On cellular decomposition of compact surfaces

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Abstract. This paper completes and continues our note [9] concerning the classification of compact surfaces admitting structural cellular decomposition used in the compact orientable two-dimensional manifolds theory. We present some definitions and results concerning cellular decomposition of the compact n-dimensional manifolds. These notions are applied in proving some results. Theorem 2.3, announced in our previous note [9], gives us a natural method for the characterization of compact orientable surfaces.

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1 Introduction

In our note [9], we provided a classification of the compact orientable 2-dimensional manifolds using the standard n-simplex concept or the n-dimensional cell. In the present study we rely on the following cornerstone result, which emerges from the Morse Functions Theory ([13]):

All the manifolds of class C^∞ admit both a triangulation, and a cellular decomposition as the finite disjoint union of cells which are diffeomorphic to open sets.

For a compact surface, one may consider a cellular decomposition into a union of cells of the following types:

- vertices (0-dimension cells);
- edges (1-dimension cells);
- faces (2-dimension cells).

In this respect, the following established facts will be further used:

i) Each face is homeomorphic to a polygonal domain, such that each one of the edges of the boundary of the face is applied on a side of the polygonal domain.

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ii) Being given a surface decomposition, we can obtain new decompositions by decomposing a face or more, using new edges for several new faces or by grouping more faces into one. Our goal is to obtain, through such changes, the fundamental decomposition of a surface, such that it permits the description and comparison to other surfaces.

iii) If we admit that the surface S is connected, we can assign numbers to the faces, and, using a homeomorphism, we can obtain a polygonal domain P of the real plane, such that each edge of a face is the image of exact two polygon sides. Consequently, it results that the boundary of the polygonal domain has an even number of sides.

Definition 1.1. A symbolic product of r factors $a_1...a_{r-1}a_r$, representing sides of the polygonal domain P, is called a symbol A, if it contains each letter $a_1, ..., a_{s-1}, a_s$ $(s = r/2)$ as factors at least once and if a letter a_i of this symbol appears only once, then the symbol also contains its reverse a_i^{-1} letter.

Remark 1.2. The surface S is defined, excepting a homeomorphism, by a pair (P, A), which is composed of a polygonal domain P and of the symbol A, which indicates the manner of identification of the sides of the polygon in order to obtain the surface S.

Definition 1.3. a) Two pairs, each composed of a polygonal domain and a symbol, which generate homeomorphic surfaces, are called equivalent pairs.[1]

b) Each product of the factors of the symbol A, in order of their occurence, is called monomial of the symbol A.

We further recall a previous key-result, which was proved in [9].

Proposition 1.1. *Excepting the polygonal domains, the following assertions are true:*

$$LxMNx^{-1}T \sim LyNMy^{-1}T\ ; \tag{1.1}$$

$$LxMNxT \quad \sim \quad LyNyM^{-1}T; \tag{1.2}$$

$$Lxx^{-1}M \quad \sim \quad LM; \tag{1.3}$$

where we used: capital letters for monomials of a symbol and small letters for the factors of a symbol.

The monomial M^{-1} is determined by reversing the order of the factors and inverting each of its elements.

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Definition 2.1. i) the letters x and y are called *separated* if they appear separated by other letters inside the symbol A, in this order: x, y, x^{-1}, y^{-1};

ii) the monomial $xyx^{-1}y^{-1}$ is called *a commutator*;

iii) the number of commutators of a symbol is called *the genus of the surface.*

Proposition 2.1. *Each symbol is equivalent to a symbol in which all separating pairs appear grouped into commutators.*

[1] In our note [9] we study such pairs and we define their equivalence criteria.

Proof. Let us assume that in the symbol A appear q commutators and a pair of letters that are separated. Then, the symbol A is equivalent to a symbol that has the same number of sides, but $q+1$ commutators. Consider the symbol $A = PxQyRx^{-1}Sy^{-1}T$. By a circular permutation of the letters of S, we obtain a symbol $A^{'}$ equivalent to the symbol A, namely

$$A^{'} = PxQyRx^{-1}Ty^{-1}.$$

Each of these q - given commutators appears in one of these different monomials P, Q, R, T. Using the equivalences given in relation (1.1) of the Proposition 1.1, we obtain the following relations:

$$\begin{aligned} A \sim A^{'} &= PxQyRx^{-1}Ty^{-1} \sim PxyRQx^{-1}Ty^{-1} \\ &\sim Pxyx^{-1}TRQy^{-1} \sim yx^{-1}TRQy^{-1}Px \\ &\sim yx^{-1}y^{-1}TRQx \sim xyx^{-1}y^{-1}PTRQ. \end{aligned}$$

Through these rearrangements, the previous commutators remain unchanged. If in the last symbol obtained still exist pairs that are separated, by a similar reason, we obtain an equivalent symbol in which the separating pairs group in a commutator.

Since the number of separating pairs is finite, after a finite number of changes, we can obtain a symbol with the properties of the assertion. □

Proposition 2.2. *If in a symbol there exist some separating pairs or pairs of the form x, x^{-1}, then the symbol A is equivalent to a symbol which is obtained from the symbol A, by removing all pairs x, x^{-1} which can not group in commutators.*

Proof. Let us consider $A = xPx^{-1}Q$. If the pair (x, x^{-1}) does not separate from any other pair (y, y^{-1}), the monomials P and Q have no common side, therefore, a vertex of the polygonal line P is never identified with any vertex of the polygonal line Q. Let B and $B^{'}$ be the endpoints of the side x, and C and $C^{'}$ the endpoints of the side x^{-1}, such that B identifies with C and $B^{'}$ with $C^{'}$. By introducing the sides BC and $B^{'}C^{'}$, the polygonal domain corresponding to the symbol A will consist of two polygonal domains $P^{'}, P^{''}$ and a rectangle $D = BB^{'}C^{'}C$. These ones, by the identifications indicated by A, provide two surfaces $S^{'}$, $S^{''}$, both having a circle $C^{'}$, $C^{''}$, respectively, as boundary and a cylinder C, limited by those two circles $C^{'}$ and $C^{''}$.

It can be shown that on the surfaces $S^{'}$ and $S^{''}$ one can determine two collars for the circles $C^{'}$ and $C^{''}$, i.e. two neighborhoods of the circles, homeomorphic to a cylinder. The suppression of sides x and x^{-1} leads to the cylinder C embedding in these two collars. □

As consequence of the preceding results, we infer that any symbol is equivalent to a symbol in which only commutators and pairs of the form x, x appear. This is be proved in the following result.

Theorem 2.3. *Any symbol is equivalent to one of the following symbols:*

$$\prod_{i=1}^{g} x_i y_i x_i^{-1} y_i^{-1}, \quad g \geq 1; \tag{2.1}$$

$$xyy^{-1}x^{-1}, \tag{2.2}$$

$$\prod_{i=1}^{g} x_i x_i, \quad g \geq 1. \tag{2.3}$$

Proof. The proof splits into two cases:
1) the symbol A contains pair of the form x, x;
2) the symbol A does not contain any pair x, x.
For proving 1), we assume that the symbol A contains the pair of the form x, x. As a result of the equivalence given by relation (1.2) of Proposition 1.1, we infer that the symbol A is equivalent to a symbol that contains the monomial xx and has the same number of sides like the symbol A.

Indeed, using the equivalences of Proposition 1, we obtain the relations:

$$\begin{array}{lclclcl} LxMNxT & \overset{(1.2)}{\sim} & LxNxM^{-1}T & \overset{(1.3)}{\sim} & LxNxM^{-1}x^{-1}xT & \overset{(1.1)}{\sim} \\ & \overset{(1.1)}{\sim} & LxxM^{-1}Nx^{-1}xT & \overset{(1.3)}{\sim} & LxxM^{-1}NT & \overset{(1.3)}{\sim} \\ & \overset{(1.3)}{\sim} & xx^{-1}LxxM^{-1}NT & \overset{(1.1)}{\sim} & xx^{-1}xLxM^{-1}NT & \overset{(1.3)}{\sim} \\ & \overset{(1.3)}{\sim} & xLxM^{-1}NT & \overset{(1.3)}{\sim} & xLxM^{-1}x^{-1}xNT & \overset{(1.1)}{\sim} \\ & \overset{(1.1)}{\sim} & xxM^{-1}Lx^{-1}xNT & \overset{(1.3)}{\sim} & xxM^{-1}LNT. & \end{array}$$

Therefore, let $A^{'} = xxyzy^{-1}z^{-1}P$ be this last symbol. By using the equivalence (1.2) of Proposition 1.1 for several times, we obtain:

$$\begin{aligned} A^{'} = xxyzy^{-1}z^{-1}P \quad &\sim \quad xyzy^{-1}z^{-1}Px \quad \sim \quad xy^{-1}z^{-1}Pxz^{-1}y^{-1} \quad \sim \\ \sim \quad xy^{-1}y^{-1}zx^{-1}P^{-1}z \quad &\sim \quad xy^{-1}y^{-1}zzPx \quad \sim \quad xxy^{'}y^{'}zzP. \end{aligned}$$

This gives us the following assertion: if the symbol A contains a pair xx, then all commutators of A can be replaced by products of the form $yyzz$. Thus, any symbol that contains at least one pair x, x is equivalent to a symbol of type (2.3) of Theorem 2.3.

2) If the symbol A does not contain any pair x, x, the following alternatives hold true:

2a) the symbol A contains pairs who separate, then the symbol A is equivalent to a symbol of type (2.1) of Theorem 2.3 (Proposition 2.1.);

2b) the symbol A contains no separating pairs, then the symbol A is equivalent to a symbol of type (2.2) of Theorem 2.3. □

Corollary 2.4. *Any connected, closed, compact surface without boundary is homeomorphic either to S^2 or to a n-hold torus ($n \geq 1$).*

3 Examples

This section contains illustrative examples for the latter developed theory.

1. The orientable surfaces of genus $g \geq 1$ - also called generalized tori - correspond to the symbol of type (2.1) of Theorem 2.3. The common torus has genus 1 and it is homeomorphic to the direct product $S^1 \times S^1$.

2. The surfaces corresponding to the symbol of type (2.2) of Theorem 2.3 are homeomorphic to the sphere S^2 (the sphere is considered to have the genus 0).

3. The non-orientable surfaces which are defined by the symbol of type (2.3) of Theorem 2.3.

The sphere, the torus and the n-hold torus are compact orientable surfaces.

In other words, we can set the directions on the boundaries of the faces of a decomposition, of any of these surfaces, such that each edge of the decomposition is driven in round trips from opposite directions, as we consider that it belongs to one or the other of the boundaries of the faces which contain it.

4 Conclusions

Theorem 2.3 gives us a natural method for the characterization of compact orientable surfaces. The classification of surfaces which admit cellular decomposition provides a valuable insight regarding the relation between the genus and the Euler-Poincaré characteristic of the surface ([13]). This classification is very important for the compact and orientable 2-dimensional differential manifold theory. E.g., the famous Gauss-Bonnet theorem which uses this classification directly on the triangulation of the surface, exemplifies the relation between the geometry and the topology of surfaces.

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