Equivalent projective representations on Hilbert C^*-modules and their multipliers

Tania-Luminiţa Costache

Abstract. The goal of this paper is to study equivalent projective representations on a Hilbert C^*-module and to present conditions for the associated multipliers of two projective representations to be equivalent. We also prove the existence of a projective representation π_2 projectively equivalent with a given projective unitary representation π_1 with the associated multiplier ω_1, which is equivalent with the associated multiplier ω_2 of π_2.

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1 Introduction and preliminaries

Hilbert C^*-modules were first introduced by Kaplansky in 1953 in [7]. His idea was to generalize Hilbert space by allowing the inner product to take values in a commutative unital C^*-algebra rather than in the field of complex numbers. In 1973, Paschke showed, in his PhD thesis [13], that, contrary to Kaplansky's misgivings, most of the basic properties of Hilbert C^*-modules are valid for modules over an arbitrary C^*-algebra. At about the same time, Rieffel [14] independently developed much of the same theory and used Hilbert C^*-modules as the technical basis for his theory of induced representations of C^*-algebras.

Definition 1.1. ([9]) A *pre-Hilbert A-module* is a complex vector space E which is also a right A-module, compatible with the complex algebra structure, equipped with an A-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to A$ which is $\mathbb{C}$ -and A-linear in its second variable and satisfies the following relations:

1. $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$ for every $\xi, \eta \in E$;
2. $\langle \xi, \xi \rangle \geq 0$ for every $\xi \in E$;
3. $\langle \xi, \xi \rangle = 0$ if and only if $\xi = 0$.

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We say that E is a *Hilbert A-module* if E is complete with respect to the topology determined by the norm $\|\cdot\|$ given by $\|\xi\| = \sqrt{\|\langle \xi, \xi \rangle\|}$.

The theory of projective representations of finite groups was founded by I. Schur [15]. Projective representations help us understand numerous physical systems. For example, they are used to describe the symmetry operations of a crystal lattice and to label the energy levels of quantum systems. Mackey remarked in [10] that a problem arising in quantum field theory can be formulated as a problem of finding certain projective representations. Brown has used projective representations of translation groups to discuss the energy-level degeneracy occurring when a crystal is subjected to a uniform magnetic field. Projective representations of abelian groups arise naturally in the study of energy bands in the presence of a magnetic field. A projective representation is also relevant for describing the symmetries of quantum mechanical systems.

Let A be a C^*-algebra, let E be a Hilbert C^*-module over A and let $\mathcal{L}_A(E)$ be the Banach space of all adjointable module homomorphisms from E to E (that is, T is a module homomorphism such that there is T^* a bounded module homomorphism from E to E satisfying $\langle \eta, T\xi \rangle = \langle T^*\eta, \xi \rangle$ for all $\xi, \eta \in E$) and let G be a locally compact group.

Definition 1.2. A map $\omega\colon G \times G \to \mathcal{U}(\mathcal{Z}(A))$, where

$$\mathcal{U}(\mathcal{Z}(A)) = \{u \in A \mid u \text{ unitary}, \ ua = au, \ \forall \ a \in A\}$$

is called a *multiplier* on G if

i) $\omega(x, e) = \omega(e, x) = 1_A$ for all $x \in G$, where e is the identity of G;

ii) $\omega(x, y)\omega(xy, z) = \omega(x, yz)\omega(y, z)$ for all $x, y, z \in G$.

Definition 1.3. Two multipliers ω_1 and ω_2 on G are called *equivalent* if there is a map $\mu\colon G \to \mathcal{U}(\mathcal{Z}(A))$ such that $\mu(e) = 1_A$ and

$$\omega_2(x, y) = \mu(x)\mu(y)\mu(xy)^*\omega_1(x, y)$$

for all $x, y \in G$.

Definition 1.4. A *projective representation* of G on E with the associated multiplier ω is a map $\pi\colon G \to \mathcal{L}_A(E)$ such that

i) $\pi(xy) = \omega(x, y)\pi(x)\pi(y)$ for all $x, y \in G$;

ii) $\pi(e) = I_E$, where I_E is the identity operator on E.

2 Equivalent projective representations on Hilbert C^*-modules

In this section we give conditions for the associated multipliers of two projective representations to be equivalent and prove the existence of a projective representation π_2 projectively equivalent with a given projective unitary representation π_1 with the associated multiplier ω_1, which is equivalent with the associated multiplier ω_2 of π_2.

Definition 2.1. Let π_1 and π_2 two projective representations from G to $\mathcal{L}_A(E)$. We say that π_1 and π_2 are *projectively equivalent* if there is an unitary operator $U \in \mathcal{L}_A(E)$ such that

$$\{U^*\pi(x)U\xi / x \in G, \xi \in E, \pi \text{ projective representation of } G \text{ on } E \text{ with the multiplier } \omega\}$$

is a dense submodule of E and there is a map $\mu\colon G \to \mathcal{U}(\mathcal{Z}(A))$ such that $\mu(e) = 1_A$ and $\pi_2(x)\xi = \mu(x)U^*\pi_1(x)U\xi$ for all $x \in G, \xi \in E$.

Proposition 2.1. *Let π_1 and π_2 be two projectively equivalent projective representations with the associated multipliers ω_1, respectively ω_2. Then ω_1 and ω_2 are equivalent.*

Proof. Since π_1 and π_2 are projectively equivalent, there are an unitary operator $U \in \mathcal{L}_A(E)$ and a map $\mu\colon G \to \mathcal{U}(\mathcal{Z}(A))$ such that

$$\pi_2(x)\xi = \mu(x)U^*\pi_1(x)U\xi$$

$$\pi_2(y)\xi = \mu(y)U^*\pi_1(y)U\xi$$

$$\pi_2(xy)\xi = \mu(xy)U^*\pi_1(xy)U\xi$$

for all $x, y \in G$ and $\xi \in E$.

Since π_1 and π_2 are projective representations, we have

$$\mu(xy)U^*\pi_1(xy)U\xi = \pi_2(xy)\xi = \omega_2(x,y)\pi_2(x)\pi_2(y)\xi =$$

$$\omega_2(x,y)\mu(x)U^*\pi_1(x)U\mu(y)U^*\pi_1(y)U\xi =$$

$$\omega_2(x,y)\mu(x)\mu(y)U^*\pi_1(x)\pi_1(y)U\xi = \omega_2(x,y)\mu(x)\mu(y)U^*\omega_1(x,y)^*\pi_1(xy)U\xi =$$

$$\omega_2(x,y)\omega_1(x,y)^*\mu(x)\mu(y)U^*\pi_1(xy)U\xi,$$

for all $x, y \in G$ and $\xi \in E$. □

Proposition 2.2. *Let π_1 and π_2 be two projectively equivalent projective representations with the same associated multiplier ω. Then the map μ from Definition 2.1 is a homomorphism from G to $\mathcal{U}(\mathcal{Z}(A))$.*

Proof. By Definition 2.1 results that there are an unitary operator $U \in \mathcal{L}_A(E)$ and a map $\mu\colon G \to \mathcal{U}(\mathcal{Z}(A))$ such that

$$\pi_2(x)\xi = \mu(x)U^*\pi_1(x)U\xi$$

$$\pi_2(y)\xi = \mu(y)U^*\pi_1(y)U\xi$$

$$\pi_2(xy)\xi = \mu(xy)U^*\pi_1(xy)U\xi$$

for all $x, y \in G$ and $\xi \in E$.

Since π_2 is a projective representation with the associated multiplier ω and π_1 and π_2 have the same associated multiplier, for all $x, y \in G$ and $\xi \in E$ we obtain :

$$\pi_2(x)\pi_2(y)\xi = \omega(x,y)^*\pi_2(xy)\xi = \omega(x,y)^*\mu(xy)U^*\pi_1(xy)U\xi.$$

On the other hand,

$$\pi_2(x)\pi_2(y)\xi = \mu(x)U^*\pi_1(x)U\mu(y)U^*\pi_1(y)U\xi =$$

$$\mu(x)\mu(y)U^*\pi_1(x)\pi_1(y)U\xi = \mu(x)\mu(y)U^*\omega(x,y)^*\pi_1(xy)U\xi =$$

$$\omega(x,y)^*\mu(x)\mu(y)U^*\pi_1(xy)U\xi.$$

Therefore, $\mu(xy)U^*\pi_1(xy)U\xi = \mu(x)\mu(y)U^*\pi_1(xy)U\xi$. So $\mu(xy) = \mu(x)\mu(y)$ for all $x, y \in G$. □

Definition 2.2. A *projective unitary representation* of G on a Hilbert C^*-module E over a C^*-algebra B with the associated multiplier ω is a map $\pi\colon G \to \mathcal{L}_B(E)$ such that :

i) $\pi(x)$ is a unitary element of $\mathcal{L}_B(E)$ for all $x \in G$;

ii) $\pi(xy) = \omega(x,y)\pi(x)\pi(y)$ for all $x, y \in G$.

Theorem 2.3. *(a) Let $\pi_i, i = 1,2$ be two projective representations with the associated multiplier $\omega_i, i = 1,2$. If π_1 is projectively equivalent with π_2, then ω_1 is equivalent with ω_2. Moreover, if $\omega_1 = \omega_2$, then every map $\mu\colon G \to \mathcal{U}(\mathcal{Z}(A))$ which satisfies the relation in Definition 2.1 is a homomorphism.*

(b) Let π_1 be a projective unitary representation with the associated multiplier ω_1. Then for every multiplier ω_2 which is equivalent with ω_1 there is a projective representation π_2 (with the associated multiplier ω_2) which is projectively equivalent with π_1.

Proof. (a) This assertion results by Proposition 2.1 and Proposition 2.2.

(b) Since ω_2 is equivalent with ω_1, By Definition 1.3, there is a map $\mu\colon G \to \mathcal{U}(\mathcal{Z}(A))$ such that $\mu(e) = 1_A$ and

$$\omega_1(x,y) = \mu(x)\mu(y)\mu(xy)^*\omega_2(x,y)$$

for all $x, y \in G$.

For every $x \in G$, we define $\pi_2\colon G \to \mathcal{L}_A(E)$ by $\pi_2(x)\xi = \mu(x)\pi_1(x)\xi$, for all $\xi \in E$.

Since π_1 is a projective unitary representation with the associated multiplier ω_1, we have $\omega_1(x,y)\pi_1(x)\pi_1(y)\xi = \pi_1(xy)\xi$ for all $x, y \in G$ and $\xi \in E$, that is $\omega_1(x,y)I_E\xi = \pi_1(xy)\pi_1(y)^*\pi_1(x)^*\xi$ for all $x, y \in G$ and $\xi \in E$.

Hence, $\omega_2(x,y)I_E\xi = \mu(x)^*\mu(y)^*\mu(xy)\pi_1(xy)\pi_1^*(y)\pi_1^*(x)\xi =$
$\mu(xy)\pi_1(xy)\mu(y)^*\pi_1^*(y)\mu(x)^*\pi_1^*(x)\xi = \pi_2(xy)\pi_2^*(y)\pi_2^*(x)\xi$.

Therefore, $\omega_2(x,y)\pi_2(x)\pi_2(y)\xi = \pi_2(xy)\xi$.

This means that π_2 is a a projective unitary representation with the associated multiplier ω_2. By the its definition, the representation π_2 is projectively equivalent with π_1, taking the unitary operator $U \in \mathcal{L}_A(E)$ equal with the identity operator on E, I_E. □

References

[1] B. Bagchi, G. Misra, *A note on the multipliers and projective representations of semi-simple Lie groups*, The Indian J. of Statistics 62, Series A (2000).

[2] T.-L. Costache, *Extensions on twisted crossed products of completely positive invariant projective u-covariant multi-linear maps*, BSG Proceedings 17, Geometry Balkan Press, Bucharest, 2010, 56-67.

[3] T.-L. Costache, *On the projective covariant representations of C*- dynamical systems associated with completely multi-positive projective u-covariant maps*, UPB Sci. Bull., Series A 72 (4) (2010), 185-196.

[4] T.-L. Costache, Mariana Zamfir, Mircea Olteanu, *On projective regular representations of discrete groups and their infinite tensor products*, Applied Sciences 13 (2011), 22-27.

[5] J. Heo, *Hilbert C^*-modules and projective representations associated with multipliers*, J. Math. Anal. Appl. 331 (2007), 499-505.

[6] M. Joiţa, T.-L. Costache and M. Zamfir, *Dilations on Hilbert C^*- modules for C^*-dynamical systems*, BSG Proceedings 14, Geometry Balkan Press, Bucharest 2007, 81-86.

[7] I. Kaplansky, *Modules over operator algebras*, Amer. J. Math. 75 (1953), 839-853.

[8] A. Kleppner, *Continuity and measurability of multiplier and projective representations*, J. Funct. Analysis 17 (1974), 214-226.

[9] E. C. Lance, *Hilbert C^*-modules. A toolkit for operator algebraists*, London Mathematical Society Lecture Note Series 210, Cambridge University Press, Cambridge, 1995.

[10] G. W. Mackey, *Unitary representations of group extensions I*, Acta Math. 99 (1958), 265-311.

[11] M. Olteanu, T.-L. Costache, M. Zamfir, *An algorithm for computing the spectra of mutiplication systems*, BSG Proceedings 14, Geometry Balkan Press, Bucharest 2007, 126-130.

[12] M. Olteanu, T.-L. Costache, M. Zamfir, *The spectra of a class of convolution operators*, BSG Proceedings 16, Geometry Balkan Press, Bucharest 2009, 108-113.

[13] W. L. Paschke, *Inner product modules over B^*-algebras*, Trans. Amer. Math. Soc. 182 (1973), 443-468.

[14] M. A. Rieffel, *Induced representations of C^*-algebras*, Adv. in Math. 13 (1974), 176-257.

[15] I. Schur, *Untersuchungen uber die Darstellung der endlichen Gruppe durch gebrochene lineare Substitutionen*, J. fur Math. 132 (1907).

Author's address:

Tania-Luminiţa Costache
Faculty of Applied Sciences,
Department of Mathematical Models and Methods,
University Politehnica of Bucharest, Bucharest, Romania.
E-mail: lumycos1@yahoo.com