Hyperbolic matrix type geometric structures

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Abstract. In this paper multidimensional hyperbolic matrix algebras are investigated from geometric point of view. A 2-dimensional sphere equipped with a multiplicative structure with zero-divisors is constructed as well.

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1 Introduction

In analysis and differential geometry one studies functions, tensors, connections, defined on the classical fields $\mathbb{R}$ and $\mathbb{C}$. These two fields can be considered as algebras without zero divisors. An exception is made by the algebra of bi-complex numbers $\mathbb{C}_2$ (according to the notation of G. B. Price [3]). This algebra admits zero divisors, but it was used in function theory as new base by many authors ([3], [8], [9]).

As well, the algebra of hyperbolic (complex) numbers (Vignaux (1935) [11], G. Sobczyk (1995) [10]), admits geometric interpretation. The interest for this geometric interpretation is rather old, emerging with Clifford (19-th century), and developed in [6], [5], [11], [10], etc.

Our paper is inspired by the book of Vyshnevskiy, Shirokov and Shurygin [12], where one studies differential geometric structures over general algebras; however, we search more concrete geometric aspects, initially for algebras of small dimensions. For instance, we are interested to interpret the role of zero divisors - if such divisors exist. The existence of zero divisors suggests the problem of their description, which is in generally a difficult one. In geometric interpretations there are of interest the open connected subdomains of missing zero divisors, in view of the possible construction of a locally defined structure on these subdomains.

2 Real finite-dimensional algebras with matrix base

We shall consider the n-dimensional algebra $\mathfrak{A}(n)$ of real $n \times n$ matrices with cyclic matrix basis (to be defined later)

$$1, j, j^2, \ldots, j^{n-1}, j^n = \mathrm{I}_n,\ j^k \in \mathfrak{A}(n). \tag{2.1}$$

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The degrees j^k, $k = 2, \ldots, n-1$ are defined as matrix products, I_n being the unit matrix in $\mathfrak{A}(n)$. Denoting by x an element of $\mathfrak{A}(n)$, one introduces its real coordinates $\{x_0, x_1, \ldots, x_{n-1}\}$ in the basis (2.1),

$$x = x_0 + jx_1 + j^2 x_2 + \ldots + j^{n-1} x_{n-1}, \quad x_k \in \mathbb{R}, \quad j^n = \mathrm{I}_n = 1. \tag{2.2}$$

One easily remarks that the meaning of the equality $j^2 = +1$ is understood in matrix sense, namely as

$$x + jy = x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + y\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; j^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = 1, \; j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We shall show the meaning of j^k, $k \neq 2$ in §2.

As illustration we consider the following examples.

Example 2.1. The real 2 dimensional algebra $\mathfrak{A}(2)$ is known as the algebra of planar hyperbolic complex numbers. It is denoted by us by $\widetilde{\mathbb{R}}^2$. Algebraically, it consists (in our notations) of the matrices

$$x = \begin{pmatrix} x_0 & x_1 \\ x_1 & x_0 \end{pmatrix}, \quad x_0, x_1 \in \mathbb{R}, \tag{2.4}$$

with the basis

$$j^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad j^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1. \tag{2.5}$$

Hence we obtain that

$$\mathfrak{A}(2) = \{x = x_0 + jx_1; \quad j^2 = +1\}. \tag{2.6}$$

Example 2.2. The 3-dimensional algebra $\mathfrak{A}(3)$ with the basis $\mathrm{I}_3, j, j^2, j^3$, $j^3 = \mathrm{I}_3 = 1$,

$$x = x_0 + jx_1 + j^2 x_3, \tag{2.7}$$

where the 3-dimensional matrices are

$$j = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad j^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad j^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Example 2.3. The 4-dimensional real algebra $\mathfrak{A}(4)$ with the basis $1, j, j^2, j^3, j^4 = \mathrm{I}_4$,

$$x = x_0 + jx_1 + j^2 x_2 + j^3 x_3, \tag{2.8}$$

$$j = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \; j^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \; j^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; j^4 = 1.$$

Remark 2.4. Every n-dimensional real vector space V_n

$$V_n = \{a_1e_1 + a_2e_2 + \ldots + a_ne_n : a_k \in \mathbb{R}\},$$

e_j being basic vector, $j = 1, 2, \ldots, n$, can be considered as a set of $1 \times n$ matrices $(a_1, a_2, \ldots, a_n)$, with the matrix basis $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, \ldots, 0) \ldots$, $e_n = (0, 0, \ldots, 1)$. This matrix basis is not cyclic.

Now we turn out to the general case.

Definition 2.5. We denote by j a special $n \times n$ matrix $j = \|j_{rs}\|$, of the algebra $\mathfrak{A}(n)$. We further assume that the following properties of j are satisfied

1. $j_{rs} = 0 \wedge 1$ (0 or 1);
2. On the all n different intersections of one r-row and one s-column [with $(r, s) \neq (r', s')$ in the sense that $r \neq r'$ and $s \neq s'$] one accept $j_{rs} = 1$ and all other $j_{rs} = 0$. (See the examples from above).
3. The degree n of the matrix j coincides with the unit matrix of $\mathfrak{A}(n)$, i.e.

$$j^n = \mathrm{I}_n.$$

When the above formulated properties are satisfied one say that the sequence (2.1) defines a matrix basis in $\mathfrak{A}(n)$.

Lemma 2.1. *The properties 1), 2) and 3) written above remain valid for all positive degrees j^k of j.*

Proof. One can use induction by n. □

Lemma 2.2. *If $\mathfrak{A}(n)$ is matrix algebra with matrix n-basis, then each element A of $\mathfrak{A}(n)$ determines $n!$ different elements in the algebra of all real $n \times n$ matrices.*

Proof. We use induction by n. □

We shall consider the set of all expressions of the type (2.2). It is not difficult to see that the mentioned set can be equipped in a natural way by a structure of a commutative algebra.

Imitating (2.2) we can take arbitrary 3 elements x, y, z in the considered matrix basis. Then

$$x + y = y + x = x_0 + y_0 + j(x_1 + y_1) + \ldots + j^{n-1}(x_{n-1} + y_{n-1}),$$

$$xy = yx = (x_0 + jx_1 + \ldots + j^{n-1}x_{n-1})(y_0 + jy_1 + \ldots + j^{n-1}y_{n-1}),$$

One may compute this, for instance, for $n = 5$

$$xy = x_0y_0 - x_1y_4 - x_2y_3 - x_3y_2 - x_4y_4 + (x_0y_1 + x_1y_0 - x_2y_4 - x_3y_3 - x_4y_2)j + \ldots +$$

$$+(x_0y_4 + x_1y_3 + x_2y_2 + x_3y_1 + x_4y_0)j^4$$

$$z(x + y) = zx + zy.$$

One may prove that this algebra is n-dimensional too. We denote it by $\mathfrak{A}^*(n)$ and we call it the dual algebra of $\mathfrak{A}(n)$.

Theorem 2.3. *The dual algebra $\mathfrak{A}^*(n)$ is isomorphic to $\mathfrak{A}(n)$, i.e.*

$$\mathfrak{A}^*(n) \simeq \mathfrak{A}(n).$$

Proof. The proof is based on the Lemmas from above. □

Remark 2.6. In fact the elements of $\mathfrak{A}^*(n)$ provide another description for the elements of $\mathfrak{A}(n)$.

Example 2.7. The n-dimensional algebra of all circular matrices is a hyperbolic matrix algebra, which admits matrix n-base.

3 Singular and regular elements

A different from zero element of the algebra $\mathfrak{A}(n)$ is called singular if it is non-invertible. The set of all singular elements of $\mathfrak{A}(n)$ is denoted by $Sing\mathfrak{A}(n)$. Respectively, the regular elements are the invertible ones and, their set is denoted by $Reg\mathfrak{A}(n)$. Usually, the underlying set of the algebra $\mathfrak{A}(n)$ is equipped with the natural topology, in which the set $Reg\mathfrak{A}(n)$ is an open subset, which is generally non-connected.

3.1 Dimension 2

We first consider the 2-dimensional algebra $\mathfrak{A}(2)$ of hyperbolic complex numbers (or real double-numbers) $x = x_0 + jx_1$. In dimension 2, the algebra $\mathfrak{A}(2)$ is of signature (1, 1). Then we write

$$\mathfrak{A}^{1,1}(2) = \mathfrak{A}(2).$$

This means that the conjugate $\bar{x}$ of x is $\bar{x} = x_0 - jx_1$. For the product $x\bar{x}$ we have the formula

$$x\bar{x} = x_0^2 - x_1^2. \tag{3.1.1}$$

The solution of the equation

$$x\bar{x} = 0 \tag{3.1.2}$$

determines the zero-divisors in the considered algebra,

$$Sing\ \mathfrak{A}^{1,1}(2) = \{(x + jx, x - jx) : x \neq 0, x \in \mathbb{R}\}. \tag{3.1.3}$$

This set is represented geometrically by the bisectrix through the origin of $\mathbb{R}^2$, which separates the plane in four orthogonal quadrants. This interpretation is denoted by $\widetilde{\mathbb{R}}^2$.

The set of regular elements $Reg\ \mathfrak{A}^{1,1}(2)$ is a disjoint union of four different open components (quadrants), each of them being connected. They are separated by the mentioned bisectrix which contain the zero divisors of $\mathfrak{A}(2)$.

Using the notion of the (see the book [7, p. 176]), one obtains the following result

Proposition 3.1. *The boundary of each regular connected component of* $\widetilde{\mathbb{R}}^2$*, consists of two zero semi-lines, all points of which are zero divisors. All directions between them consist only of regular elements.*

Proof. Let us take the component, which lyes in the first and the fourth quadrants. We remark that the boundary directions are defined by the equation $x\bar{x} = 0$. For a regular element x, we have $x\bar{x} \neq 0$. □

3.2 Dimension 3

We shall consider the algebra $\mathfrak{A}(3)$ with the signature (1,2), i.e. $\mathfrak{A}(3) = \mathfrak{A}^{1,2}(3)$.

Proposition 3.2. *The algebra* $\mathfrak{A}^{1,2}(3)$ *does not admit singular elements different from 0.*

Proof. The proof is straightforward. Let $x, \bar{x} \in \mathfrak{A}^{1,2}(3)$ be two conjugate elements

(3.2.1) $$x = x_0 + jx_1 + j^2x_2, \quad x = x_0 - jx_1 - j^2x_2.$$

Then for all such elements we have

(3.2.2) $$x\bar{x} = 0.$$

Since $x\bar{x} = 0$ implies $x\bar{x} = x_0^2 - 2x_1x_2 - j^4x_2^2 - j^2x_1^2$, $j^4 = j$ if $x\bar{x} = 0$, then $x_1^2 = x_2^2 = x_0^2 - 2x_1x_2 = 0$. □

We are interested now in introducing a family $(O_{x_0}, \widetilde{\mathbb{R}}^2)$ of planar hyperbolic numbers on the orthogonal planes ($x_0 = const$) to the x_0-axis, namely, defined on the planes ($x_0 = const$) which are parallel to the basic plane $(0, \widetilde{\mathbb{R}}^2)$. On each such plane one takes the bisectrix type 0-semilines boundary directions of the corresponding regular component. The union of all such 1-dimensional boundaries defines a type of 2-dimensional wall boundary for the union of all regular 2-dimensional open components. The union of all these regular components is considered as an open 3-dimensional regular component in $\mathbb{R}^3$.

Regarding the directions through the origin in $\mathbb{R}^3$, one receives a 3-dimensional decomposition of 4 regular open components (quadrants) in $\mathbb{R}^3$, which are separated by 2-dimensional wall-boundaries. Each direction in the wall-boundary is projected on the basic 1-dimensional boundary direction.

We shall further present some questions on a hyperbolic version of Pauli matrices.

We consider the vector space $\mathbb{R} \times \widetilde{\mathbb{R}}^2$, where $\widetilde{\mathbb{R}}^2$ denotes the plane $\mathbb{R}^2$ equipped with the hyperbolic number structure j, $j^2 = 1$. The following sequence of 3 matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{with the condition } j^2 = 1$$

form a matrix basis for the space $\mathbb{R} \times \widetilde{\mathbb{R}}^2$.

When $j^2 = -1$, the above written basis is this one of Pauli matrices and it is an orthonormal one. In the case $j^2 = +1$, this is not true.

3.3 Dimension 4

Now we take the real algebra $\mathfrak{A}(4)$ and an element x of $\mathfrak{A}^*(4)$,

$$x = x_0 + jx_1 + j^2x_2 + j^3x_3,\ x_k \in \mathbb{R},\ j^4 = +1. \tag{3.3.1}$$

Imitating the Minkowski 4-dimensional signature (1, 3) we consider the possible four Minkowski type conjugates

$$\bar{x} = x_0 - j(x_1 + jx_2 + j^2x_3), \tag{3.3.2}$$

$$\bar{x} = x_0 + j^2x_2 + j^3x_3 - jx_1, \tag{3.3.3}$$

$$\bar{x} = x_0 + jx_1 + j^3x_3 - j^2x_2, \tag{3.3.4}$$

$$\bar{x} = x_0 + jx_1 + j^2x_2 - j^3x_3. \tag{3.3.5}$$

We further obtain by direct calculation

$$x\bar{x} = x_0^2 - x_2^2 - 2x_1x_3 - 2jx_2x_3 - j^2x_3^2 - 2j^3x_1x_2, \tag{3.3.2$'$}$$

$$x\bar{x} = x_0^2 + x_2^2 + 2jx_2x_3 + j^2(x_3^2 + 2x_0x_2 - x_1^2) + 2j^3x_1x_2, \tag{3.3.3$'$}$$

$$x\bar{x} = x_0^2 - x_2^2 + 2x_1x_3 + 2jx_0x_1 + j^2x_3^2 + 2j^3x_0x_3, \tag{3.3.4$'$}$$

$$x\bar{x} = x_0^2 + x_2^2 + 2jx_0x_2 + j^2(x_1^2 - x_3^2 + 2x_0x_2) + 2j^3x_1x_2. \tag{3.3.5$'$}$$

Proposition 3.3. *The following equivalence relations are valid:*

- *In the case (3.3.2') $x\bar{x} = 0$ is equivalent to $x_0^2 - x_2^2 = 0,\ x_1 = x_3 = 0.$*
- *In the case (3.3.3')*
  $x\bar{x} = 0$ *is equivalent to* $x_1^2 - x_3^2 = 0,\ x_0 = x_2 = 0.$
- *In the case (3.3.4')*
  $x\bar{x} = 0$ *is equivalent to* $x_0^2 - x_2^2 = 0,\ x_1 = x_3 = 0.$
- *In the case (3.3.5')*
  $x\bar{x} = 0$ *is equivalent to* $x_1^2 - x_3^2 = 0,\ x_0 = x_2 = 0.$

Proof. The proof is easy. □

Remark 3.1. Clearly, there exist two type of equivalences $x_0^2 - x_2^2 = 0$, $x_1 = x_3 = 0$ and $x_1^2 - x_3^2 = 0$, $x_0 = x_2 = 0$. This means that, in the first case, x reduces to $x_0 + j^2x_2$, i.e. $x = x_0 + j^2x_2$, and in the second type x reduces to $jx_1 + j^3x_3$, i.e. $x = j(x_1 + j^2x_3)$.

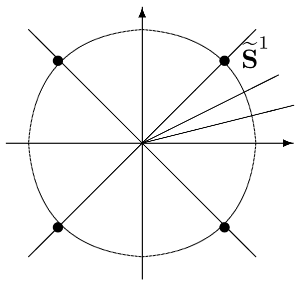

Figure 1: The special circle.

4 Special geometric construction

Let us take the constructed in the previous paragraph decomposition of $\mathbb{R}^3$ in four open regular components separated by the two orthogonal 2-dimensional boundaries (called 0-wall-boundaries).

Now let us consider all directions in $\mathbb{R}^3$ defined by semilines through the origin. We remark that all such directions which lye in a 2-dimensional wall are projected on a boundary of 0-semilines in the basis plane $(0, \widetilde{z})$.

The intersection of this direction with the unit sphere S^2 in $\mathbb{R}^3$ is a full meridian (a great circle) all points of which define zero divisors. All other intersections of semilines with S^2 of direction in $\mathbb{R}^3$ lye between two boundary meridians and define a regular point.

Remark 4.1. The simplest picture appears for the algebra $\mathfrak{A}(2)$ of the hyperbolic complex numbers (double numbers according Yaglom [13]). The Decartes circle S^1 is decomposed in four open regular arcs separated by four points which define zero-divisors in $\mathfrak{A}(2)$. The circle S^1 equipped with the described decomposition is denoted by $\widetilde{S}^1$. We call it special circle, see Figure 1. Analogously, $\widetilde{S}^2$, described above, is called special two-dimensional sphere.

In the 4-dimensional case, we consider the two equations

$$x_0^2 - x_2^2 = 0 \qquad \text{and} \qquad x_1^2 - x_3^2 = 0.$$

These two equations define a special bi-planar geometric structure

$$\widetilde{\mathbb{R}}^2_{02} \oplus j\widetilde{\mathbb{R}}^2_{13} \subset \mathbb{R}^5, \tag{4.1}$$

where by $\widetilde{\mathbb{R}}^2_{02}$ and $\widetilde{\mathbb{R}}^2_{13}$ are denoted the multiplicative decomposition in singular and regular elements in the ambient algebra, respectively, for the 2-dimensional subspaces $\mathbb{R}^2(x_0, x_2)$ and $\mathbb{R}^2(x_1, x_3)$ of $\mathbb{R}^4(x_0, x_1, x_2, x_3)$,

$$\widetilde{\mathbb{R}}^2_{02} = \widetilde{\mathbb{R}}^2(x_0, x_2), \quad \widetilde{\mathbb{R}}^2_{13} = \widetilde{\mathbb{R}}^2(x_1, x_3). \tag{4.2}$$

5 Elements of differential geometry

In this short paragraph we present some basic notions from differential geometry, which can be adapted for the constructed above geometric structures, by standard techniques.

First, we remark that one can consider $\widetilde{\mathbb{R}}^2$ as 2-dimensional Minkowski space with signature (1, 1), where the product $x\bar{x}$ is a Minkowski non-positive metric: $x\bar{x} = x_0^2 - x_1^2$. In fact, this metric defines a fibration of hyperbolas ($x_0^2 - x_1^2 = const.$) on each open connected component (quadrant) of $\widetilde{\mathbb{R}}^2$. So, the group $Aut(\widetilde{\mathbb{R}}^2)$ of automorphisms $\varphi : \widetilde{\mathbb{R}}^2 \to \widetilde{\mathbb{R}}^2$ which preserves the Minkowski metric, is regarded as a group of transformations which transforms hyperbolas in hyperbolas, on each component of $\widetilde{\mathbb{R}}^2$.

An analogous description is valid for the bi-planar hyperbolic structure $\widetilde{\mathbb{R}}^2_{02} \oplus \widetilde{\mathbb{R}}^2_{13}$ endowed with a pair of Minkowski metrics [1], [2].

Secondly, on each open connected component is induced a structure of differentiable manifold from the one of the ambient space. In particular, we have the tangent bundle and related with its tensor fields, vector fields, differential forms. The coherence on the boundaries (defined 0-semilines, or 0-meridians) is possible to be proven by standard techniques.

6 Problems and questions, conclusion

Further investigations concern the following topics:

6.1 To develop complex matrix geometry and to clarify possible interconnection with some generalized Pauli matrices with hyperbolic imaginary units j, $j^2 = +1$ (this is, most likely, not possible).

6.2 To describe the special spheres $\widetilde{S}^n \subset \mathbb{R}^{n+1}$ with 0-meridians (great circles) and some multi-planar geometric structures in higher dimensions.

6.3 To investigate the vector fields on the special spheres with 0-meridians.

We conclude by recalling that the present work represents a contribution which applies the remarkable theory of vector fields on Euclidean spheres $S^n \subset \mathbb{R}^{n+1}$ and studies the problem of exhibiting the full role of zero-divisors in matrix hyperbolic geometry.

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