $\{\delta_k := \frac{\delta}{\delta z^k}, \dot{\partial}_k := \frac{\partial}{\partial \eta^k}\}$ will be called the adapted frame of the (*c.n.c.*), which obey the change rules $\delta_k = \frac{\partial z'^j}{\partial z^k}\delta'_j$ and $\dot{\partial}_k = \frac{\partial z'^j}{\partial z^k}\dot{\partial}'_j$. By conjugation everywhere we obtain an adapted frame $\{\delta_{\bar{k}}, \dot{\partial}_{\bar{k}}\}$ on $T''_u(T'M)$. The dual adapted frames are $\{dz^k, \delta\eta^k\}$ and $\{d\bar{z}^k, \delta\bar{\eta}^k\}$.

A (*c.n.c.*) related only to the fundamental function of the complex Finsler space (M, F) is the so-called Chern-Finsler (*c.n.c.*), (see. [1]), with the local coefficients $N_j^i := g^{\overline{m}i}\frac{\partial g_{l\overline{m}}}{\partial z^j}\eta^l$. Subsequently, δ_k is the adapted frame with respect to the Chern-Finsler (*c.n.c.*). A Hermitian connection D, of $(1,0)-$ type, which satisfies in addition $D_{JX}Y = JD_XY$, for all X horizontal vectors and J the natural complex structure of the manifold, is the Chern-Finsler connection, [1]. It is locally given by the following coefficients (see [16]):

$$L_{jk}^i := g^{\bar{l}i}\delta_k g_{j\bar{l}} = \dot{\partial}_j N_k^i \;\; ; \; C_{jk}^i := g^{\bar{l}i}\dot{\partial}_k g_{j\bar{l}}. \tag{2.1}$$

In [1] are used the following terminologies. A complex Finsler space (M, F) is called: *strongly Kähler* iff $T_{jk}^i = 0$, *Kähler* iff $T_{jk}^i\eta^j = 0$ and *weakly Kähler* iff $g_{i\bar{l}}T_{jk}^i\eta^j\overline{\eta}^l = 0$, where $T_{jk}^i := L_{jk}^i - L_{kj}^i$. In [10] is proved that strongly Kähler and Kähler notions actually coincide. We notice that in the particular case of the complex Finsler metrics which come from Hermitian metrics on M, so-called *purely Hermitian metrics* in [16], (i.e. $g_{i\bar{j}} = g_{i\bar{j}}(z)$), all those nuances of Kähler are same. On the other hand, as in Aikou's work [2], a complex Finsler space which is Kähler and $L_{jk}^i = L_{jk}^i(z)$ is named *complex Berwald* space.

Chern-Finsler (*c.n.c.*) does not generally come from a complex spray, but, its local coefficients N_j^i always determine a complex spray with coefficients $G^i = \frac{1}{2}N_j^i\eta^j$. Further, G^i induce a (*c.n.c.*) denoted by $\overset{c}{N_j^i} := \dot{\partial}_j G^i$ which is called *canonical* in [16], where it is proved that it coincides with Chern-Finsler (*c.n.c.*) if and only if the complex Finsler metric is Kähler. With respect to the canonical (*c.n.c.*), we consider the frame $\{\overset{c}{\delta}_k, \dot{\partial}_k\}$, where $\overset{c}{\delta}_k := \frac{\partial}{\partial z^k} - \overset{c}{N_k^j}\dot{\partial}_j$, and its dual coframe $\{dz^k, \overset{c}{\delta}\eta^k\}$, where $\overset{c}{\delta}\eta^k := d\eta^k + \overset{c}{N_j^k}dz^j$. Moreover, we associate to the canonical (*c.n.c.*), a complex linear connection of Berwald type $B\Gamma$ with its connection form

$$\omega_j^i(z,\eta) = G_{jk}^i dz^k + G_{j\bar{k}}^i d\bar{z}^k, \tag{2.2}$$

where $G_{jk}^i := \dot{\partial}_k \overset{c}{N_j^i} = G_{kj}^i$ and $G_{j\bar{k}}^i := \dot{\partial}_{\bar{k}}\overset{c}{N_j^i}$.

Note that the spray coefficients perform $2G^i = N_j^i\eta^j = \overset{c}{N_j^i}\eta^j = G_{jk}^i\eta^j\eta^k = L_{jk}^i\eta^j\eta^k$. We denote by $G_{jkh}^i := \dot{\partial}_h G_{jk}^i$, $G_{j\bar{k}\bar{h}}^i = \dot{\partial}_{\bar{h}}G_{j\bar{k}}^i$ and $G_{j\bar{k}h}^i := \dot{\partial}_h G_{j\bar{k}}^i$ the hv-, $\bar{h}\bar{v}$- and $h\bar{v}$- curvature tensors, respectively.

An extension of the complex Berwald spaces, directly related to the $B\Gamma$ connection, is called by us *generalized Berwald* in [5]. For a such space the coefficients G_{jk}^i depend only on the position z. This condition is equivalent with either $\dot{\partial}_{\bar{h}}G^i = 0$ or the complex linear connection $B\Gamma$ is of $(1,0)$ - type. Since in the Kähler case $G_{jk}^i = L_{jk}^i$, any complex Berwald space is generalized Berwald. Moreover, in [3] we have proved that any generalized Berwald space, which is weakly Kähler, is a complex Berwald space.

3 Complex Douglas spaces

In Abate-Patrizio's sense, (see [1] p. 101), the equations of a complex geodesic curve $z = z(s)$ of (M, F), with s a real parameter, can be expressed as follows

$$\frac{d^2 z^i}{ds^2} + 2G^i(z(s), \frac{dz}{ds}) = \theta^{*i}(z(s), \frac{dz}{ds}) \; ; \; i = \overline{1,n}, \tag{3.1}$$

where by $z^i(s)$, $i = \overline{1,n}$, we denote the coordinates along of curve $z = z(s)$ and $\theta^{*k} := 2g^{\bar{j}k}\overset{c}{\delta}_{\bar{j}}L$. Note that θ^{*i} identically vanishes if and only if the space is weakly Kähler.

Taking an arbitrary transformation of the parameter $t = t(s)$, with $\frac{dt}{ds} > 0$, the equations (3.1) cannot in general be preserved. It follows that the complex geodesic curve $z = z(t(s))$ of (M, F) satisfies the differential equations

$$\frac{d^2 z^j}{dt^2}\eta^k - \frac{d^2 z^k}{dt^2}\eta^j + 2D^{jk} = 0, \tag{3.2}$$

where $D^{jk} := G^j\eta^k - G^k\eta^j - \frac{1}{2}(\theta^{*j}\eta^k - \theta^{*k}\eta^j)$.

The homogeneity property of the spray coefficients G^i and of the functions θ^{*i} leads to

$$D^{jk}_{r}\eta^r + D^{jk}_{\bar{r}}\bar{\eta}^r = 3D^{jk} \text{ and } D^{jk}_{\bar{r}}\bar{\eta}^r = -\frac{1}{2}(\theta^{*j}\eta^k - \theta^{*k}\eta^j), \tag{3.3}$$

where $D^{jk}_{r} := \dot{\partial}_r D^{jk}$ and $D^{jk}_{\bar{r}} := \dot{\partial}_{\bar{r}} D^{jk}$. Further on, differentiating (3.3) with respect to η, it gives

$$\begin{aligned} D^{jk}_{rh}\eta^r + D^{jk}_{\bar{r}h}\bar{\eta}^r &= 2D^{jk}_{h} \; ; \; D^{jk}_{rhl}\eta^r + D^{jk}_{\bar{r}hl}\bar{\eta}^r = D^{jk}_{hl}; \\ D^{jk}_{rhlm}\eta^r + D^{jk}_{\bar{r}hlm}\bar{\eta}^r &= 0 \; ; \; D^{jk}_{\bar{m}rhl}\eta^r + D^{jk}_{\bar{m}\bar{r}hl}\bar{\eta}^r = 0, \end{aligned} \tag{3.4}$$

where $D^{jk}_{rh} := \dot{\partial}_h D^{jk}_{r}$, $D^{jk}_{\bar{r}h} := \dot{\partial}_h D^{jk}_{\bar{r}}$, $D^{jk}_{rhl} := \dot{\partial}_l D^{jk}_{rh}$, and all that.

Let $\tilde{F}$ be another complex Finsler metric on the underlying manifold M. Corresponding to the metric $\tilde{F}$, we have the spray coefficients $\tilde{G}^i$ and the functions $\tilde{\theta}^{*i}$. The complex Finsler metrics F and $\tilde{F}$ on the manifold M, are called *projectively related* if they have the same complex geodesics as point sets. This means that for any complex geodesic curve $z = z(s)$ of (M, F) there is a transformation of its parameter s, $\tilde{s} = \tilde{s}(s)$, with $\frac{d\tilde{s}}{ds} > 0$, such that $z = z(\tilde{s}(s))$ is a geodesic of $(M, \tilde{F})$, and conversely.

Theorem 3.1. *[4]. Let F and $\tilde{F}$ be complex Finsler metrics on the manifold M. Then F and $\tilde{F}$ are projectively related if and only if there is a smooth function P on $T'M$ with complex values, such that*

$$\tilde{G}^i = G^i + B^i + P\eta^i; \; i = \overline{1,n}, \tag{3.5}$$

where $B^i := \frac{1}{2}(\tilde{\theta}^{*i} - \theta^{*i})$.

The relations (3.5) between the spray coefficients $\tilde{G}^i$ and G^i of the projectively related complex Finsler metrics F and $\tilde{F}$ is called *projective change.* Further on, the projective change (3.5) gives rise to various projective invariants, (for more details see [3]). Indeed, some successive differentiations of (3.5) with respect to η and $\bar{\eta}$ give three *projective curvature invariants of Douglas type*

(3.6)
$$\begin{aligned} D^i_{jkh} &= G^i_{jkh} - \frac{1}{n+1}[(\dot{\partial}_h D_{jk})\eta^i + \sum_{(j,k,h)} D_{jh}\delta^i_k] \\ &\quad -\frac{1}{2}\{\theta^{*i}_{jkh} - \frac{1}{n}[(\dot{\partial}_h \theta^{*l}_{ljk})\eta^i + \sum_{(j,k,h)} \theta^{*l}_{ljh}\delta^i_k]\}; \\ D^i_{j\bar{k}\bar{h}} &= G^i_{j\bar{k}\bar{h}} - \frac{1}{n+1}[(\dot{\partial}_j D_{\bar{k}\bar{h}})\eta^i + D_{\bar{k}\bar{h}}\delta^i_j] \\ &\quad -\frac{1}{2}\{\theta^{*i}_{j\bar{k}\bar{h}} - \frac{1}{n}[(\dot{\partial}_{\bar{h}} \theta^{*l}_{l\bar{k}j})\eta^i + \theta^{*l}_{l\bar{k}\bar{h}}\delta^i_j]\}; \\ D^i_{j\bar{k}h} &= G^i_{j\bar{k}h} - \frac{1}{n+1}[(\dot{\partial}_h D_{\bar{k}j})\eta^i + D_{\bar{k}j}\delta^i_h + D_{\bar{k}h}\delta^i_j] \\ &\quad -\frac{1}{2}\{\theta^{*i}_{j\bar{k}h} - \frac{1}{n}[(\dot{\partial}_h \theta^{*l}_{l\bar{k}j})\eta^i + \theta^{*l}_{l\bar{k}j}\delta^i_h + \theta^{*l}_{l\bar{k}h}\delta^i_j]\}, \end{aligned}$$

where $D_{kh} := G^i_{ikh}$, $D_{\bar{k}\bar{h}} := G^i_{i\bar{k}\bar{h}}$ and $D_{\bar{k}h} := G^i_{i\bar{k}h}$ are respectively, hv-, $\bar{h}\bar{v}$- and $h\bar{v}$- *Ricci tensors* and $\theta^{*i}_{jkh} := \dot{\partial}_h \theta^{*i}_{jk}$, $\theta^{*i}_{jk} := \dot{\partial}_k \theta^{*i}_j$, $\theta^{*i}_{j\bar{k}h} := \dot{\partial}_{\bar{k}} \theta^{*i}_{jh}$, $\theta^{*i}_{j\bar{k}\bar{h}} := \dot{\partial}_{\bar{k}} \theta^{*i}_{j\bar{h}}$ and $\theta^{*i}_{j\bar{h}} := \dot{\partial}_{\bar{h}} \theta^{*i}_j = \dot{\partial}_j \theta^{*i}_{\bar{h}}$. In (3.6), $\sum_{(j,k,h)}$ is the cyclic sum.

Definition 3.1. [3]. A complex Finsler space (M, F) is called complex Douglas space if the invariants (3.6) vanish identically.

Note that any complex Berwald space is a complex Douglas space.

Theorem 3.2. *Let F and $\tilde{F}$ be projectively related complex Finsler metrics on the manifold M. Then, F is a Douglas metric if and only if $\tilde{F}$ is also a Douglas metric.*

Theorem 3.3. *Let (M, F) be a complex Finsler space. The following statements are equivalent:*

i) (M, F) is a complex Douglas space;

ii) (M, F) is a generalized Berwald space and the functions $K^i := \theta^{*i} - \frac{1}{n}\theta^{*l}_l \eta^i$ are homogeneous polynomials in η and in $\bar{\eta}$ of first degree;

iii) (M, F) is a generalized Berwald space with D^{jk}_{hrm} and $D^{jk}_{h\bar{l}m}$ depending only on z and $\bar{z}$.

Any purely Hermitian metric is a complex Douglas metric. Indeed, considering the purely Hermitian metric $g_{i\bar{j}} = g_{i\bar{j}}(z)$, we obtain

$$G^i = \frac{1}{2} g^{\bar{m}i} \frac{\partial g_{l\bar{m}}}{\partial z^j} \eta^l \eta^j \ ; \ \theta^{*i} = -g^{\bar{m}i}(\frac{\partial g_{l\bar{m}}}{\partial \bar{z}^k} - \frac{\partial g_{l\bar{k}}}{\partial \bar{z}^m})\eta^l \bar{\eta}^k.$$

From these we deduce $\dot{\partial}_{\bar{h}} G^i = 0$ and so, any purely Hermitian metric is generalized Berwald. Moreover, because the functions $g^{\bar{m}i}(\frac{\partial g_{l\bar{m}}}{\partial \bar{z}^k} - \frac{\partial g_{l\bar{k}}}{\partial \bar{z}^m})$ depend only on z and $\bar{z}$, it gives that K^i are homogeneous polynomials in η and in $\bar{\eta}$ of first degree.

Theorem 3.4. *Let (M, F) be a complex Finsler space. The following statements are equivalent:*

i) (M,F) *is a complex Berwald space;*
ii) (M,F) *is a generalized Berwald space and the functions* K^i *vanish identically;*
iii) the functions D^{jk} *are holomorphic with respect to* η.

4 Conformal complex Douglas spaces

Let (M,F) be a complex Finsler space and $\tilde{F}$ be another complex Finsler metric on the manifold M. The complex Finsler metrics F and $\tilde{F}$ on a common underlying manifold M, are called *conformal* if there exists a real valued function ρ, which depends only on the position z, satisfying $\tilde{F}^2 = e^{\rho(z)}F^2$. This implies $\tilde{g}_{i\bar{j}} = e^{\rho(z)}g_{i\bar{j}}$ and the following relations are hold between the geometrical objects of the conformal complex Finsler spaces $(M,\tilde{F})$ and (M,F).

Lemma 4.1. *Let* F *and* $\tilde{F}$ *be two conformal complex Finsler metrics. Then,*
i) $\tilde{G}^i = G^i + \frac{1}{2}\frac{\partial\rho}{\partial z^j}\eta^j\eta^i$; $\dot{\partial}_{\bar{h}}\tilde{G}^i = \dot{\partial}_{\bar{h}}G^i$; $\tilde{C}^i_{jk} = C^i_{jk}$.
ii) $\tilde{\overset{c}{N}}{}^i_j = \overset{c}{N}{}^i_j + \frac{1}{2}(\frac{\partial\rho}{\partial z^j}\eta^i + \frac{\partial\rho}{\partial z^l}\eta^l\delta^i_j)$; $\tilde{\overset{c}{\delta}}_j = \overset{c}{\delta}_j - \frac{1}{2}(\frac{\partial\rho}{\partial z^j}\eta^i + \frac{\partial\rho}{\partial z^l}\eta^l\delta^i_j)\dot{\partial}_i$.

Theorem 4.2. *Let* (M,F) *and* $(M,\tilde{F})$ *be two conformal complex Finsler spaces.* (M,F) *is generalized Berwald if and only if* $(M,\tilde{F})$ *is also generalized Berwald.*

The proof immediately results by Lemma 4.1 i).

The conformal function ρ is called *homothetic* if $\rho_i := \frac{\partial\rho}{\partial z^i}$. Also, ρ is called C - *conformal* if it is not homothetic, but satisfies $C^i_{jk}\rho^k = 0$, where $\rho^k := g^{\bar{j}k}\rho_{\bar{j}}$ and $\rho_{\bar{j}} := \frac{\partial\rho}{\partial\bar{z}^j}$.

By Lemma 4.1 it follows

$$\begin{aligned} \tilde{\theta}^{*i} &= \theta^{*i} + \rho^i L - \rho_{\bar{l}}\bar{\eta}^l\eta^i; \\ \tilde{\theta}^{*k}_k &= \theta^{*k}_k - C^k_{kh}\rho^h - (n-1)\rho_{\bar{l}}\bar{\eta}^l. \end{aligned}$$

Thus, we obtain

$$\tilde{K}^i = K^i + \rho^i L + \frac{1}{n}(C^k_{kh}\rho^h - \rho_{\bar{l}}\bar{\eta}^l)\eta^i. \tag{4.1}$$

Theorem 4.3. *Let* (M,F) *and* $(M,\tilde{F})$ *be two conformal complex Finsler spaces. If one of the metrics* F *and* $\tilde{F}$ *is weakly Kähler then the other is also weakly Kähler if and only if the function* ρ *is homothetic.*

Proof. Indeed, F and $\tilde{F}$ are weakly Kähler iff $\rho^i = \frac{1}{L}\rho_{\bar{l}}\bar{\eta}^l\eta^i$ which gives $C^k_{kh}\rho^h = -\frac{n-1}{L}\rho_{\bar{l}}\bar{\eta}^l$ and so ρ compulsory needs be a homothetic function. □

Theorem 4.4. *Let* (M,F) *and* $(M,\tilde{F})$ *be two conformal complex Finsler spaces. If one of* F *and* $\tilde{F}$ *is a complex Douglas metric then the other is also a complex Douglas metric if and only if* $\rho^i L + \frac{1}{n}(C^k_{kh}\rho^h - \rho_{\bar{l}}\bar{\eta}^l)\eta^i$ *are homogeneous polynomials in* η *and in* $\bar{\eta}$ *of first degree.*

Proof. It results by Theorem 3.2 ii) and by the formula (4.1). □

Due to (4.1), if the conformal function ρ is homothetic and one of the space (M,F) and $(M,\tilde{F})$ is complex Douglas, then the other is also a complex Douglas space. Likewise, by (4.1) it follows: if (M,F) or $(M,\tilde{F})$ is purely Hermitian then both are purely Hermitian.

Theorem 4.5. *If the conformal function ρ between the conformal complex Douglas spaces (M,F) and $(M,\tilde{F})$ is C - conformal, then F and $\tilde{F}$ are purely Hermitian.*

Proof. Since F and $\tilde{F}$ are complex Douglas metrics and ρ is C - conformal, by Theorem 4.2 it follows that $\rho^i L - \frac{1}{n}\rho_{\bar{l}}\bar{\eta}^l\eta^i$ are homogeneous polynomials in η and in $\bar{\eta}$ of first degree. Two successive differentiations of the last relation with respect to η lead to $C^i_{jk}=0$ which means that F is purely Hermitian. By Lemma 4.1 i) it results that $\tilde{F}$ is also purely Hermitian. □

As a consequence of Theorems 4.1 and 4.2 we have the following result.

Theorem 4.6. *Let (M,F) and $(M,\tilde{F})$ be two conformal complex Finsler spaces. If one of the metrics F and $\tilde{F}$ is complex Berwald then the other is also complex Berwald if and only if the function ρ is homothetic.*

5 Two-dimensional complex Douglas spaces

An exhaustive study of 2 - dimensional complex Finsler spaces with respect to the local complex Berwald frames is made in [6]. Here, we shall summarize some basic results which are needed.

Let M be a complex manifold of complex dimension two. Everywhere in this section the indices $i,j,k,...$ run over $\{1,2\}$. Consider $l := l^i\dot{\partial}_i$ the radial vertical vector field with its dual form $\omega = l_i\delta\eta^i$, where $l^i = \frac{1}{F}\eta^i$ and $l_i := \frac{1}{F}g_{i\bar{j}}\bar{\eta}^j = g_{i\bar{j}}l^{\bar{j}}$.

The vertical bundle $VT'M$, which is 2 - dimensional in any point, is decomposed into $VT'M = \{l\}\oplus\{l\}^\perp$, where $\{l\}^\perp$ is spanned by a complex vector m. If l and m are orthogonal and m is a unit vector then these get the following linear system $\begin{cases} l_1m^1 + l_2m^2 = 0 \\ m_1m^1 + m_2m^2 = 1 \end{cases}$, where $m_i := g_{i\bar{j}}m^{\bar{j}}$, Formally, solving this system as one linear, it is obtained the solutions $m^1 = \frac{-l_2}{\Delta}$, $m^2 = \frac{l_1}{\Delta}$, $m_1 = -\Delta l^2$ and $m_2 = \Delta l^1$, where $\Delta = l_1m_2 - l_2m_1$, which indeed are not completely determined because Δ depends on m_i. A straightforward computation proves that $|\Delta| = \sqrt{g}$ and $\Delta' = \mathcal{T}\Delta$, under a change of the local coordinates $(z^k,\eta^k)_{k=\overline{1,2}}$ into $(z'^k,\eta'^k)_{k=\overline{1,2}}$, where $g := \det(g_{i\bar{j}})$ and $\mathcal{T} := \det\left(\frac{\partial z^i}{\partial z'^j}\right)$. But, when we work in a fixed local chart, we can choose $\Delta = \sqrt{g}$, i.e. Δ is real, which produces the unique solutions $m^1 = \frac{-l_2}{\sqrt{g}}$, $m^2 = \frac{l_1}{\sqrt{g}}$, $m_1 = -\sqrt{g}l^2$ and $m_2 = \sqrt{g}l^1$. Thus, we have

$$m = \frac{1}{\sqrt{g}}(-l_2\dot{\partial}_1 + l_1\dot{\partial}_2), \tag{5.1}$$

in a fixed chart. Then $\{l,m,\bar{l},\bar{m}\}$, with m given by (5.1) is called the *local complex Berwald frame* of the space. Indeed, since the local frame is orthonormal we have: $l^il_i = m^im_i = 1$ and $l^im_i = l_im^i = 0$.

We specify that (5.1) provides only a local frame, because the set of natural local basis in every chart does not have tensorial character. More precisely, we can check that $m' = \frac{\mathcal{I}}{|\mathcal{I}|}m$, at the local change of charts. This show that m is not a vector, it depends on the local change. Therefore, it will say that m from (5.1) is a pseudo-vector. Although m from (5.1) depends on the local changes of the coordinates, it is very important in our study, in a fixed chart. Note that all next results are globally valid.

With respect to the local complex Berwald frame, $\dot{\partial}_k$ and $g_{i\bar{j}}$ are decomposed as follows $\dot{\partial}_i = l_i l + m_i m$ and hence, $g_{i\bar{j}} = l_i l_{\bar{j}} + m_i m_{\bar{j}}$. From here it is deduced

$$G^i = \frac{L}{2}(Jl^i + Om^i)\ ;\ \overset{c}{N}{}^i_j = N^i_j + \frac{F}{2}(U-V)l^i m_j + \frac{F}{2}(Y-E)m^i m_j, \tag{5.2}$$

where $J := l^j l^k l_i L^i_{jk}$, $O := l^j l^k m_i L^i_{jk}$, $U := m^j l^k l_i L^i_{jk}$, $V := l^j m^k l_i L^i_{jk}$, $Y := m^j l^k m_i L^i_{jk}$ and $E := l^j m^k m_i L^i_{jk}$.

Note that $U = V$ and $Y = E$ characterize the Kähler spaces and for any weakly Kähler space we have $U = V$. Moreover, we obtain

$$\overset{c}{\delta}_{\bar{j}} L = -\frac{L}{2}(\bar{U} - \bar{V})m_{\bar{j}}$$

which implies

$$\theta^{*k} = -L(\bar{U} - \bar{V})m^k,\ k = 1, 2. \tag{5.3}$$

Owing to (5.2) and (5.3), it follows

$$D^{12} = -\frac{F^3}{\sqrt{g}}[\frac{O}{2} - (\bar{U} - \bar{V})]. \tag{5.4}$$

Using now Theorem 3.4 iii), we have proven the following.

Theorem 5.1. *Let (M, F) be a 2 - dimensional complex Finsler space. Then, it is a complex Berwald space if and only if $D^{12} = -\frac{F^3}{\sqrt{g}}[\frac{O}{2} - (\bar{U} - \bar{V})]$ is holomorphic with respect to η. Given any of them, we have $D^{12} = -\frac{F^3 O}{2\sqrt{g}}$.*

Subsequently, our goal is to find the conditions when a 2 - dimensional complex Finsler space is a complex Douglas spaces.

The local complex Berwald frames also satisfy important properties. We mention here only some, which are needed in our study, (for more, see [6]),

$$\begin{aligned} l(m^i) &= -\frac{1}{2F}m^i\ ;\ m(m^i) = -\frac{1}{2}Bm^i - Al^i\ ;\ l(\bar{U}) = \frac{1}{2F}\bar{U}; \\ m(\bar{U}) &= \frac{1}{F}(\bar{Y} - \bar{J}) - \frac{1}{2}B\bar{U} - F\bar{A}[m(\bar{O}) - \frac{1}{2}B\bar{O}]; \\ l(\bar{V}) &= \frac{1}{2F}\bar{V}\ ;\ m(\bar{V}) = \frac{1}{F}(\bar{E} - \bar{J}) - \frac{1}{2}B\bar{V}, \end{aligned} \tag{5.5}$$

where $A := m^j m^k l_h C^h_{kj}$ and $B := m_h m^k m^j C^h_{jk}$.

With respect to the local complex Berwald frames, some trivial computations lead to

$$\theta_l^{*l} = -F[(\bar{Y} - \bar{E}) - BF(\bar{U} - \bar{V}) - L\bar{A}[m(\bar{O}) - \frac{1}{2}B\bar{O}] \tag{5.6}$$

and hence

$$K^i = -L(\bar{U} - \bar{V})m^i + \frac{L}{2}[(\bar{Y} - \bar{E}) - BF(\bar{U} - \bar{V}) - L\bar{A}[m(\bar{O}) - \frac{1}{2}B\bar{O}]l^i \tag{5.7}$$

Not that a 2 - dimensional complex Finsler space is generalized Berwald iff $m(\bar{O}) - \frac{1}{2}B\bar{O} = 0$, (see [6]). According to Theorem 3.3 ii) we have substantiated the following result.

Theorem 5.2. *Let (M, F) be a 2 - dimensional complex Finsler space. It is a complex Douglas space if and only if $m(\bar{O}) - \frac{1}{2}B\bar{O} = 0$ and $K^i = -L(\bar{U} - \bar{V})m^i + \frac{L}{2}[(\bar{Y} - \bar{E}) - BF(\bar{U} - \bar{V})]l^i$, $i = 1, 2$, are homogeneous polynomials in η and in $\bar{\eta}$ of first degree.*

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Author's address:

Nicoleta Aldea
Transilvania Univ., Department of Mathematics and Informatics,
Iuliu Maniu 50, Braşov 500091, Romania.
E-mail: nicoleta.aldea@lycos.com