Deformation algebras on jet prolongations of projectable vector fields

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Abstract. Our aim is to develop the canonical structures of the deformation algebras on the jet prolongations of projectable vector fields and properties related to deformation algebras.

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1 Fibered manifolds

A *fibered manifold* is a triple (Y, π, X), where Y and X are finite dimensional differentiable manifolds and $\pi : Y \to X$ is a surjective submersion with dim $X = n$ and dim$Y = m+n$. At every point $y \in Y$, the following two equivalent conditions defining a submersion are satisfied:

(1) the tangent mapping $T_y\pi : T_yY \to T_{\pi(y)}X$ is surjective,

(2) there exist a chart (V, ψ), $\psi = (u^i, y^\alpha)$, at y, where $1 \le i \le n$, $1 \le \sigma \le m$ and a chart (U, φ), $\varphi = (x^i)$, at $x = \pi(y)$ such that $U = \pi(V)$ and $x^i\pi = u^i$.

We write x^i instead of u^i, and call (V, ψ), $\psi = (x_i, y^\sigma)$, a *fibered chart.* The chart (U, φ), $\varphi = (x^i)$, on X is unique, and is said to be *associated with* (V, ψ), $\psi = (x^i, y^\sigma)$.

A *section* of a fibered manifold (Y, π, X), is a mapping $\gamma : U \to Y$, where $U \subset X$, is an open set, such that

$$\pi \circ \gamma = id_U. \tag{1.1}$$

A vector field Ξ on Y is said to be π-*projectable*, or simply *projectable*, if there exists a vector field ξ on X such that

$$T\pi \cdot \Xi = \xi \circ \pi. \tag{1.2}$$

If ξ exists, it is unique, and is called the π-*projection* of Ξ. In a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, a π-projectable vector field Ξ is expressed by

$$\Xi = \xi^i \frac{\partial}{\partial x^i} + \Xi^\sigma \frac{\partial}{\partial y^\sigma}, \tag{1.3}$$

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where $\xi^i = xi^i(x^j)$ and $\Xi^\sigma = \Xi^\sigma(x^j, y^\sigma)$.

2 Jet prolongations of a fibered manifold

Let $y \in Y$ be a point, $x = \pi(y)$, and let $\Gamma^r_{x,y}$ be the set of smooth sections γ of Y defined at x such that $\gamma(x) = y$. Let $r > 0$ be an integer. The binary relation $\gamma_1 \sim \gamma_2$ if there exists a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, at y such that

$$D_{i_1} D_{i_2} \dots D_{i_k}(y^\sigma \gamma_1 \varphi^{-1})(\varphi(x)) = D_{i_1} D_{i_2} \dots D_{i_k}(y^\sigma \gamma_2 \varphi^{-1})(\varphi(x)) \tag{2.1}$$

for all $k = 1, 2, \dots, r$ and all $i_1, i_2, \dots, i_k$ such that $1 < i_1 < i_2 < \dots < i_k < n$ is an equivalence on the set $\Gamma^r_{x,y}$. The equivalence class containing a section γ, is called an r-jet with *source* x and *target* y or the r-jet of y at x, and is denoted by $J^r_x\gamma$. We denote by J^rY the set of r-jets with source in X and target in Y. The *canonical jet projections* are the mappings $\pi^{r,s}$ (respectively π^r) of J^rY onto J^sY, where $0 < s < r$ (respectively on X), defined by $\pi^{r,s}(J^r_x\gamma) = (J^s_x\gamma)$ (respectively $\pi^r(J^r_x\gamma) = x$).

The *smooth structure* of J^rY associated with the smooth structure of Y is defined as follows. Let (V, ψ), $\psi = (x^i, y^\sigma)$, where $1 \le i \le n$, $1 \le \sigma \le m$, be a fibered chart on Y, (U, φ), $\varphi = (x^i)$, the associated chart on X. Then the *associated fibered chart* (V^r, ψ^r), $\psi^r = (x^i, y^\sigma, y^{\sigma j_1}, \dots, y^{\sigma j_1 j_2 \cdots j_r})$ on J^rY is defined by the following two conditions:

$$V^r = (\pi^{r,0})^{-l}(V), \tag{2.2}$$

and if

$$J^r_x\gamma \in V^r, \tag{2.3}$$

then

$$y^{\sigma j_1 j_2 \cdots j_k}(J^r_x\gamma) = D_{j_1} D_{j_2} \dots D_{j_k}(y^\sigma \gamma \varphi^{-1})(\varphi(x)) \tag{2.4}$$

where $k = 1, 2, \dots, r$ and $1 < j_1 < j_2 < \dots < j_k < n$. If (V', ψ'), $\psi' = (x'^i, y'\sigma)$, is another fibered chart such that $V \cap V' \neq \varnothing$, then writing $y'^\sigma \gamma \varphi'^{-1} = y' \psi^{-1} \circ \psi \gamma \varphi^{-1} \circ \varphi \varphi'^{-1}$ we get, using the chain rule, the *transformation formula* in a recurrent form

$$y'^{\sigma j_1 j_2 \cdots j_k} = D_{j_1} D_{j_2} \dots D_{j_k}(y'^\sigma \psi^{-1} \circ \psi \gamma \varphi^{-1} \circ \varphi \varphi'^{-1})(\varphi'(x)). \tag{2.5}$$

The dimension of J^rY is given by $\dim J^rY = n + m \begin{pmatrix} n+r \\ n \end{pmatrix}$.

3 The horizontalization of tangent vectors

A vector bundle morphism acts on tangent spaces to the jet prolongations of a fibered manifold. Similarly as in the case of differential forms, this vector bundle morphism is induced by the structure of the jet prolongations.

Let $r > 0$ be an integer. One can assign to every tangent vector $\xi \in TJ^{r+1}Y$ at a point $J^{r+1}_x\gamma \in J^{r+1}Y$ a tangent vector $h\xi \in TJ^rY$ at $J^r_x\gamma = \pi^{r+1,r}(J^{r+1}_x\gamma) \in J^rY$ by

$$h\xi = T_xJ^r\gamma \circ T\pi^{r+1} \cdot \xi. \tag{3.1}$$

The mapping $h : TJ^{r+1}Y \to TJ^rY$ defined by this formula is a vector bundle morphism over the jet projection $\pi^{r+1,r}$; we call h the π-*horizontalization*, or simply the *horizontalization* and

$$h\xi = \xi^j \left(\frac{\partial}{\partial x^i} + \sum_{k=0}^{r} \sum_{j_1 \leq j_2 \dots j_k} y^\sigma_{j_1 j_2 \dots j_k j} \frac{\partial}{\partial y^\sigma}_{j_1 j_2 \dots j_k} \right). \tag{3.2}$$

Lemma 1 [**4**]. *Let* Ξ *be a* π*-projectable vector field on* Y*,* (V, ψ)*,* $\psi = (x^i, y^\sigma)$*, a fibered chart on* Y*, and let* Ξ *be expressed by* $\Xi = \xi^i \frac{\partial}{\partial x^i} + \Xi^\sigma \frac{\partial}{\partial y^\sigma}$*. Then* $J^r\Xi$ *is expressed with respect to the associated chart* $(V^r, \psi r)$

$$J^r\Xi = \xi^i \frac{\partial}{\partial x^i} + \left(\sum_{k=0}^{r} \sum_{j_1 \leq j_2 \dots j_k} \Xi^\sigma_{j_1 j_2 \dots j_k} \frac{\partial}{\partial y^\alpha_{j_1 j_2 \dots j_k}} \right), \tag{3.3}$$

where the components $\Xi^\sigma_{j_1 j_2 \dots j_k}$ are determined by the recurrent formula

$$\Xi^\sigma_{j_1 j_2 \dots j_k} = d_{j_k} \Xi^\sigma_{j_1 j_2 \dots j_{k-1}} - y^\sigma_{j_1 j_2 \dots j_{k-1} i} \frac{\partial \xi^i}{\partial x^{j_k}}. \tag{3.4}$$

Lemma 2 [**4**]. *Let* Ξ_1 *and* Ξ_2 *be two -projectable vector fields on* Y*. Then the Lie bracket* $[\Xi_1, \Xi_2]$ *is also* π*-projectable vector fields on* Y*, and* $J^r[\Xi_1, \Xi_2] = [J^r\Xi_1, J^r\Xi_2]$.

Let $A(\Xi_1, \Xi_2) = \nabla_{\Xi_1}\Xi_2 - \nabla'_{\Xi_1}\Xi_2$ be (1,2) tensorial field which defines the deformation algebra associated to the pair of linear connections (∇, ∇') on TJ^rY, noted by $U(J^rY, A)$. These exist from [3, Prop. 3, p. 226].

Definition 1 [**6**]. A point $J^r_x\gamma = c(t)$ is called ∇-stationary of the curve c if there is a real number α such that $\nabla_{\dot{c}}\dot{c}(t) = \alpha\dot{c}(t)$.

Theorem 1. *Let* $U(J^rY, A)$ *be the deformation algebra associated to the pair of linear connections* (∇, ∇') *on* TJ^rY *and let* Ξ *be a vector field that does not vanish at any point. The following statements are equivalents*:

1) Ξ *be a characteristic vector field in* $U(J^rY, A)$;

2) *For any* $J^r_x\gamma$ *of* J^rY*, if* $c : I \to J^rY$ *is a curve with the property that there exists* $t_0 \in I$ *such that* $c(t_0) = J^r_x\gamma$ *and* $\dot{c}(t_0) = \Xi^r_{J_x\gamma}$ *and the point* $J^r_x\gamma$ *is* ∇*-stationary then the point* $J^r_x\gamma$ *is* ∇'*-stationary*;

3) *There is* Ξ' *a characteristic vector field in* $U(J^{r+1}Y, A)$ *such that* $h\Xi' = \Xi$;

4) *There is a curve* $c' : I \to J^{r+1}Y$ *such that* $c = \pi^{r+1,r} \circ c'$.

Proof. We use the [6, Theorem, Def. 1, pp.: 60, 54,58] we get that 1) is equivalent to 2) and from the definition of h π-horizontalization [4] and Definition 1) we get that 1) is equivalent to 2) and 1) is equivalent to 3). □

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