# Elliptic variational inequalities in nonreflexive Banach spaces

B. Aylaj and J. Lahrache

**Abstract.** In this paper, we give sufficient conditions for the existence and stability results of solutions to elliptic variational inequalities in nonreflexive Banach spaces.

**M.S.C. 2010**: 35R35; 49J40.
**Key words**: elliptic; variational inequality; pseudomontone operator; Attouch-Wets convergence; nonreflexive Banach spaces.

## 1 Introduction

Let $V$ be a real Banach space, $\Gamma(V)$ the set of proper lower semicontinuous convex functions on $V$, $A : V \to V^*$ a pseudomonotone operator, $\Phi$ in $\Gamma(V)$ and $f \in V^*$. The elliptic variational inequality problem is to find an $u \in V$ such that

$$\langle Au - f, v - u\rangle \ \geq \ \Phi(u) - \Phi(v) \quad \text{for all } v \in V. \tag{1.1}$$

Many authors have studied elliptic variational inequalities in reflexive Banach spaces (see for example [6, 5, 3, 8] and also [7]). For results in nonreflexive Banach spaces, Watson [10] studied the existence and stability of solutions to variational inequality problem

$$\langle Au, v - u\rangle \ \geq \ 0 \quad \text{for all } v \in K, \tag{1.2}$$

where $K$ is a nonempty closed bounded convex subset of $V$. We can show that the elliptic variational inequalities (1.1) are equivalent to inequalities of type (1.2) in the space $\widetilde{V} = V \times \mathbb{R}$

$$\widetilde{u} \in \widetilde{K} : \ \langle \widetilde{A}\widetilde{u} - \widetilde{f}, \widetilde{v} - \widetilde{u}\rangle \geq 0 \ \text{ for all } \widetilde{v} \in \widetilde{K} \tag{1.3}$$

such that, $\widetilde{V}$ normed by $\| \ \|_* = \| \ \| + | \ |$, $\widetilde{A} = (A, 1)$ in $\widetilde{V}$ to $\widetilde{V}^*$, $\widetilde{f} = (f, -1)$ and $\widetilde{K} = \text{epi}\Phi$. This makes it possible to deduce the properties of (1.1) from the corresponding properties of (1.3). The sufficient conditions are given in [10] for the existence and stability results cannot be applied in the elliptic variational inequalities (1.1) $\big($since, $\widetilde{K}$ not bounded subset of $\widetilde{V}\big)$. It is the purpose of this paper to prove the existence and stability results of solutions to elliptic variational inequalities (1.1) under weaker conditions than those needed in [4, 10].

BSG Proceedings, Vol. 19, 2012, pp. 11-17.

## 2 Main results

Throughout the sequel, we assume $V$ is a real nonreflexive Banach space and we denote by $CB(V)$ the set of nonempty closed norm bounded convex subsets of $V$. First of all, we recall some definitions.

- An operator $A : V^{**} \to V^*$ is said to be

  $(i)$ - hemicontinuous if for any $u$ and $v$ in $V^{**}$ the map $t \mapsto A(u + tv)$ of $[0, 1]$ to $V^*$ is continuous for the natural topology of $[0, 1]$ and the weak topology of $V^*$.

  $(ii)$ - pseudomonotone if $\langle Au, v - u\rangle \geq 0$ implies $\langle Av, v - u\rangle \geq 0$, for all $v, u \in V^{**}$.

- We define the Hausdorff distance between two elements $C$ and $B$ of nonempty closed subsets, i.e. $CL(V)$, of $V$, by

$$H(B, C) = sup_{v\in V} |d^1(v, B) - d(v, C)|$$

- For any $\mu > 0$, the $\mu-$Hausdorff distance between $B$ and $C$ is given by

$$\text{haus}_\mu(B, C) = max\{e(B \cap \mu U, C), e(C \cap \mu U, B)\},$$

  where $U$ is the closed unit ball of $V$ and $e(B, C) = \sup_{c\in C} d(c, B)$ is the excess of $B$ over $C$.

- Let $K, K_1, K_2, ...$ be a sequence of $CL(V)$. Then $K_n$ is declared Attouch-Wets convergent to $K$, and we write $K = \tau_{AW} - \lim K_n$ if and only if for each $\mu > 0$, we have $\lim_{n\to+\infty} haus_\mu(K, K_n) = 0$.

Next, we give some results which will be needed in the proof of the main results.

**Lemma 2.1.** *[2] Let $\Phi \in \Gamma(V)$, and $(\Phi_n)$ be a sequence in $\Gamma(V)$, $(\Phi_n)$ is slice convergent to $\Phi$ if and only if both of the following conditions are satisfied*

$(i)$ *for each $v \in V$, there exists a sequence $(v_n)$ strongly convergent to $v$ such that $\Phi(v) = \lim_{n\to+\infty} \Phi_n(v_n)$;*

$(ii)$ *for each $f \in V^*$, there exists a sequence $(f_n)$ strongly convergent to $f$ such that $\Phi^*(f) = \lim_{n\to+\infty} \Phi^*_n(f_n)$.*

**Lemma 2.2.** *[10] Let $A : K \to V^*$ be pseudomonotone hemicontinuous and suppose $K$ is a subset of $CB(V^{**})$. Then, the variational inequality*
*$\langle Au, v - u\rangle \geq 0$, for all $v \in K$, has a solution $u \in K$.*

*We now prove the main result of this paper.*

[1] d be the metric determined by the norm.

**Theorem 2.3.** *Let $A : V^{**} \to V^*$ be a bounded pseudomonotone hemicontinuous operator, suppose $\Phi \in \Gamma(V^{**})$. Further assume*

$$(2.1)\qquad \begin{cases} \textit{there exists } v_0 \quad \textit{such that} \quad \Phi(v_0) < +\infty \quad \textit{and} \\ \dfrac{\langle Au, u - v_0\rangle + \Phi(u)}{\|u\|} \to +\infty \qquad \textit{if } \|u\| \to +\infty. \end{cases}$$

*then, for $f$ in $V^*$, there exists an $u \in V^{**}$ such that*

$$\langle Au - f, v - u\rangle \geq \Phi(u) - \Phi(v) \quad \textit{for all } v \in V^{**}.$$

*Proof.* It suffices to deal with the variational inequality (1.3) in the space $\widetilde{V}^{**}$. Clearly, $\widetilde{A}$ is pseudomonotone, hemicontinuous operator, provided $A$ is such.
Let $R \geq 0$ and

$$\widetilde{K}_R = \{\widetilde{v} | \widetilde{v} = (v, \xi) \in \widetilde{K},\ \|v - v_0\| + |\xi - \Phi| \leq R\}.$$

Then, by Lemma 2.2, there exists $\widetilde{u}_R$ in $\widetilde{K}_R$ such that

$$(2.2)\qquad \langle \widetilde{A}\widetilde{u}_R - \widetilde{f}, \widetilde{v} - \widetilde{u}_R\rangle \geq 0 \quad \text{for all } \widetilde{v} \in \widetilde{K}_R$$

For this purpose, it suffices to take $\widetilde{v} = \widetilde{v}_0 = (v_0, \Phi(v_0))$. If $\widetilde{u}_R = (u_R, \alpha_R)$; then setting $\widetilde{v}_0$ in (2.2) we get

$$(2.3)\qquad \langle Au_R, u_R - v_0\rangle + \alpha_R \leq \langle f, u_R - v_0\rangle + \Phi(v_0)$$

since $\alpha_R \geq \Phi(u_R)$ hence

$$\begin{aligned} \langle Au_R, u_R - v_0\rangle + \Phi(u_R) &\leq \langle f, u_R - v_0\rangle + \Phi(v_0) \\ &\leq c(1 + \|u_R\|); \end{aligned}$$

By the hypothesis (2.1), with $\Phi$ replaced by $\Phi - c$, we may conclude that $\|u_R\| \leq const$. Thus from (2.3) we obtain that : $\alpha_R \leq const$.
Let $\chi$ subgradient to $\Phi$ at $v_0$; it follows that

$$\alpha_R \geq \Phi(u_R) \geq \langle \chi, u_R - v_0\rangle + \Phi(v_0)$$

then $\alpha_R \geq \Phi(u_R) \geq -c_1$ from this, we obtain that $\|u_R\| + |\alpha_R| \leq c_2$ with $c_2$ is constant independent of $R$, we may conclude that $\|u_R - v_0\| + |\alpha_R - \Phi(v_0)| \leq c_3$. If the constant $R$ is chosen as $R > c_3$, then $\widetilde{u}_R$ is solution of inequality (1.3). Indeed, if $\widetilde{w}$ is taken in $\widetilde{K}$, thanks to the fact that $\|\widetilde{u}_R\|_{\widetilde{K}_R} < R$, we have

$$\widetilde{v} = (1 - \theta)\widetilde{u}_R + \theta\widetilde{w} \in \widetilde{K}_R$$

where $\theta$ is a positive constant sufficiently small, then setting $\widetilde{v}$ in (2.2), we get

$$\begin{aligned} \theta\langle \widetilde{A}\widetilde{u}_R, \widetilde{w} - \widetilde{u}_R\rangle &\geq \theta\langle \widetilde{f}, \widetilde{w} - \widetilde{u}_R\rangle, \\ \text{then } \langle \widetilde{A}\widetilde{u}_R, \widetilde{w} - \widetilde{u}_R\rangle &\geq \langle \widetilde{f}, \widetilde{w} - \widetilde{u}_R\rangle \quad \forall \widetilde{w} \in \widetilde{K}. \end{aligned}$$

This implies that $u_R$ is solution of inequality (1.1).

□

**Remark 2.1.** This proof use the weakens conditions required in [10] and combined with the method needed in [6] to prove the existence of solutions to elliptic inequalities (1.1).

## 3 Perturbation problems for elliptic variational inequalities

We consider a sequence of elliptic variational inequalities $(1_n)$, i.e., find $u_n \in V$ such that

$$\langle A_n u_n - f_n, v - u_n \rangle \geq \Phi_n(u_n) - \Phi_n(v), \text{ for all } v \in K_n.$$

**Theorem 3.1.** *Let $A,\ A_n : V \to V^*$ be pseudomonotone and hemicontinuous. Let $\Phi \in \Gamma(V)$, $\Phi_n$ be a sequence in $\Gamma(V)$, such that $\Phi \leq \Phi_n$ for each $n$. Further assume there exists a nonempty subset $X_0$ contained in a compact convex subset $X_1$ of $\widetilde{K}_n$ for each $n$ such that the set*

$$D = \{\widetilde{v} \in \widetilde{K} : \langle \widetilde{A}\widetilde{u} - \widetilde{f}, \widetilde{u} - \widetilde{v} \rangle \geq 0, \text{ for all } \widetilde{u} \in X_0\}$$

*is compact or empty, and let us suppose that*

*(i)- $f_n \to f$ in $V^*$;*

*(ii)- $A_n v \to Av$ in $V^*$ for all $v \in V$;*

*(iii)- $\Phi_n \overset{\tau_s}{\to} \Phi$; then,*

*(a)- there exists, for $f; f_n$ in $V^*$, the solutions $u$ and $u_n$ of* (1.1) *and* $(1_n)$, *respectively and*
*(b)- if $u_n \rightharpoonup v$ in $V$ then $v$ solution to the elliptic variational inequality* (1.1).

*Proof.* For each $n$, the condition $\widetilde{K}_n \subset \widetilde{K}$ is trivially satisfied if $\Phi \leq \Phi_n$, then we shall obtain the desired result (a), by applying [10, Theorem 1] (see also [9]) and, combined with the proof of [10, Theorem 2].
It is seen that the elliptic variational inequality $(1_n)$ is equivalent to the search of $u_n$ in $V$ such as[2]

$$-(A_n u_n - f_n) \in \partial\Phi_n(u_n),$$

let $\chi \in V^*$, by lemma 2.1, there exists a sequence $(\chi_n)$ strongly convergent to $\chi$ such that $\lim \Phi_n^*(\chi_n) = \Phi^*(\chi)$. Let $w$ be an element of $V$, again using the Minty's Lemma we conclude that

$$u_n \in \partial\Phi_n^*(-A_n w + f_n) \tag{3.1}$$

since, there exists an $u_0 \in V$ such that

$$\Phi^*(-Aw + f) < \langle f - Aw, u_0 \rangle - \Phi(u_0),$$

by lemma 2.1, there exists a sequence $(u_{0n})$ strongly convergent to $u_0$ such that $\lim \Phi_n(u_{0n}) = \Phi(u_0)$. Hence, from (3.1) we may deduce that

$$\langle \chi_n + A_n w - f_n, u_n \rangle \leq \Phi_n^*(\chi_n) - \langle A_n w + f_n, u_{0n} \rangle + \Phi_n(u_{0n}) \tag{3.2}$$

[2] $\partial\Phi_n(u_n)$ is the set of subgradients of $\Phi_n$ at $u_n$.

to prove then $v \in \partial\Phi^*(-Aw+f)$ for all $w \in V$, it suffices to pass to the limit in (3.2), by the Minty's lemma we conclude that $v \in \partial\Phi^*(-Av+f)$. Moreover, taking into account that

$$-Av + f \in \partial\Phi(v),$$

we obtain the desired result (b). □

**Theorem 3.2.** *Under the assumptions of Theorem 3.1 relative to the operators $A, A_n : V^{**} \to V^*$, with $(A_n)$ a sequence of uniformly bounded operators, if the following conditions are satisfied*

*(a)-* $f_n \to f$ *in* $V^*$;

*(b)-* $(A_n)$ *continuously converges to* $A$, *that is for all* $w \in V^{**}$ *and all sequence* $(w_n)$ *converging to* $w$, *the sequence* $(A_n w_n)$ *converges to* $Aw$;

*(c)-* $\Phi_n \overset{AW}{\to} \Phi$;

*(d)-* $\Phi_n(v_0) = \Phi(v_0)$ *and* $\Phi_n \geq \Phi$ *in subset of solutions the variational inequality* $(1_n)$;

*(e)-* $$\begin{cases} \dfrac{\langle A_n u, u - v_0\rangle + \Phi_n(u)}{\|u\|} \to +\infty \qquad \textit{uniformly} \\ \textit{if } \|u\| \to +\infty. \end{cases}$$

*then, the solution* $(u_n)$ *to the variational inequality* $(1_n)$ *has a subsequence which converges to the solution* $u$ *to* (1.1) *in the weak-star topology.*

*Proof.* Consider a sequence of elliptic variational inequalities $(3_n)$, corresponding to (1.3), and let $R \geq 0$. Under the assumptions listed in Theorem, we may apply Theorem 2.3 and deduce the existence of the solutions $\widetilde{u}$ and $\widetilde{u}_n$ of (1.3) and $(3_n)$ respectively, such that $\|\widetilde{u}_n\| \leq c$ with $c$ constant independent of $R$. For $R > c$, and under the assumption that $(A_n)$ be a sequence of uniformly bounded operators and hypothesis (a),(c) and (e) and, by taking into account the proof of Theorem 2.3, we obtain the constant $c$ independent to $n$. Let,

$$\widetilde{B}_R = \{\widetilde{v} = (v, \xi) \in \widetilde{V}^{**} |\ \ \|v - v_0\| + |\xi - \Phi(v_0)| \leq R\},$$

since $\widetilde{K}_n \cap \widetilde{B}_R$ is a nonempty closed convex subset of $\widetilde{V}^{**}$, then $\widetilde{K}_n \cap \widetilde{B}_R$ converges to $\widetilde{K} \cap \widetilde{B}_R$, in the sense of definition of Attouch-Wets convergence [2], for $\mu$ sufficiently large, we have $\widetilde{K}_n \cap \widetilde{B}_R \overset{H}{\to} \widetilde{K} \cap \widetilde{B}_R$. Since,

$$d(\widetilde{u}_n, \widetilde{K} \cap \widetilde{B}_R) \leq H(\widetilde{K}_n \cap \widetilde{B}_R, \widetilde{K} \cap \widetilde{B}_R) \to 0 \text{ as } n \to +\infty$$

there exists a subsequence of $(\widetilde{u}_n)$, denoted by $(\widetilde{u}_k)$; such that $\widetilde{u}_k \rightharpoonup \widetilde{u} \in \widetilde{K} \cap \widetilde{B}_R$ in the weak-star topology. We now prove that

$$\langle \widetilde{A}\widetilde{u} - \widetilde{f}, \widetilde{v} - \widetilde{u}\rangle \geq 0,\ \forall \widetilde{v} \in \widetilde{K}.$$

It suffices to prove that $\widetilde{u}$ is a solution of $\langle \widetilde{A}\widetilde{u} - \widetilde{f}, \widetilde{v} - \widetilde{u} \rangle \geq 0$, $\forall \widetilde{v} \in \widetilde{K} \cap \widetilde{B}_R$, Indeed, given $\widetilde{v} \in \widetilde{K} \cap \widetilde{B}_R$, there is a sequence $\widetilde{v}_n \in \widetilde{K}_n \cap \widetilde{B}_R$ converging in norm to $\widetilde{v}$ as $n \to +\infty$, by the Minty's lemma, we have

$$\langle \widetilde{A}_k \widetilde{v}_k - \widetilde{f}_k, \widetilde{v}_k - \widetilde{u}_k \rangle \geq 0;$$

letting $k \to +\infty$, and using the assumptions (a) and (b), we get

$$\langle \widetilde{A}\widetilde{v} - \widetilde{f}, \widetilde{v} - \widetilde{u} \rangle \geq 0,$$

and as $\widetilde{v} \in \widetilde{K} \cap \widetilde{B}_R$ is arbitrary, then thanks to the proof of theorem 2.3, we have

$$\big(R > c \geq \liminf \|\widetilde{u}_n\| \geq \|\widetilde{u}\|\big)$$

and using Minty's lemma, we get the desired conclusion. □

**Remark 3.1.** In reflexive Banach space, a closed convex bounded set is compact. Then, it is possible to prove a result similar to Theorem 3.1 by applying the lemma 2.2 [10], because in this case $X_1 = K$ is a nonempty compact convex subset of $V$.

## 4 Conclusions

We have extended a well-studied and generic problem of variational inequalities in reflexive Banach spaces to encompass nonreflexive Banach spaces, under weaker conditions for the existence of solutions.

## References

[1] H. Attouch, *Partial Differential Equations associates to a family of subdifferentials* (in French), Ph.D. Thesis, Paris 1976.

[2] G. Beer, *Topologies on Closed and Closed Convex Sets,* Kluwer Academic Publishers, Dordrecht 1993.

[3] L. Boccardo and F. Donati, *Existence and stability results for solutions of some strongly nonlinear constrained problems,* Nonlinear Analysis, Theory Methods and Applications 5, 9 (1981), 975–988.

[4] S. S. Chang, B. S. Lee and Y. Q. Chen, *Variational inequalities for monotone operators in nonreflexive Banach Spaces*, Appl. Math. Lett. 8, 6 (1995), 29–34.

[5] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems* (in French), Dunod-Gauthier Villars, Paris 1974.

[6] J. L. Lions, *Some methods of resolutions of nonlinear boundaries problems* (in French), Dunod, Paris 1968.

[7] S. Mititelu and M. Postolache, *Efficiency and duality for multitime vector fractional variational problems on monifolds*, Balkan J. Geom. Appl. 16, 2 (2011), 90–101.

[8] U. Mosco, *Convergence of convex sets and of solutions of variational inequalities,* Advances in Mathematics. 3, 4 (1969), 510–585.

[9] E. Tarafdar, *A fixed point theorem equivalent to the Fan-Knaster-Kuratowski-Mazurkiewicz theorem*, Journal of Mathematical Analysis and Applications, 128, 2 (1987), 475–479.

[10] P. J. Watson, *Variational inequalities in nonreflexive Banach spaces,* Applied Mathematics Letters 10, 2 (1997), 45–48.

*Authors' address:*

Bouchra Aylaj, Jaafar Lahrache
Department of Mathematics, Faculty of Sciences,
University of Chouab Doukkali, B.P. 20, El Jadida, Morocco.
E-mails: bouchra.aylaj@gmail.com ; jaafarlahrache@yahoo.fr