Numerical integration and stability problems for special cases of Lotka-Volterra equation

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Abstract. The goals of our paper is to study the Lyapunov stability of the equilibrium points using energy-Casimir method, numerical integration problems via Kahan and Lie-Trotter algorithms and to find the Lax formulation of some special Lotka-Volterra systems.

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Key words: Lotka-Volterra system; nonlinear stability; energy-Casimir method; Lax formulation; numerical integration; Kahan integrator; Lie-Trotter integrator.

1 Introduction

It is known that the three dimensional Lotka-Volterra system of competing species, given by the formula

$$(1.1)\qquad \begin{cases} \dot{x} = x(-\frac{1}{ab}y + z + \lambda) \\ \dot{y} = y(az + x + \mu) \\ \dot{z} = z(bx + y + \mu b - \lambda ab), \end{cases}$$

where $a, b \in \mathbb{R}^*$ and $\lambda, \mu \in \mathbb{R}$, has a Hamilton-Poisson realization (see [2]).

In this paper, let us consider the following cases

1. $b = -1$ and $\mu = \lambda = 0$, so the system (1.1) becomes

$$(1.2)\qquad \begin{cases} \dot{x} = x(\frac{1}{a}y + z) \\ \dot{y} = y(az + x) \\ \dot{z} = z(-x + y), \end{cases}$$

where $a \in \mathbb{R}^*$.

2. $a = -1$ and $\mu = \lambda = 0$, so the system (1.1) becomes

$$(1.3)\qquad \begin{cases} \dot{x} = x(\frac{1}{b}y + z) \\ \dot{y} = y(-z + x) \\ \dot{z} = z(bx + y), \end{cases}$$

where $b \in \mathbb{R}^*$.

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For these systems we are interested to study some dynamical properties.

2 The Poisson geometry of the system (1.2)

The system (1.2) has the Hamilton-Poisson realization (see [2])

$$(\mathbb{R}^3, \Pi_-, H),$$

where

$$\Pi_- = \begin{pmatrix} 0 & \frac{1}{a}xy & -\frac{1}{a}xz \\ -\frac{1}{a}xy & 0 & -yz \\ \frac{1}{a}xz & yz & 0 \end{pmatrix}$$

is the minus Lie-Poisson structure and the Hamiltonian $H \in C^\infty(\mathbb{R}^3, \mathbb{R})$ is given by

$$H(x, y, z) = -ax + y - az, \quad x, y, z \in \mathbb{R}, \quad a \in \mathbb{R}^*.$$

Also, the function $C : \mathbb{R}^3_+ \to \mathbb{R}$,

$$C(x, y, z) = -a \ln x + \ln y + \ln z, \quad x, y, z > 0,$$

is a Casimir of our Poisson configuration (see [3]).

Remark 2.1. The phase curves of dynamics (1.2) are the intersections of the surfaces

$$-ax + y - az = const.$$

and

$$-a \ln x + \ln y + \ln z = const.,$$

see the *Figure 1.*

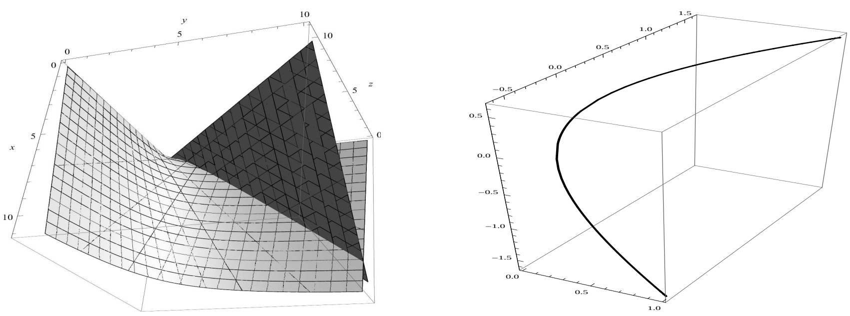

Figure 1: The phase curves of the system (1.2)

Let us continue with a discussion concerning the nonlinear stability of equilibrium states of our system (1.2) (see [4], for details).

It is obviously to see that the equilibrium points of our dynamics are given by $e_1^M = (M, 0, 0)$, $e_2^M = (0, M, 0)$, $e_3^M = (0, 0, M)$ and $e_4^{a,M} = (-aM, -aM, M)$, $M \in \mathbb{R}$.

Let A be the matrix of linear part of our system (1.2), that is

$$A = \begin{pmatrix} \frac{1}{a}y + z & \frac{1}{a}x & x \\ y & az + x & ay \\ -z & z & -x + y \end{pmatrix}.$$

If we consider $M > 0$ and $a < 0$, then the characteristic roots of $A(e_1^M)$ [resp. $A(e_2^M)$, resp. $A(e_3^M)$, resp. $A(e_4^{a,M})$] are given by

$$\lambda_1 = 0, \quad \lambda_2 = M, \quad \lambda_3 = aM$$

$$[\text{ resp. } \lambda_1 = 0, \quad \lambda_2 = M, \quad \lambda_3 = \frac{1}{a}M,$$

$$\text{resp.} \quad \lambda_1 = 0, \quad \lambda_{2,3} = \pm M,$$

$$\text{resp.} \quad \lambda_1 = 0, \quad \lambda_{2,3} = \pm iM\sqrt{a(a-2)}\ ],$$

so we can conclude that the equilibrium states e_1^M, e_2^M and e_3^M are unstable and $e_4^{a,M}$ are spectrally stable.

Proposition 2.1. *The equilibrium states $e_4^{a,M}$ are nonlinearly stable for any $M > 0$ and $a < 0$.*

Proof. We shall use energy-Casimir method, see [1] for details. Let

$$H_\varphi(x, y, z) = -ax + y - az + \varphi(-a\ln x + \ln y + \ln z)$$

be the energy-Casimir function, where $x, y, z > 0$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is a smooth real valued function defined on $\mathbb{R}$.

Now, the first variation of H_φ is given by

$$\delta H_\varphi(x, y, z) = -a\delta x + \delta y - a\delta z + \dot{\varphi}(-a\ln x + \ln y + \ln z)\left(-\frac{a}{x}\delta x + \frac{1}{y}\delta y + \frac{1}{z}\delta z\right).$$

This equals zero at the equilibrium of interest if and only if

$$\dot{\varphi}[-a\ln(-aM) + \ln(-aM) + \ln M] = aM. \tag{2.1}$$

The second variation of H_φ is given by

$$\delta^2 H_\varphi(x, y, z) = \ddot{\varphi}(-a\ln x + \ln y + \ln z)\left(-\frac{a}{x}\delta x + \frac{1}{y}\delta y + \frac{1}{z}\delta z\right)^2 +$$

$$+\dot{\varphi}(-a\ln x + \ln y + \ln z)\left[\frac{a}{x^2}(\delta x)^2 - \frac{1}{y^2}(\delta y)^2 - \frac{1}{z^2}(\delta z)^2\right]$$

and, taking into account the relation (2.1), we obtain

$$\delta^2 H_\varphi(-aM, -aM, M) =$$

$$= \ddot{\varphi}[-a\ln(-aM) + \ln(-aM) + \ln M]\left(\frac{1}{M}\delta x - \frac{1}{aM}\delta y + \frac{1}{M}\delta z\right)^2 +$$

$$+\left[\frac{1}{M}(\delta x)^2-\frac{1}{aM}(\delta y)^2-\frac{a}{M}(\delta z)^2\right].$$

If we choose now φ such that the relation (2.1) is valid and

$$\ddot{\varphi}[-a\ln(-aM)+\ln(-aM)+\ln M]>0,$$

then the second variation of H_φ at the equilibrium of interest is positive definite and so our equilibrium states are nonlinearly stable. □

As a consequence, we can find the periodic orbits of some equilibrium points $e_4^{a,M}$, $M>0$ and $a<0$. More exactly, we have

Proposition 2.2. *Near to $e_4^{-1,M}=(M,M,M)$, $M>0$, the reduced dynamics has, for each sufficiently small value of the reduced energy, at least 1-periodic solution whose period is close to*

$$\frac{2\pi}{\sqrt{3}M}\ .$$

For proof, see [6].

Proposition 2.3. *The dynamics* (1.2) *allows a formulation in terms of Lax pairs.*

Proof. Let us take

$$L=\begin{pmatrix}0&\alpha&\beta&0&0&0\\-\alpha&0&\gamma&0&0&0\\-\beta&-\gamma&0&0&0&0\\0&0&0&0&\delta&\epsilon\\0&0&0&-\delta&0&\varepsilon\\0&0&0&-\epsilon&-\varepsilon&0\end{pmatrix},$$

$$B=\begin{pmatrix}0&\zeta&\eta&0&0&0\\-\zeta&0&\zeta&0&0&0\\-\eta&-\zeta&0&0&0&0\\0&0&0&0&\theta&0\\0&0&0&-\theta&0&\vartheta\\0&0&0&0&-\vartheta&0\end{pmatrix},$$

where

$$\alpha=x-\frac{y}{a}+z\ ,\quad \beta=-ix+\frac{iy}{a}\ ,\quad \gamma=x-\frac{y}{a}\ ,$$

$$\delta=(\frac{2i}{\sqrt{5}}+\frac{4ia}{\sqrt{5}}-i\sqrt{5}a)x+i\sqrt{5}y-i\sqrt{5}az,$$

$$\epsilon=\frac{-i(1+2a)}{\sqrt{5}}x,\quad \varepsilon=x+2y-2az,\quad \zeta=-i(x-y),$$

$$\eta=-x+y,\quad \theta=\frac{i\sqrt{5}}{a}y+i\sqrt{5}z,\quad \vartheta=\frac{2}{a}y+2z,$$

$i=\sqrt{-1}$. Then, using MATHEMATICA 8.0, we can put the system (1.2) in the equivalent form

$$\dot{L}=[L,B]$$

as desired. □

Let us pass now to the numerical integration of the equations (1.2). We purpose to make comparison between the Kahan integrator, the Lie-Trotter integrator and the 4^{th}-step Runge-Kutta method and to point out some of their properties.

It is easy to see that for the system (1.2), Kahan integrator can be written in the following form (see [5], for details)

$$\begin{cases} x^{n+1}-x^n=\frac{h}{2}(\frac{1}{a}x^{n+1}y^n+\frac{1}{a}y^{n+1}x^n+z^{n+1}x^n+x^{n+1}z^n) \\ y^{n+1}-y^n=\frac{h}{2}(x^{n+1}y^n+y^{n+1}x^n+az^{n+1}y^n+ay^{n+1}z^n) \\ z^{n+1}-z^n=\frac{h}{2}(-x^{n+1}z^n-z^{n+1}x^n+z^{n+1}y^n+y^{n+1}z^n). \end{cases} \tag{2.2}$$

The solutions of this system are presented in the Figure 2.

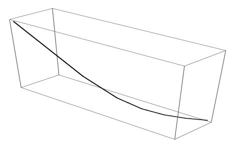

Figure 2: Kahan integrator for the system (1.2)

After some long but straightforward computations or using MATHEMATICA 8.0, we get the following proposition, which shows the incompatibility of the Kahan integrator with the Poisson geometric structure of the our system.

Proposition 2.4. *Kahan integrator* (2.2) *does not preserve the minus Lie-Poisson structure* Π_-*, the Hamiltonian* H *and the Casimir* C *of our Poisson configuration.*

We shall discuss now the numerical integrator of the dynamics (1.2) via the Lie-Trotter integrator, see for details [9]. For the beginning, let us observe that the Hamiltonian vector field X_H splits as follows

$$X_H = X_{H_1} + X_{H_2} + X_{H_3},$$

where

$$H_1(x,y,z) = -ax, \quad H_2(x,y,z) = y, \quad H_3(x,y,z) = -az.$$

Following [9], we obtain the Lie-Trotter integrator (see the Figure 3)

$$\begin{cases} x^{n+1} = e^{[\frac{1}{a}y(0)+z(0)]t}x^n \\ y^{n+1} = e^{[x(0)+az(0)]t}y^n \\ z^{n+1} = e^{[-x(0)+y(0)]t}z^n, \end{cases} \tag{2.3}$$

that has the following properties.

Proposition 2.5. *Lie-Trotter integrator* (2.3) *is a integrator Poisson that does not preserve the Hamiltonian* H *of our system* (1.2) *and it restriction to the regular coadjoint orbits*

$$\{(x,y,z)\in\mathbb{R}^3_+ \mid -a\ln x+\ln y+\ln z = const.\}$$

gives rise to a symplectic integrator.

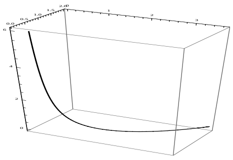

Figure 3: Lie-Trotter integrator for the system (1.2)

Remark 2.2. If we make a comparison with the 4^{th}-step Runge-Kutta method *(see the Figure 4)* we obtain almost the same results. However, Kahan integrator and Lie-Trotter integrator have the advantage to be easier implemented.

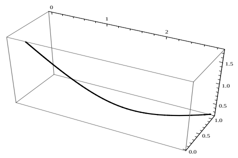

Figure 4: The 4^{th}-step Runge-Kutta for the system (1.2)

3 The Poisson geometry of the system (1.3)

The system (1.3) has the Hamilton-Poisson realization (see [2])

$$(\mathbb{R}^3, \Pi_-, H),$$

where

$$\Pi_- = \begin{pmatrix} 0 & \frac{1}{b}xy & xz \\ -\frac{1}{b}xy & 0 & -yz \\ -xz & yz & 0 \end{pmatrix}$$

is the minus Lie-Poisson structure and the Hamiltonian $H \in C^\infty(\mathbb{R}^3, \mathbb{R})$ is given by

$$H(x, y, z) = -bx + y + z, \quad x, y, z \in \mathbb{R}, \quad b \in \mathbb{R}^*.$$

Also, the function $C : \mathbb{R}^3_+ \to \mathbb{R}$,

$$C(x, y, z) = -b \ln x - b \ln y + \ln z, \quad x, y, z > 0.$$

is a Casimir of our Poisson configuration (see [3]).

Remark 3.1. The phase curves of dynamics (1.3) are the intersections of the surfaces

$$-bx + y + z = const.$$

and

$$-b \ln x - b \ln y + \ln z = const.,$$

see the *Figure 5*.

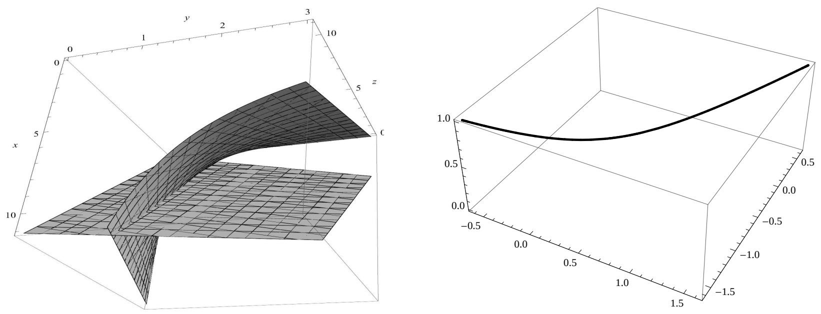

Figure 5: The phase curves of the system (1.3)

The equilibrium points of our dynamics (1.3) are given by $e_1^M = (M, 0, 0)$, $e_2^M = (0, M, 0)$, $e_3^M = (0, 0, M)$ and $e_4^{b,M} = (M, -bM, M)$, $M \in \mathbb{R}$.

Let A be the matrix of linear part of our system (1.3), that is

$$A = \begin{pmatrix} \frac{1}{b}y + z & \frac{1}{b}x & x \\ y & -z + x & -y \\ bz & z & bx + y \end{pmatrix}.$$

If we consider $M > 0$ and $b < 0$, then the characteristic roots of $A(e_1^M)$ [resp. $A(e_2^M)$, resp. $A(e_3^M)$, resp. $A(e_4^{b,M})$] are given by

$$\lambda_1 = 0, \quad \lambda_2 = M, \quad \lambda_3 = bM$$

$$(\text{ resp. } \quad \lambda_1 = 0, \quad \lambda_2 = M, \quad \lambda_3 = \frac{1}{b}M,$$

$$\text{resp. } \quad \lambda_1 = 0, \quad \lambda_{2,3} = \pm M,$$

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so we can conclude that the equilibrium states e_1^M, e_2^M and e_3^M are unstable and $e_4^{b,M}$ are spectrally stable.

Proposition 3.1. *The equilibrium states* $e_4^{b,M}$ *are nonlinearly stable for any* $M > 0$ *and* $b < 0$.

Proof. We shall use the energy-Casimir method, see [1] for details. Let

$$H_\varphi(x, y, z) = -bx + y + z + \varphi(-blnx - blny + lnz),$$

be the energy-Casimir function, where $x, y, z > 0$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is a smooth real valued function defined on $\mathbb{R}$.

Now, the first variation of H_φ is given by

$$\delta H_\varphi(x, y, z) = -b\delta x + \delta y + \delta z + \dot{\varphi}(-blnx - blny + lnz)\left(-\frac{b}{x}\delta x - \frac{b}{y}\delta y + \frac{1}{z}\delta z\right).$$

This equals zero at the equilibrium of interest if and only if

$$\dot{\varphi}[-blnM - bln(-bM) + lnM] = -M. \tag{3.1}$$

The second variation of H_φ is given by

$$\delta^2 H_\varphi(x,y,z) = \ddot{\varphi}(-blnx - blny + lnz)\left(-\frac{b}{x}\delta x - \frac{b}{y}\delta y + \frac{1}{z}\delta z\right)^2 +$$

$$+\dot{\varphi}(-blnx - blny + lnz)\left[\frac{b}{x^2}(\delta x)^2 + \frac{b}{y^2}(\delta y)^2 - \frac{1}{z^2}(\delta z)^2\right]$$

and, taking into account the relation (3.1), we obtain

$$\delta^2 H_\varphi(M, -bM, M) =$$

$$= \ddot{\varphi}[-blnM - bln(-bM) + lnM]\left(-\frac{b}{M}\delta x + \frac{1}{M}\delta y + \frac{1}{M}\delta z\right)^2 +$$

$$+\left[-\frac{b}{M}(\delta x)^2 - \frac{1}{bM}(\delta y)^2 + \frac{1}{M}(\delta z)^2\right].$$

If we choose now φ such that the relation (3.1) is valid and

$$\ddot{\varphi}[-blnM - b\ln(-bM) + lnM] > 0,$$

then the second variation of H_φ at the equilibrium of interest is positive definited and so our equilibrium states are nonlinearly stable. □

As a consequence, we can find the periodic orbits of some equilibrium points $e_4^{b,M}$, $M > 0$ and $b < 0$. More exactly, we have

Proposition 3.2. *Near to* $e_4^{-1,M} = (M, M, M)$, $M > 0$, *the reduced dynamics has, for each sufficiently small value of the reduced energy, at least* 1*-periodic solution whose period is close to*

$$\frac{2\pi}{\sqrt{3}M}.$$

For proof, see [6].

Proposition 3.3. *The dynamics* (1.3) *allows a formulation in terms of Lax pairs.*

Proof. Let us take

$$L = \begin{pmatrix} 0 & \alpha & \beta & 0 & 0 & 0 \\ -\alpha & 0 & \gamma & 0 & 0 & 0 \\ -\beta & -\gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta & \epsilon \\ 0 & 0 & 0 & -\delta & 0 & \varepsilon \\ 0 & 0 & 0 & -\epsilon & -\varepsilon & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & \zeta & \eta & 0 & 0 & 0 \\ -\zeta & 0 & \theta & 0 & 0 & 0 \\ -\eta & -\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \vartheta & \kappa \\ 0 & 0 & 0 & -\vartheta & 0 & \xi \\ 0 & 0 & 0 & -\kappa & -\xi & 0 \end{pmatrix},$$

where

$$\alpha = x - \frac{y}{b} - \frac{z}{b}\ , \quad \beta = bx - y - z\ , \quad \gamma = -ibx + iy + iz\ ,$$

$$\delta = 2\sqrt{2}b(bx - y - z), \quad \epsilon = i(-bx + y - 4bz), \quad \varepsilon = -bx + y - 4bz,$$

$$\zeta = \frac{i}{b}(y + z), \quad \eta = i(y + z), \quad \theta = y + z,$$

$$\vartheta = -i(2y + z), \quad \kappa = \frac{-y + 2bz}{\sqrt{2}b}\ , \quad \xi = \frac{i(y - 2bz)}{\sqrt{2}b}\ ,$$

$i = \sqrt{-1}$. Then, using MATHEMATICA 8.0, we can put the system (1.3) in the equivalent form

$$\dot{L} = [L, B]$$

as desired. □

Let us pass now to the numerical integration of the equations (1.3).

It is easy to see that for the system (1.3), Kahan's integrator can be written in the following form (see [5], for details)

$$\text{(3.2)} \qquad \begin{cases} x^{n+1} - x^n = \frac{h}{2}(\frac{1}{b}x^{n+1}y^n + \frac{1}{b}y^{n+1}x^n + z^{n+1}x^n + x^{n+1}z^n) \\ y^{n+1} - y^n = \frac{h}{2}(x^{n+1}y^n + y^{n+1}x^n - z^{n+1}y^n - y^{n+1}z^n) \\ z^{n+1} - z^n = \frac{h}{2}(bx^{n+1}z^n + bz^{n+1}x^n + z^{n+1}y^n + y^{n+1}z^n). \end{cases}$$

The solutions of this system are presented in the Figure 6.

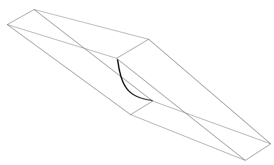

Figure 6: Kahan integrator for the system (1.3)

A long but straightforward computation or using MATHEMATICA 8.0 leads us to

Proposition 3.4. *Kahan integrator* (2.2) *does not preserve the minus Lie-Poisson structure* Π_-, *the Hamiltonian* H *and the Casimir* C *of our Poisson configuration.*

We shall discuss now the numerical integrator of the dynamics (1.3) via the Lie-Trotter integrator, see for details [9]. For the beginning, let us observe that the Hamiltonian vector field X_H splits as follows

$$X_H = X_{H_1} + X_{H_2} + X_{H_3},$$

where

$$H_1(x,y,z) = -bx, \quad H_2(x,y,z) = y, \quad H_3(x,y,z) = z.$$

Following [9], we obtain the Lie-Trotter integrator (see the Figure 7)

$$\left\{ \begin{array}{l} x^{n+1} = e^{[\frac{1}{b}y(0)+z(0)]t}x^n \\ y^{n+1} = e^{[x(0)-z(0)]t}y^n \\ z^{n+1} = e^{[bx(0)+y(0)]t}z^n, \end{array} \right. \tag{3.3}$$

that has the following properties.

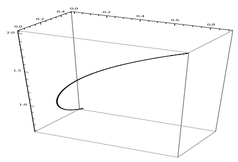

Figure 7: Lie-Trotter integrator for the system (1.3)

Proposition 3.5. *Lie-Trotter integrator* (3.3) *is a integrator Poisson that does not preserve the Hamiltonian H of our system* (1.3) *and it restriction to the regular coadjoint orbits*

$$\{(x,y,z) \in \mathbb{R}^3_+ \mid -b\ln x - b\ln y + \ln z = const.\}$$

gives rise to a symplectic integrator.

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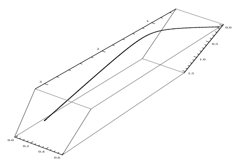

Figure 8: The 4^{th}-step Runge-Kutta for the system (1.3)

4 Conclusion

The paper presents some special Lotka-Volterra systems from the mechanical geometry point of view. The Hamilton-Poisson realization was proved to be important in many cases (see [7] and [8]). In the first part we study for each system the nonlinear stability of their equilibrium points using energy-Casimir method and we find the Lax formulation. In addition we have presented a comparison between three numerical integration methods: 4^{th}-step Runge-Kutta, Lie-Trotter algorithm and Kahan algorithm.

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