A note on semi-symmetric spaces with metric F-connection

Fatma Özdemir

Abstract. In this work, we consider semi-symmetric spaces with metric F-connection and examine the curvature properties of the spaces having such a connection. We also several conditions for these spaces to have the same curvature with the Riemannian connection and to have conformally flat curvature. Furthermore, a special recurrent torsion tensor is found so that the space with F-connection becomes an Einstein space. Finally, a condition is given for every path of the connection to be a geodesic.

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1 Introduction

Let (M_n, g) be an n-dimensional $(n > 2)$ differentiable manifold with metric tensor g and let ∇ be the Riemannian connection. A linear connection D is said to be a *semi-symmetric connection* on M_n if the torsion tensor S of D satisfies

$$S(X,Y) = p\,(Y)\,X - p\,(X)\,Y,$$

where p is a smooth linear differential form [3, 6, 7].

In a Kaehlerian manifold with Hermitian metric tensor g_{ij} and complex structure tensor $F_i{}^h$, Yano and Imai [7] constructed an affine connection D such that

$$D_k\, g_{ij} = 0\,, \qquad D_j\, F_i{}^h = 0$$

with the connection coefficients

$$\Gamma_{ji}{}^h = \begin{Bmatrix} h \\ ji \end{Bmatrix} + \delta_j^h\, p_i - g_{ji}\, p^h + F_j{}^h\, q_i + F_i{}^h\, q_j - F_{ji}\, q^h\,.$$

Here p_i is a 1-form and q^h is a vector field.

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In theoretical physics, especially in the general theory of relativity, spaces with Riemannian and non-Riemannian connections have been studied widely. Among these spaces, certain type of spaces are of interest such as spaces with constant curvatures, conformal flat spaces and spaces having recurrent curvature tensor [5]. Einstein spaces are one of the most studied ones.

Semi-symmetric spaces are generalization of the Riemaninan spaces and therefore, we consider semi-symmetric spaces with metric F-connection and exhibit some conditions to obtain certain type of curvatures of these spaces.

2 Preliminaries

Let M_n be an n-dimensional Kaehlerian manifold covered by a system of coordinate neighborhoods $\{U;\xi^i\}$, $(n \geq 4)$, and denote by g_{ij} and $F_i^{\ j}$ components of the Hermitian metric tensor and those of the complex structure tensor of M_n respectively.

Let ∇ denote the covariant differentiation with respect to Christoffel symbol $\left\{{h \atop ji}\right\}$ determined by g_{ij}; then we have

$$\nabla_k\, g_{ij} = 0\,, \qquad \nabla_k\, F_i^{\ j} = 0, \quad \nabla_k\, F_{ij} = 0,$$

where $F_{ij} = F_i^{\ t} g_{tj}$, and consequently $F_{ji} = -F_{ij}$ [8].

3 Semi-symmetric metric F-connections

Definition 3.1. Let $\left\{{h \atop ij}\right\}$ be the coefficients of Riemannian connection ∇ and $\Gamma_{ij}^{\ \ h}$ be the coefficients of an affine connection D. If D satisfies

$$D_k\, g_{ij} = 0\,, \qquad D_j\, F_i^{\ h} = 0 \tag{3.1}$$

then D is called a metric F-connection [7].

If the coefficients of D have the form

$$\Gamma_{ji}^{\ \ h} = \left\{{h \atop ji}\right\} + U_{ji}^{\ \ h} \tag{3.2}$$

then the torsion tensor $S_{ji}^{\ \ h}$ of the connection D is given by

$$S_{ji}^{\ \ h} = \Gamma_{ji}^{\ \ h} - \Gamma_{ij}^{\ \ h} = U_{ji}^{\ \ h} - U_{ij}^{\ \ h}\,, \tag{3.3}$$

where the tensor U_{kji} is defined by the metric as $U_{kji} = U_{kj}^{\ \ t}\, g_{ti}$ and S_{ijh} is defined by $S_{ijh} = S_{ij}^{\ \ t}\, g_{th}$.

The torsion tensor has the following property [7]

$$S_{jik} + S_{kji} + S_{ikj} = 2U_{jik} \tag{3.4}$$

or

$$U_{ji}^{\ \ h} = \frac{1}{2}(S_{ji}^{\ \ h} + S^h_{\ \ ji} + S^h_{\ \ ij})\,. \tag{3.5}$$

By using (3.1) and (3.2) it can be shown that

$$U_{ji}{}^{t} F_{t}{}^{h} - U_{jt}{}^{h} F_{i}{}^{t} = 0 \,. \tag{3.6}$$

In this work, it is assumed that the torsion tensor $S_{ji}{}^{h}$ has the form

$$S_{ji}{}^{h} = \delta_j^h \, p_i - \delta_i^h \, p_j - 2F_{ji} \, q^h \,, \tag{3.7}$$

where p_i is an 1-form, q^i is a contravariant vector field and F_{ij} are the components of a skew-symmetric tensor.

If D is a semi-symmetric and metric F-connection then, from (3.5) and (3.7), $U_{ji}{}^{h}$ is obtained as

$$U_{ji}{}^{h} = \delta_j^h \, p_i - g_{ji} \, p^h + F_{i}{}^{h} \, q_j + F_{j}{}^{h} \, q_i - F_{ji} \, q^h \,, \tag{3.8}$$

where $p_i = \partial_i p$ is gradient of a scalar function, $p^i = p_t \, g^{ti}$ and $q_i = q^t \, g_{ti}$.

Then, the coefficients of the connections satisfying (3.2) are obtained [7]

$$\Gamma_{ji}{}^{h} = \left\{ {h \atop ji} \right\} + \delta_j^h \, p_i - g_{ji} \, p^h + F_{j}{}^{h} \, q_i + F_{i}{}^{h} \, q_j - F_{ji} \, q^h \,. \tag{3.9}$$

From (3.6) and (3.8), it is obtained that the vector fields p_i and q_i are related to each other by

$$p_i = -F_{ti} \, q^t, \qquad q_i = F_{ti} \, p^t \,. \tag{3.10}$$

By using the relations in (3.10) we find that

$$q_i p^i = -q_t p^t \quad \text{and then} \quad q_i p^i = 0 \,.$$

In general, the curvature tensor of a manifold M_n is defined by [1]

$$L_{kji}{}^{h} = \partial_k \Gamma_{ji}{}^{h} - \partial_j \Gamma_{ki}{}^{h} + \Gamma_{kt}{}^{h} \, \Gamma_{ji}{}^{t} - \Gamma_{jt}{}^{h} \, \Gamma_{ki}{}^{t} \,. \tag{3.11}$$

For the connection coefficients $\Gamma_{ji}{}^{h}$ given by (3.9), the curvature tensor $L_{kji}{}^{h}$ of spaces with semi-symmetric metric-F connection becomes

$$\begin{aligned} L_{kji}{}^{h} = {} & R_{kji}{}^{h} + \delta_k^h (p_i p_j - g_{ji} p_t p^t - q_i q_j - F_{ji} p_t q^t - \nabla_j p_i) \\ & + \delta_j^h (-p_i p_k + g_{ki} p_t p^t + q_i q_k + F_{ki} p_t q^t + \nabla_k p_i) \\ & + g_{ij} (p^h p_k - q_k q^h - \nabla_k p^h) + g_{ik} (q^h q_j - p^h p_j + \nabla_j p^h) + 2F_{jk} (-p^h q_i + q^h p_i) \\ & + F_{ik} (p^h q_j + p_j q^h - \nabla_j q^h) + F_{ji} (p_k q^h + p^h q_k - \nabla_k q^h) \\ & + F_{k}{}^{h} (q_i p_j + q_j p_i - g_{ji} p^t q_t - F_{ji} q_t q^t - \nabla_j q_i) \\ & + F_{j}{}^{h} (-q_i p_k - q_k p_i + g_{ki} q_t p^t + F_{ki} q_t q^t + \nabla_k q_i) + F_{i}{}^{h} (\nabla_k q_j - \nabla_j q_k) \,, \end{aligned} \tag{3.12}$$

where the tensors p_{ij} and q_{ij} are of the form

$$p_{ji} = \nabla_j p_i - p_j \, p_i + q_j \, q_i + \frac{1}{2} p_t \, p^t g_{ji} \,, \tag{3.13}$$

(3.14) $$q_{ji} = \nabla_j q_i - p_j\, q_i - p_i\, q_j + \frac{1}{2} p_t\, p^t F_{ji}\,,$$

and $R_{kji}{}^h$ is the curvature tensor of the Riemannian manifold.

Multiplying (3.12) by g_{lh}, the curvature tensor is obtained in the form

(3.15) $$\begin{aligned} L_{kjih} &= R_{kjih} - p_{ji}g_{hk} + p_{ki}g_{jh} - p_{kh}g_{ji} + p_{jh}g_{ki} \\ &\quad - F_{kh}q_{ji} + F_{jh}q_{ki} - F_{ji}q_{kh} + F_{ki}q_{jh} - \alpha_{kj}F_{ih} - \beta_{ih}F_{kj}\,, \end{aligned}$$

where

(3.16) $$\alpha_{kj} = -(\nabla_k q_j - \nabla_j q_k)\,, \quad \beta_{ih} = 2(p_i q_h - p_h q_i)\,.$$

From (3.16) it is seen that $\alpha_{kj} = -\alpha_{jk}$, $\beta_{ih} = -\beta_{hi}$.

Now we examine the properties of the curvature tensor L_{kjih} of the connection D in detail, and give some conditions to get special type of curvatures.

The curvature tensor L_{kjih} has the following properties:

i) $L_{kjih} = -L_{jkih}$;
ii) $L_{kjih} = -L_{kjhi}$;
iii) $L_{kkih} = L_{kjhh} = 0$

Let $L_{ji} = L_{kjih}g^{kh}$ denote the Ricci tensor of the connection D and R_{ji} be the Ricci tensor of the connection ∇. Then, multiplying (3.15) by g^{kh} we obtain

(3.17) $$\begin{aligned} L_{ji} = \ & R_{ji} - (2n-1)p_{ji} - g_{ji}p_{kh}g^{kh} \\ & + F_j{}^k q_{ki} - F_{ji}q_{kh}\, g^{kh} - \alpha_{kj}F_i{}^k - \beta_{ih}F^h_{\ j}. \end{aligned}$$

Using (3.13), (3.14) and (3.16) we have

(3.18) $$\begin{aligned} L_{ji} &= R_{ji} - 2n(\nabla_j p_i - p_i p_j + g_{ji}p_k p^k) - 2(n-1)q_i q_j - g_{ji}\nabla_k p^k \\ &\quad + F_j{}^k(\nabla_k q_i) - F_{ji}(\nabla_k q^k) + F_i{}^k(\nabla_k q_j)\,. \end{aligned}$$

Let $L = L_{ji}g^{ji}$ be the scalar curvature of the connection D. Multiplying (3.18) by g^{ij}, we obtain the scalar curvature

(3.19) $$\begin{aligned} L &= R - (2n-1)p_{ji}g^{ij} - 2np_{kh}g^{kh} + F_j{}^k g^{ji}q_{ki} - \alpha_{kj}F_i{}^k g^{ij} - \beta_{ih}F^h_{\ j}g^{ij} \\ &= R - \alpha_{kj}F^{jk} - \beta_{ih}F^{hi} = R + (\nabla_k p^k + \nabla_j p^j) - 2(p_i p^i + p_i p^i), \end{aligned}$$

where R is the scalar curvature with respect to the connection ∇.

Then, we get the scalar curvature of the semi-symmetric metric F-connection in terms of the divergence and the norm of the vector field p as

(3.20) $$L = R + 2\left(\nabla_k p^k - 2p_k p^k\right)\,.$$

Theorem 3.1. *A semi-symmetric metric F-connection D and the Riemannian connection ∇ on M_n have the same curvature tensor if the divergence of the vector field q vanishes with respect to ∇, i.e., $\nabla_k q^k = 0$.*

Proof. For the curvature tensor of the semi-symmetric connection D we have

$$\begin{aligned} L_{kjih} &= R_{kjih} + p_{ji}g_{hk} - p_{ki}g_{jh} + p_{kh}g_{ji} - p_{jh}g_{ki} \\ &\quad + F_{kh}q_{ji} - F_{jh}q_{ki} + F_{ji}q_{kh} - F_{ki}q_{jh} - \alpha_{kj}F_{ih} - \beta_{ih}F_{kj}\,. \end{aligned} \tag{3.21}$$

By assumption $L_{kjih} = R_{kjih}$ and multiplying (3.21) by F^{hk}, we get

$$F^k_{\ j}p_{ki} + g_{ji}(q_{km}g^{km}) - (2n+3)q_{ji} + \alpha_{ij} + \beta_{ij} - F_{ji}(p_{km}g^{km}) = 0\,. \tag{3.22}$$

Also, multiplying the equation (3.22) by g^{ji}, we obtain

$$F^{ki}p_{ki} + 2n(q_{km}g^{km}) - (2n+3)(q_{ji}g^{ij}) = 0\,. \tag{3.23}$$

Since

$$q_{km}g^{km} = \nabla_k q^k - 2p_k q^k + \frac{1}{2}p_t p^t (F_{km}g^{km})\,, \tag{3.24}$$

and from equation (3.23) and (3.24) we find

$$q_{km}g^{km} = \nabla_k q^k - 2p_k q^k + \frac{1}{2}p_t p^t (F_{km}g^{km}) = 0.$$

Consequently, we obtain $\nabla_k q^k = 0$. □

In the next section, we define the conformal curvature tensor of the connection D and examine the conformal flatness of D.

4 Conformal curvature tensor of the semi-symmetric metric F-connection

The conformal curvature C_{kjih} tensor of the connection D is defined by

$$\begin{aligned} C_{kjih} = \; & L_{kjih} - \tfrac{1}{n-2}(g_{kh}L_{ji} - g_{jh}L_{ki} + L_{kh}g_{ji} - L_{jh}g_{ki}) \\ & + \tfrac{L}{(n-1)(n-2)}(g_{kh}g_{ji} - g_{jh}g_{ki}), \end{aligned} \tag{4.1}$$

where L_{kjih}, L_{ij} and L are the curvature tensor, Ricci tensor and the scalar curvature of the connection D, respectively. We can state the following theorem related to the conformal flatness of the connection D.

Theorem 4.1. *A semi-symmetric space with metric F-connection D is conformal flat if the equation*

$$a\,\nabla_k\, p^k + b\, p_k\, p^k = 0$$

is satisfied, with the coefficients $a = (10n^3 - 9n^2 + 6n - 3)$ and $b = n(10n^3 - 11n^2 - 6n - 1)$.

Proof. By using (3.15), (3.17) and (3.20), the conformal curvature tensor (4.1) becomes

$$
\begin{aligned}
C_{kjih} = \;& C^0_{kjih} + \tfrac{(n+1)}{(n-2)}(p_{ji}g_{hk} - p_{ki}g_{jh} + p_{kh}g_{ji} - p_{jh}g_{ki}) \\
& -F_{kh}q_{ji} + F_{jh}q_{ki} - F_{ji}q_{kh} + F_{ki}q_{jh} - \alpha_{kj}F_{ih} - \beta_{ih}F_{kj} \\
& +\tfrac{2}{n-2}p_{lm}g^{lm}(g_{kh}g_{ji} - g_{jh}g_{ki}) - \tfrac{1}{n-2}F_j{}^m(g_{ki}q_{mh} - g_{kh}q_{mi}) \\
& -\tfrac{1}{n-2}F_k{}^m(g_{jh}q_{mi} - g_{ji}q_{mh}) + \tfrac{1}{n-2}F_i{}^m(g_{kh}\alpha_{mj} \\
& -g_{jh}\alpha_{mk}) + \tfrac{1}{n-2}F_{hl}(g_{ji}\alpha_{lk} - g_{ki}\alpha_{lj}) + \tfrac{1}{n-2}F^m_j(g_{kh}\beta_{im} - g_{ki}\beta_{hm}) \\
& +\tfrac{1}{n-2}p_{lm}F^{lm}(g_{ki}F_{jh} - g_{kh}F_{ji} - g_{ji}F_{kh} + g_{jh}F_{ki})
\end{aligned}
\tag{4.2}
$$

where C^0_{kjih} is the conformal curvature tensor of the Riemannian connection. Since the conformal curvature tensor is traceless in Riemannian spaces, that is $C^0_{kjih}g^{kh} = 0$, by using conformal flatness assumption in (4.2), we obtain

$$
\begin{aligned}
0 = \;& \tfrac{10n^2}{(n-2)}(p_{ji}g^{ji}) - \tfrac{(n+1)}{(n-2)}p_{jh}g_{ki}g^{ij}g^{kh} \\
& +F_{jh}q_{ki}g^{kh}g^{ji} + F_{ki}q_{jh}g^{ji}g^{kh} - \alpha_{kj}g^{ji}F_{ih}g^{kh} - \beta_{ih}g^{ij}F_{kj}g^{kh} \\
& -\tfrac{(2-2n)}{(n-2)}q_{mi}F^{mi} - \tfrac{(4n-2)}{(n-2)}\alpha_{mj}F^{mj} - \tfrac{(4n-2)}{(n-2)}\beta_{im}F^{mj} \\
& +\tfrac{2n(2n-1)}{(n-1)(n-2)}\left(2\nabla_k p^k - 4p_k p^k\right).
\end{aligned}
\tag{4.3}
$$

Rearranging the terms in (4.3), we obtain

$$a\,\nabla_k\,p^k + b\,p_k\,p^k = 0$$

where $a = (10n^3 - 9n^2 + 6n - 3)$ and $b = n(10n^3 - 11n^2 - 6n - 1)$. □

5 Einstein spaces with semi-symmetric metric F-connection

Definition 5.1. Let M_n be a manifold with semi-symmetric metric F-connection and let L_{ij} be the Ricci tensor of the connection D. If the symmetric part of the Ricci curvature L_{ij} satisfies the relation

$$L_{(ij)} = \lambda\, g_{ij}\,,$$

where λ is a function, then the space is called Einstein space with the semi-symmetric metric F-connection.

Theorem 5.1. *An Einstein-Riemannian space is an Einstein space with semi-symmetric metric F-connection if the following condition*

$$(\lambda - \gamma) = -\frac{1}{2n}\left(e\,\nabla_k p^k + f\,p_k p^k\right)$$

is satisfied. Here the coefficients are $e = (3n+1)$ and $f = (n-1)^2$, and γ is a scalar function coming from the Einstein property of Riemann space, that is, $R_{ij} = \gamma g_{ij}$.

Proof. Using (3.17), we obtain the symmetric part of the Ricci tensor L_{ij} of the connection D

$$\begin{aligned} L_{(ij)} &= \frac{1}{2}(L_{ji}+L_{ij}) \\ &= \frac{1}{2}[2R_{ij}-(2n-1)(p_{ji}+p_{ij})-2g_{ji}p_{kh}g^{kh}+F_i{}^k q_{kj}+F_j{}^k q_{ki} \\ &\quad -\alpha_{ki}F_j{}^k-\alpha_{kj}F_i{}^k-\beta_{jh}F^h_i-\beta_{ih}F^h_j]\,. \end{aligned} \tag{5.1}$$

By transvecting (5.1) with g^{ij} and using (3.13), (3.14) and (3.16), we get

$$4n(\lambda-\gamma)=-2(3n+1)\nabla_k p^k-2(n-1)^2 p_k p^k\,.$$

From (3.1) we reach the following condition for semi-symmetric space with metric F-connection space being Einstein

$$(\lambda-\gamma)=-\frac{1}{2n}\left((3n+1)\nabla_k p^k+(n-1)^2 p_k p^k\right)\,.$$

But the the space is Einstein if the following relation holds:

$$L=R-2\left((3n+1)\nabla_k p^k+(n-1)^2 p_k p^k\right).$$ □

We shall further consider semi-symmetric spaces with metric F-connection having recurrent torsion tensor.

Definition 5.2. A tensor S on M is said to be *recurrent* with respect to a given linear connection, if it is not a zero tensor and if the covariant derivative of the tensor S is equal to the tensor product of a non-zero covariant vector w_h with S itself [4]. In local coordinates, we have

$$\nabla_h S_{kl\cdots}{}^{ij\cdots}=w_h S_{kl\cdots}{}^{ij\cdots}\,.$$

Theorem 5.2. *A semi-symmetric space with metric F-connection has a recurrent torsion tensor S, i.e., $D_k S_{ji}{}^h = w_k\, S_{ji}{}^h$ if and only if, the vectors p_k and w_k are collinear and q_k is recurrent, where p and q are vectors contained in the definition of torsion tensor in (3.7) and w is the recurrence vector field.*

Proof. Firstly, we assume that the torsion tensor $S_{ji}{}^h$ is recurrent

$$S_{ji}{}^h = w_k\, S_{ji}{}^h \tag{5.2}$$

$$= w_k(\delta^h_j\, p_i-\delta^h_i\, p_j-2F_{ji}\, q^h)\,. \tag{5.3}$$

By taking the covariant derivative of the torsion tensor $S_{ji}{}^h$ we get

$$D_k S_{ji}{}^h = D_k(\delta_j{}^h p_i-\delta_i{}^h p_j-2F_{ji}q^h)=(\delta_j{}^h D_k p_i-\delta_i{}^h D_k p_j-2F_{ji}D_k q^h)\,. \tag{5.4}$$

By contracting h and i in (5.3), we get the following relations

$$S_{hj}{}^h=(2n+1)p_j \qquad \text{and} \qquad D_k\, S_{hj}{}^h = w_k\, S_{hj}{}^h.$$

Then recurrence condition of p is obtained, $D_k\, p_j = w_k\, p_j$. On the other hand, to calculate the difference $(D_k\, p_j - D_j\, p_k)$ we need

$$D_k p_j = \partial_k p_j - p_h \begin{Bmatrix} h \\ kj \end{Bmatrix} - p_h \delta_k^h p_j + p_h g_{kj} p^h - p_h F_k{}^h q_j - p_h F_j{}^h q_k\,, \tag{5.5}$$

and

$$D_j p_k = \partial_j p_k - p_h \begin{Bmatrix} h \\ jk \end{Bmatrix} - p_h \delta_j^h p_k + p_h g_{jk} p^h - p_h F_j{}^h q_k - p_h F_k{}^h q_j\,. \tag{5.6}$$

From (5.5) and (5.6), we get

$$\begin{aligned} D_k p_j - D_j p_k &= \partial_k p_i - \partial_i p_k \\ &= w_k p_j - w_j p_k\,. \end{aligned} \tag{5.7}$$

Since p_k is gradient, that is $p_k = \partial_k p$, from (5.7) we obtain

$$(\partial_k p_i - \partial_i p_k) = (w_k p_j - w_j p_k) = 0 \qquad \text{or} \qquad w_k p_j - w_j p_k = 0$$

which means that vectors w_k and p_k are collinear, $w_k = a\, p_k$, where a is an arbitrary constant. Also, from (5.4), we conclude that q^h is recurrent.

Conversely, using that the vectors p_k and w_k are collinear and q_k is recurrent in (5.4), by assumption, we get $D_k\, S_{hj}{}^h = w_k\, S_{hj}{}^h$ which completes the proof. □

Theorem 5.3. *Let M_n be an Einstein-Riemannian space. If M_n has a recurrent torsion tensor with respect to the connection D then, the space is Einstein under the condition*

$$w_i\, p_j + w_i\, p_j = 2\left(q_i\, q_j - \frac{1}{(n-1)} p_i\, p_j\right) \tag{5.8}$$

where w is the recurrence vector, p and q are vectors defined by torsion tensor.

Proof. We know that if the torsion tensor is recurrent, then $D_k\, p_i = w_k\, p_i$ and $D_k\, q_i = w_k\, q_i$. When we substitute recurrence conditions into curvature tensor (5.1) we get

$$\begin{aligned} L_{(ij)} = \frac{1}{2}[&2R_{ij} - (2n-1)(p_{ji} + p_{ij}) - 2g_{ji} p_{kh} g^{kh} + F_i{}^k q_{kj} + F_j{}^k q_{ki} \\ &-\alpha_{ki} F_j{}^k - \alpha_{kj} F_i{}^k - \beta_{jh} F^h_i - \beta_{ih} F^h_j] \end{aligned} \tag{5.9}$$

where p_{ji}, q_{kj} are given by (3.13) and (3.14), and α_{kj} and β_{jh} are given by (3.16). By using the definition of the covariant derivative D_k, we also have the relations

$$D_j p_i = \nabla_j\, p_i - pi\, p_j + g_{ij} p^h + 2q_i\, q_j \tag{5.10}$$

$$D_k q_j = \nabla_k\, q_j - 2q_k\, p_j - p_h\, q_j + F_{kj} p_t\, p^t; \tag{5.11}$$

then p_{ji}, q_{kj} become

$$p_{ji} = D_j\, p_i - q_i\, q_j - \frac{1}{2} p_t\, p^t\, g_{ji} \tag{5.12}$$

$$q_{kj} = D_k\, q_j + q_k\, p_j - \frac{1}{2} p_t\, p^t\, F_{kj}\,. \tag{5.13}$$

Since the recurrence condition requires that $D_k\, p_i = w_k\, p_i$ and $D_k\, q_i = w_k\, q_i$, we have

$$p_{ji} = w_j\, p_i - q_i\, q_j - \frac{1}{2} p_t\, p^t\, g_{ji} \tag{5.14}$$

$$q_{kj} = w_k\, q_j + q_k\, p_j - \frac{1}{2} p_t\, p^t\, F_{kj}\,. \tag{5.15}$$

After some calculations we get

$$L_{(ij)} = \frac{1}{2}[2R_{ij} - 2(2n-1)(w_i\, p_j + w_j\, p_i) + 4(n-1)q_i\, q_j + 4p_i\, p_j + 2(n+2)p_t\, p^t\, g_{ij}].$$

Since $R_{ij} = \gamma\, g_{ij}$, to get $L_{(ij)} = \lambda\, g_{ij}$ we conclude that any semi-symmetric space with metric F-connection is Einstein if

$$w_i\, p_j + w_i\, p_j = 2[q_i\, q_j - \frac{1}{(n-1)} p_i\, p_j]$$

holds. □

We further point out sufficient conditions under which the paths of the connection D are geodesics.

Definition 5.3. A path of ${\Gamma_{\lambda\mu}}^{\nu}$ is a curve $x^{\nu}(t)$ whose tangent vector dx^{ν}/dt satisfies the relations [2]

$$\frac{dx^{[\lambda}}{dt}\left(\frac{d^2x^{\nu]}}{dt^2} + {\Gamma_{\alpha\beta}}^{\nu]}\frac{dx^{\alpha}}{dt}\frac{dx^{\beta}}{dt}\right) = 0\,.$$

Proposition 5.4. *Every path of a D-connection ${\Gamma_{\lambda\mu}}^{\nu}$ whose tangent vector dx^{ν}/dt coincides with the D-recurrent vector p^{ν}, is a geodesic.*

Proof. The equations of D-paths are

$$\begin{aligned}&\tfrac{dx^{\lambda}}{dt}\left(\tfrac{d^2x^{\nu}}{dt^2} + \left(\left\{{\nu \atop \alpha\beta}\right\} + {U_{\alpha\beta}}^{\nu}\right)\tfrac{dx^{\alpha}}{dt}\tfrac{dx^{\beta}}{dt}\right)\\ &\quad -\tfrac{dx^{\nu}}{dt}\left(\tfrac{d^2x^{\lambda}}{dt^2} + \left(\left\{{\lambda \atop \alpha\beta}\right\} + {U_{\alpha\beta}}^{\lambda}\right)\tfrac{dx^{\alpha}}{dt}\tfrac{dx^{\beta}}{dt}\right) = 0\,,\end{aligned}$$

where

$${U_{\alpha\beta}}^{\nu} = \delta^{\nu}_{\alpha}p_{\beta} - g_{\alpha\beta}p^{\nu} + F_{\alpha}{}^{\nu}q_{\beta} + F_{\beta}{}^{\nu}q_{\alpha} - F_{\alpha\beta}q^{h}\,.$$

If $p^{\nu} = dx^{\nu}/dt$, then these reduce to the geodesic equations,

$$\frac{dx^{\lambda}}{dt}\left(\frac{d^2x^{\nu}}{dt^2} + \left\{{\nu \atop \alpha\beta}\right\}\frac{dx^{\alpha}}{dt}\frac{dx^{\beta}}{dt}\right) - \frac{dx^{\nu}}{dt}\left(\frac{d^2x^{\lambda}}{dt^2} + \left\{{\lambda \atop \alpha\beta}\right\}\frac{dx^{\alpha}}{dt}\frac{dx^{\beta}}{dt}\right) = 0.$$ □

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Author's address:

Fatma Özdemir
Istanbul Technical University,
Faculty of Science and Letters,
34469 Maslak-Istanbul, Turkey.
E-mail: fozdemir@itu.edu.tr