

On the rigidity of hypersurfaces which are complete

Yoshio Matsuyama

Abstract. The notion of the rigidity is important for classifying the hypersurfaces in a real space form. It is well known that a hypersurface M^n in a real space form has the rigidity in the case of the type number ≥ 3 at each point of M^n . The purpose of the present paper is to prove that M^n has the rigidity in the case of M^n being complete, $c \neq 0$ and $n \geq 3$.

M.S.C. 2010: 53C40, 53B25.

Key words: rigidity; hypersurfaces; real space form.

1 Introduction

Let $\tilde{M}^{n+1}(c)$ be an $(n+1)$ -dimensional real space form of constant curvature c (i.e. complete, simply connected Riemannian manifold of constant sectional curvature, say c). For each real number c and each integer $n > 1$ there is (up to isometry) exactly one n -dimensional real space form of constant curvature c .

The real space forms are:

- (1) If $c = 0$, then $\tilde{M}^{n+1}(c)$ is a Euclidean space E^{n+1} .
- (2) If $c < 0$, then $\tilde{M}^{n+1}(c)$ is a real hyperbolic space $H^{n+1}(c)$.
- (3) If $c > 0$, then $\tilde{M}^{n+1}(c)$ is a Euclidean sphere $S^{n+1}(c)$.

Let f be an isometric immersion of a Riemannian manifold M^n in $\tilde{M}^{n+1}(c)$. Then we call a hypersurface such a M^n . Let $t(x)$ denote the rank of the second fundamental form of $x \in M^n$. $t(x)$ is called the type number at x . Let \bar{f} be the other isometric immersion of M^n in $\tilde{M}^{n+1}(c)$. If there exists an isometry α of $\tilde{M}^{n+1}(c)$ such that $\alpha \circ \bar{f} = f$, then it is called that M^n has the rigidity. Then it is well known (Beez [1] and Thomas [12]): If the type number $t(x) \geq 3$ at each point of M^n in $\tilde{M}^{n+1}(c)$, then M^n has the rigidity. Sacksteder [11] proved : If M^n is a complete convex hypersurface in E^{n+1} , $n \geq 3$, and $t(x) \geq 3$ at one point, then M^n has the rigidity. Harle [3] showed : If the scalar curvature of M^n is constant, $c \neq 0$ and $n \geq 4$, then M has the rigidity. The author [4] showed : If the mean curvature of M^n is non-zero constant, $c \neq 0$ and $n \geq 4$, then M^n has the rigidity. Ferus [2] obtained : If M^n is complete, $t(x) \geq 2$ at

each point, $c > 0$ and $n \geq 5$, then M^n has the rigidity.

The purpose of this paper is to prove the rigidity of M^n in the case of $c \neq 0$ and complete. We prove the following theorem.

Theorem. *Let M^n be a hypersurface in $\tilde{M}^{n+1}(c)$. If M is complete, $c \neq 0$ and $n \geq 3$, then M^n has the rigidity.*

2 Preliminaries

Let $\tilde{M}^{n+1}(c)$ be an $(n+1)$ -dimensional space form, i.e., a Riemannian manifold of constant curvature, say c . Let f of M^n in $\tilde{M}^{n+1}(c)$ be an isometric immersion of an n -dimensional Riemannian manifold M^n in $\tilde{M}^{n+1}(c)$. For simplicity, we say that M^n is a hypersurface in $\tilde{M}^{n+1}(c)$, and for all local formulas and computations, we may consider f as an imbedding and thus identify $x \in M^n$ with $f(x) \in \tilde{M}^{n+1}(c)$. The tangent space $T_x(M)$ is identified with a subspace of the tangent space $T_x(\tilde{M})$, and the normal space T_x^\perp is the subspace of $T_x(\tilde{M})$ consisting of all $X \in T_x(\tilde{M})$ which are orthogonal to $T_x(M)$ with respect to the Riemannian metric g .

For an arbitrary point $x_0 \in M^n$, we may choose a field ξ of unit normal vectors defined in a neighborhood U of x_0 . The second fundamental form h and the corresponding symmetric operator A are defined and related to covariant differentiations $\tilde{\nabla}$ and ∇ in $\tilde{M}^{n+1}(c)$ and M^n , respectively, by the following formulas:

$$(2.1) \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

$$(2.2) \quad \tilde{\nabla}_X \xi = -AX,$$

where X and Y are vector fields tangent to M^n . The Gauss equation is:

$$(2.3) \quad R(X, Y) = c(X \wedge Y) + AX \wedge AY, \quad X, Y \in T_x(M),$$

where $X \wedge Y$ denotes the skew-symmetric endomorphism of $T_x(M)$. And the Codazzi equation is expressed by

$$(2.4) \quad (\nabla_X A)Y = (\nabla_Y A)X.$$

Moreover, we prepare the following lemmas to prove the Theorem. We shall assume that M^n is oriented (so that a unit normal vector field ξ is defined on the whole M^n) and that the type number $k(x)$ is greater than 0 everywhere on M^n . It is known that the function $k(x)$ is locally constant and hence is a constant, say k , since M^n is connected. We may also speak of the differentiable function $\lambda(x)$ which assigns to each $x \in M^n$ the non-zero eigenvalue of A at x . Thus, at each $x \in M^n$, $\lambda(x)$ is the non-zero eigenvalue of A with multiplicity k and 0 is the eigenvalue with multiplicity $n - k$. We define two distributions on M^n as follows:

$$T_0(x) = \{X \in T_x(M); AX = 0\},$$

$$T_\lambda(x) = \{X \in T_x(M); AX = \lambda(x)X\}.$$

We have $T_x(M) = T_0(x) + T_\lambda(x)$ (direct sum). For any $Y \in T_x(M)$, Y_0 and Y_λ will denote the components of Y in $T_0(x)$ and $T_\lambda(x)$, respectively.

Lemma 1 [6]. *T_0 and T_λ are differentiable.*

Proof. Let $\{X_1, \dots, X_k\}$ be a basis of $T_\lambda(x_0)$ and $\{X_{k+1}, \dots, X_n\}$ be a basis of $T_0(x_0)$ for any point $x_0 \in M^n$. We extend X_i 's to vector fields on M^n and define vector fields

$$\begin{aligned} Y_i &= AX_i & \text{for } 1 \leq i \leq k, \\ Y_j &= (A - \lambda I)X_j & \text{for } k+1 \leq j \leq n, \end{aligned}$$

where I denotes the identity transformation. At x_0 , we have $Y_i = \lambda X_i$ for $1 \leq i \leq k$ and $Y_j = -\lambda X_j$ for $k+1 \leq j \leq n$. Thus Y_1, \dots, Y_n are linearly independent at x_0 and hence in a neighborhood U of x_0 . At each point of U , we have

$$\begin{aligned} (A - \lambda I)Y_i &= (A - \lambda I)AX_i = 0 & \text{for } 1 \leq i \leq k, \\ AY_j &= A(A - \lambda I)X_j = 0 & \text{for } k+1 \leq j \leq n. \end{aligned}$$

Hence Y_1, \dots, Y_k form a basis of T_λ and Y_{k+1}, \dots, Y_n form a basis of T_0 .

Lemma 2 [6]. *T_0 and T_λ are involutive.*

Proof. We recall the Codazzi equation (2.4). Suppose that X and Y are vector fields belonging to T_0 . Then we obtain

$$\begin{aligned} (\nabla_X A)Y &= \nabla_X(AY) - A(\nabla_X Y) = -A(\nabla_X Y), \\ (\nabla_Y A)X &= -A(\nabla_Y X). \end{aligned}$$

Thus we get $A(\nabla_X Y) = A(\nabla_Y X)$, that is, $A([X, Y]) = A(\nabla_X Y - \nabla_Y X) = 0$, showing that $[X, Y]$ belongs to T_0 . Thus T_0 is involutive. Suppose now that X and Y belong to T_λ . Then we get

$$\begin{aligned} (\nabla_X A)Y &= \nabla_X(AY) - A(\nabla_X Y) \\ &= \nabla_X(\lambda Y) - A(\nabla_X Y) \\ &= X\lambda \cdot Y + \lambda \nabla_X Y - A(\nabla_X Y). \end{aligned}$$

Interchanging X and Y here and using the Codazzi equation, we get $(X\lambda)Y - (Y\lambda)X + (\lambda I - A)[X, Y] = 0$. Since $(X\lambda)Y - (Y\lambda)X \in T_\lambda$ and $(\lambda I - A)[X, Y] = \lambda[X, Y]_0$, we get

$$(2.5) \quad (X\lambda)Y - (Y\lambda)X = 0 \text{ and } [X, Y]_0 = 0.$$

The second identity shows that $[X, Y] \in T_\lambda$, proving that T_λ is involutive.

Lemma 3. *If $k \geq 2$ and X belongs to $T_\lambda(x)$, then $X\lambda = 0$.*

Proof. Since $\dim T_\lambda(x) \geq 2$, we may choose $Y \in T_\lambda(x)$ such that X and Y are linearly independent. Extending X and Y to vector fields belonging to T_λ , we have $(X\lambda)Y - (Y\lambda)X = 0$ at x from the first identity of (2.5). Thus $X\lambda = Y\lambda = 0$ at x .

Lemma 4 [6]. *If $X \in T_\lambda, Y \in T_0$, then $A(\nabla_X Y) = -(Y\lambda)X$.*

Proof. Let $X \in T_\lambda, Y \in T_0$, and we compute the both sides of the Codazzi equation (2.4) :

$$\begin{aligned} (\nabla_X A)Y &= \nabla_X(AY) - A(\nabla_X Y) = -A(\nabla_X Y) = -\lambda(\nabla_X Y)_\lambda, \\ (\nabla_Y A)X &= \nabla_Y(AX) - A(\nabla_Y X) = \nabla_Y(\lambda X) - A(\nabla_Y X) \\ &= Y\lambda \cdot X + \lambda(\nabla_Y X) - A(\nabla_Y X) \\ &= Y\lambda \cdot X + \lambda(\nabla_Y X)_0 \end{aligned}$$

Therefore we have $(\nabla_Y X)_0 = 0$, that is, $\nabla_Y X \in T_\lambda$ and $(Y\lambda)X = -\lambda(\nabla_X Y)_\lambda = -A(\nabla_X Y)$.

Lemma 5 [6].

(i) *If $Y \in T_0$, then $\nabla_Y(T_\lambda) \subset T_\lambda$.*

(ii) *If $Y \in T_0$, then $\nabla_Y(T_0) \subset T_0$.*

(iii) *If $Y \in T_0, X \in T_\lambda$ and $[X, Y] = 0$, then $\nabla_X Y \in T_\lambda$.*

Proof. (i) has been already shown above. (ii) follows from (i) and from the fact that T_0 and T_λ are orthogonal complements to each other. (iii) follows from $\nabla_X Y = \nabla_Y X + [X, Y] = \nabla_Y X \in T_\lambda$.

Lemma 6 [6]. *If $Y\lambda = 0$ for every $Y \in T_0$, then $X \in T_\lambda$ implies $\nabla_X(T_0) \subset T_0$ and $\nabla_X(T_\lambda) \subset T_\lambda$.*

Proof. Under the assumption Lemma 4 implies $A(\nabla_X Y) = 0$, that is, $\nabla_X Y \in T_0$ for $X \in T_\lambda$ and $Y \in T_0$. Thus $\nabla_X(T_0) \subset T_0$ for $X \in T_\lambda$. Since T_λ is the orthogonal complement of T_0 , we have $\nabla_X(T_\lambda) \subset T_\lambda$ as well.

Lemma 7. *Let Y be vector fields belonging to T_0 such that $\nabla_Y Y = 0$. If there is a non-vanishing vector field X belonging to T_λ such that $[X, Y] = 0$, then $(YY)(\frac{1}{\lambda}) + c\frac{1}{\lambda} = 0$.*

Proof. We know that $R(X, Y) = c(X \wedge Y) + AX \wedge AY = c(X \wedge Y)$, since $AY = 0$. On the other hand, we have

$$R(X, Y) \cdot Y = \nabla_X(\nabla_Y Y) - \nabla_Y(\nabla_X Y) - \nabla_{[X, Y]}Y = -\nabla_Y(\nabla_X Y)$$

in view of $\nabla_Y Y = 0$ and $[X, Y] = 0$. By Lemma 4 we have $-(Y\lambda)X = A(\nabla_X Y)$. By Lemma 5 (iii) we obtain $A(\nabla_X Y) = \lambda(\nabla_X Y)$. Thus we get $\nabla_X Y = -\frac{Y\lambda}{\lambda}X$.

Therefore $\nabla_Y(\frac{Y\lambda}{\lambda}X) = cX$, which implies

$$\frac{\lambda(YY\lambda) - (Y\lambda)(Y\lambda)}{\lambda^2}X + \frac{Y\lambda}{\lambda}\nabla_Y X = cX.$$

Since $[X, Y] = 0$, we have $\nabla_Y X = \nabla_X Y$ and this is equal to $-\frac{Y\lambda}{\lambda}X$. Hence the equation above reduces to

$$\frac{\lambda(YY\lambda) - 2(Y\lambda)(Y\lambda)}{\lambda^2}X = cX.$$

Since X is non-vanishing, we get

$$\frac{\lambda(Y\lambda) - 2(Y\lambda)(Y\lambda)}{\lambda^2} = c.$$

A simple computation shows

$$Y\lambda\left(\frac{1}{\lambda}\right) = -\frac{\lambda Y(Y\lambda) - 2(Y\lambda)(Y\lambda)}{\lambda^3} = -c\frac{1}{\lambda}.$$

Lemma 8 [9]. *If $t(x) \geq 2$, $\ker A_x = \{X \in T_x M \mid R(X, Y) = cX \wedge Y \text{ for all } Y \in T_x M\}$, where $X \wedge Y$ is defined by $(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y$.*

Proof. Denote the latter space by $T_0(x)$. If $AX = 0$, then $0 = AX \wedge AY = R(X, Y) - cX \wedge Y$ by the Gauss equation. Thus $\ker A_x \subset T_0(x)$. If we now choose X arbitrary in $T_0(x)$, there is a $Y \in T_x M$ such that $AY \neq 0$ and $g(AX, AY) = 0$ at x . Since $AX \wedge Y = 0$ at x , $g(AY, AX)AX = g(AX, AX)AY$ at x . Thus $g(AX, AX) = 0$, and hence $AX = 0$, namely, $X \in \ker A_x$.

Lemma 9 [9]. *Let f and \bar{f} be isometric immersions of M^n as a hypersurface in $M^{n+1}(c)$. If $t(x)$ for $f \geq 3$ for all x , then $A = \pm\bar{A}$, where \bar{A} is the second fundamental form corresponding to \bar{f} .*

Proof. We first observe that $t(x) = \bar{t}(x)$. For if $\bar{t}(x) \leq 1$, then $\bar{A}X \wedge \bar{A}Y = 0$ for all X and Y and hence $\dim T_0(x) = n$ contrary to the fact that $t(x) \geq 3$. Thus $\bar{t}(x) \geq 2$ for all x . Hence $\ker \bar{A}_x = T_0(x) = \ker A_x$. Since A and \bar{A} are symmetric, $\text{Im } A_x = \text{Im } \bar{A}_x = T_0(x)^\perp$. In particular $t(x) = \bar{t}(x)$.

Furthermore, for arbitrary $X \in T_x M$, $AX \wedge \bar{A}X = 0$. For if not, we may choose Y so that $AX \wedge \bar{A}X \wedge \bar{A}Y \neq 0$. But $AX \wedge AY = \bar{A}X \wedge \bar{A}Y$ since $R(X, Y) - cX \wedge Y$ is independent of the immersion. Thus $AX \wedge \bar{A}X \wedge AY \neq 0$ which is a contradiction.

Thus, for each X there is a scalar c , possibly depending on X such that $AX = c\bar{A}X$. Choose X_1 and X_2 linearly independent in $T_0(x)^\perp$. Then $AX_1 = c_1\bar{A}X_1$, $AX_2 = c_2\bar{A}X_2$ and $A(X_1 + X_2) = c_3\bar{A}(X_1 + X_2)$. But A and \bar{A} are one to one on $T_0(x)^\perp$ and so $c_1 = c_2 = c_3$. Thus $AX = c\bar{A}X$ for some c independent of X . This equation also holds of course for $X \in \ker A$. Now $AX \wedge AY = c^2\bar{A}X \wedge \bar{A}Y$ so $c = \pm 1$. We conclude that $A_x = \pm\bar{A}_x$.

3 Proof of Theorem

Now we prove the following theorem:

Theorem. *Let M^n be a hypersurface in $\tilde{M}^{n+1}(c)$. If M is complete, $c \neq 0$ and $n \geq 3$, then M has the rigidity.*

Proof. Let f and \bar{f} be isometric immersions of M^n as a hypersurface in $M^{n+1}(c)$. Assume that $t(x)$ for $f \geq 3$ for all x . Then, from Lemma 9, we see that $A = \pm\bar{A}$, where \bar{A} is the second fundamental form corresponding to \bar{f} . Therefore, from the result of Beez and Thomas we know that M^n has the rigidity. Hence, we may assume that there exists a point x_0 which has the type number ≤ 2 .

At first, we consider a neighborhood U of x_0 which has the type number = 1. Then we see that the distributions T_0 and T_λ are differentiable and involutive. Moreover,

we know from (2) of Lemma 5 that the maximal integral manifold $M_0(x)$ to $T_0(x)$ through x is totally geodesic and hence complete (See Proposition 1 of [6]). Then by the Lemma 7 we obtain that

$$YY\left(\frac{1}{\lambda}\right) + c\frac{1}{\lambda} = 0 \quad \text{for } Y \in T_0.$$

We choose $Y \in T_0$ as above. Let L be a geodesic in the maximal integral manifold $M_0(x)$ of $T_0(x)$ and Y the parallel vector field in the direction of L . Then we have $YY\left(\frac{1}{\lambda}\right) + c\frac{1}{\lambda} = 0$. This means that if s is the length parameter of L , then

$$(3.1) \quad \frac{d^2}{ds^2}\left(\frac{1}{\lambda}\right) + c\frac{1}{\lambda} = 0.$$

Assume that $c > 0$. From the equation (3.1) we have the solution

$$\frac{1}{\lambda} = a \cos(\sqrt{c}s) + b \sin(\sqrt{c}s),$$

where a and b are certain constants which are independent of s . Then there exist α and β such that $\frac{1}{\lambda} = \alpha \cos(\sqrt{c}s + \beta)$. If $\frac{1}{\lambda}$ is dependent of s , we see that it is a contradiction, since M is complete. We have thus shown that λ is equal to a constant on L . Since L can be an arbitrary geodesic in $M_0(x)$ starting from x , we conclude that λ is equal to a constant on $M_0(x)$. Thus $Y\lambda = 0$ for any $Y \in T_0$. Since $R(X, Y)Y = 0$, we obtain $c = 0$, which is a contradiction to the condition of $c > 0$. Hence, we know that the case of $c > 0$ does not occur. When $c < 0$, even complete, the solution

$$\frac{1}{\lambda} = a \cosh(\sqrt{-c}s) + b \sinh(\sqrt{-c}s)$$

is nonzero, but by the existence and uniqueness of the solutions of the ordinary differential equation and the second term of (3.1) we thus see that $A = \pm\bar{A}$ if $c < 0$.

Next, we assume that a neighborhood U of x_0 which has the type number 2.

We consider the case of U having the three distinct eigenvalues λ, μ and 0. In this case we define three distributions as follows:

$$\begin{aligned} T_\lambda &= \{X \in T_x(M); AX = \lambda X\}, \\ T_\mu &= \{X \in T_x(M); AX = \mu X\}, \\ T_0 &= \{X \in T_x(M); AX = 0\}. \end{aligned}$$

Let X and Y be unit vector fields belonging to T_λ and T_0 such that $\nabla_Y Y = 0$, $[X, Y] = 0$, respectively. Using the Codazzi equation, we obtain

$$\begin{aligned} -\lambda(\nabla_X Y)_\lambda &= (Y\lambda)X, \\ \lambda(\nabla_Y X)_0 &= 0, \\ (\lambda - \mu)(\nabla_Y X)_\mu &= -\mu(\nabla_X Y)_\mu. \end{aligned}$$

On the other hand, we have

$$(3.2) \quad \begin{aligned} c &= g(R(X, Y)Y, X) \\ &= g(\nabla_X \nabla_Y Y - \nabla_Y \nabla_X Y - \nabla_{[X, Y]}Y, X). \end{aligned}$$

From $(\nabla_Y X)_0 = 0, \lambda(\nabla_Y X)_\mu = 0$ we obtain $A\nabla_X Y = \lambda\nabla_X Y = -(Y\lambda)X$.
Hence, we get

$$(3.3) \quad c = \frac{\lambda Y Y \lambda - 2(Y\lambda)(Y\lambda)}{\lambda^2}.$$

From (3.3) we have

$$Y Y \left(\frac{1}{\lambda}\right) + c \frac{1}{\lambda} = 0$$

By the same argument as above we see that the case of $c > 0$ does not occur. Assume that $c < 0$. Let Z be a unit vector fields belonging to T_μ such that $[Z, Y] = 0$. Similarly, we can obtain

$$Y Y \left(\frac{1}{\mu}\right) + c \frac{1}{\mu} = 0.$$

By the existence and uniqueness of the solutions of the ordinary differential equation and the second term of (3.1) we thus see that this case does not occur.

Assume that U has the two distinct eigenvalues λ and 0 . In the case we consider two distributions as above:

$$\begin{aligned} T_\lambda &= \{X \in T_x(M); AX = \lambda X\}, \\ T_0 &= \{X \in T_x(M); AX = 0\}. \end{aligned}$$

Let X, Y be unit vector fields belonging to T_λ and T_0 such that $\nabla_Y Y = 0, [X, Y] = 0$, respectively. Using the Codazzi equation, we obtain

$$\begin{aligned} -\lambda(\nabla_X Y)_\lambda &= (Y\lambda)X, \\ \lambda(\nabla_Y X)_0 &= 0. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} c &= g(R(X, Y)Y, X) \\ &= g(\nabla_X \nabla_Y Y - \nabla_Y \nabla_X Y - \nabla_{[X, Y]} Y, X). \end{aligned}$$

From $(\nabla_Y X)_0 = 0$ we obtain $A\nabla_X Y = \lambda\nabla_X Y = -(Y\lambda)X$.
Hence, we get

$$(3.4) \quad c = \frac{\lambda Y Y \lambda - 2(Y\lambda)(Y\lambda)}{\lambda^2}.$$

From (3.4) we have

$$Y Y \left(\frac{1}{\lambda}\right) + c \frac{1}{\lambda} = 0.$$

By the same argument as above we see that the case of $c > 0$ does not occur. Assume that $c < 0$. Since A and \bar{A} are simultaneously diagonalizable, we see that $A = \pm \bar{A}$. Thus we see that f and \bar{f} are totally geodesic immersions if $c > 0$ and $A = \pm \bar{A}$ if $c < 0$ in the case of the typer number ≤ 2 . Hence M^n has the rigidity. This proves theorem.

References

- [1] R. Beez, *Zur theorie der Krümmungsmasses von Mannigfaltigkeiten höherer Ordnung*, Z. Math. Physik 21 (1876), 373-401.
- [2] D. Ferus, *On the type number of hypersurfaces in spaces of constant curvature*, Math. Ann. 187 (1970), 310-316.
- [3] C. E. Harle, *Rigidity of hypersurfaces of constant scalar curvature*, J. Differential Geometry 5 (1971), 85-111.
- [4] Y. Matsuyama, *Rigidity of hypersurfaces with constant mean curvature*, Tohoku Math. J. 28 (1976), 199-213.
- [5] Y. Matsuyama, *Complete hypersurfaces with $RS = 0$ in E^{n+1}* , Proc. Amer. Math. Soc. 88 (1983), 119-123.
- [6] K. Nomizu, *On hypersurfaces satisfying a certain condition on the curvature tensor*, Tohoku Math. Journ. 20 (1968), 46-59.
- [7] S. Kobayashi and K. Nomizu, *Foundations of differential geometry, Vol. II*, Interscience Tracts No. 15, John Wiley and Sons, New York, 1963.
- [8] K. Nomizu and B. Smyth, *A formula of Simons' type and hypersurfaces with constant mean curvature*, J. Differential Geometry 3 (1969), 367-377.
- [9] P. J. Ryan, *Homogeneity and some curvature conditions for hypersurfaces*, Tohoku Math. Journ. 21 (1969), 363-388.
- [10] P. J. Ryan, *Hypersurfaces with parallel Ricci tensor*, Osaka J. Math. 8 (1971), 251-259.
- [11] R. Sacksteder, *The rigidity of hypersurfaces*, J. Math. Mech. 11 (1962), 929-939.
- [12] T. Y. Thomas, *Riemannian spaces of class one and their characterizations*, Acta Math. 67 (1936), 169-211.

Author's address:

Yoshio Matsuyama
Department of Mathematics, Chuo University,
1-13-27 Kasuga Bunkyo-ku, Tokyo 112-8551, Japan.
E-mail: matuyama@math.chuo-u.ac.jp