On the geometry of a Poisson-Lie algebroid

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Abstract. In this paper we continue the study of the linear contravariant connections on Poisson-Lie algebroids. Some properties of the torsion and curvature tensors are investigated and the equations of the geodesic curves are pointed out. The compatibility conditions between the horizontal lift and the canonical Poisson bivector on the prolongation of a Lie algebroid to its dual bundle are given.

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1 Introduction

The notion of Lie algebroid [11] is a generalization of the concepts of Lie algebra and integrable distribution. The prolongations of a Lie algebroid [6] over the vector bundle projections generalize the concepts of tangent and cotangent bundle. The Poisson manifolds [10], [18] are the smooth manifolds endowed with a Poisson bracket on their ring of functions and the framework of the Poisson geometry is the tangent bundle. The Poisson-Lie algebroids are Lie algebroids equipped with a Poisson structure, which generalize the Poisson geometry. In the last years, diverse aspects of these subjects are inverstigated in a lot of papers (see for instance [2], [3], [4], [5], [13], [16], [17]). In this paper we continue the investigation of the geometry of Poisson-Lie algebroids started in [14], [15].

The paper is organized as follows. In the second section we present the preliminary results on Lie algebroids. In the section three, we introduce the notion of linear contravariant connections on Poisson-Lie algebroids (see also [14]) and some properties of the torsion and curvature tensors are investigated. The Bianchi identities are pointed out and the relation with a covariant connection on Lie algebroids is given. In the last part, the equations of the geodesic curves on Poisson-Lie algebroids are found. In the last section, we consider the prolongation of a Lie algebroid to its dual bundle and study the compatibility conditions of a horizontal lift with the canonical Poisson bivector. The particular case when a nonlinear connection is defined by a linear connection is pointed out.

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2 Lie algebroids

Let M be a differentiable, n-dimensional manifold and (TM, π_M, M) its tangent bundle. A Lie algebroid over the manifold M is the triple $(E, [\cdot,\cdot]_E, \sigma)$, where $\pi : E \to M$ is a vector bundle of rank m over M, whose $C^\infty(M)$-module of sections $\Gamma(E)$ is equipped with a Lie algebra structure $[\cdot,\cdot]_E$ and $\sigma : E \to TM$ is a vector bundle homomorphism (called *the anchor*) which induces a Lie algebra homomorphism (also denoted σ) from $\Gamma(E)$ to $\mathcal{X}(M)$, satisfying the Leibniz rule

$$[s_1, f s_2]_E = f[s_1, s_2]_E + (\sigma(s_1)f)s_2,$$

for every $f \in C^\infty(M)$ and s_1, $s_2 \in \Gamma(E)$. Therefore, we have

$$[\sigma(s_1), \sigma(s_2)] = \sigma[s_1, s_2]_E, \quad [s_1, [s_2, s_3]_E]_E + [s_2, [s_3, s_1]_E]_E + [s_3, [s_1, s_2]_E]_E = 0.$$

If $\omega \in \bigwedge^k(E^*)$ then the *exterior derivative* $d^E\omega \in \bigwedge^{k+1}(E^*)$ is given by the formula

$$\begin{aligned} d^E\omega(s_1, ..., s_{k+1}) \quad &= \quad \sum_{i=1}^{k+1}(-1)^{i+1}\sigma(s_i)\omega(s_1, ..., \hat{s}_i, ..., s_{k+1}) + \\ &\qquad + \sum_{1\le i<j\le k+1} (-1)^{i+j}\omega([s_i, s_j]_E, s_1, ..., \hat{s}_i, ..., \hat{s}_j, ...s_{k+1}). \end{aligned}$$

where $s_i \in \Gamma(E)$, $i = \overline{1, k+1}$, and it results that $(d^E)^2 = 0$. Also, for $\xi \in \Gamma(E)$ one can define the *Lie derivative* with respect to ξ by

$$\mathcal{L}_\xi = i_\xi \circ d^E + d^E \circ i_\xi,$$

where i_ξ is the contraction with ξ.

If we take the local coordinates (x^i) on an open $U \subset M$, a local basis $\{s_\alpha\}$ of the sections of the bundle $\pi^{-1}(U) \to U$ generates the local coordinates (x^i, y^α) on E. The local functions $\sigma^i_\alpha(x)$, $L^\gamma_{\alpha\beta}(x)$ on M are called the *structure functions of the Lie algebroid.* They are given by

$$\sigma(s_\alpha) = \sigma^i_\alpha \frac{\partial}{\partial x^i}, \quad [s_\alpha, s_\beta]_E = L^\gamma_{\alpha\beta} s_\gamma, \quad i = \overline{1, n}, \quad \alpha, \beta, \gamma = \overline{1, m},$$

and satisfy the so called *structure equations* on Lie algebroids

$$\text{(2.1)} \qquad \sigma^j_\alpha \frac{\partial \sigma^i_\beta}{\partial x^j} - \sigma^j_\beta \frac{\partial \sigma^i_\alpha}{\partial x^j} = \sigma^i_\gamma L^\gamma_{\alpha\beta}, \quad \sum_{(\alpha,\beta,\gamma)} \left(\sigma^i_\alpha \frac{\partial L^\delta_{\beta\gamma}}{\partial x^i} + L^\delta_{\alpha\eta} L^\eta_{\beta\gamma} \right) = 0.$$

Locally, if $f \in C^\infty(M)$ then $d^E f = \frac{\partial f}{\partial x^i}\sigma^i_\alpha s^\alpha$ and if $\theta \in \Gamma(E^*)$, $\theta = \theta_\alpha s^\alpha$ then

$$d^E\theta = (\sigma^i_\alpha \frac{\partial \theta_\beta}{\partial x^i} - \frac{1}{2}\theta_\gamma L^\gamma_{\alpha\beta}) s^\alpha \wedge s^\beta,$$

where $\{s^\alpha\}$ is the dual basis of $\{s_\alpha\}$. Particularly

$$d^E x^i = \sigma^i_\alpha s^\alpha, \quad d^E s^\alpha = -\frac{1}{2} L^\alpha_{\beta\gamma} s^\beta \wedge s^\gamma.$$

3 Poisson-Lie algebroids

We recall that the Schouten-Nijenhuis bracket on a Lie algebroid E is given by [16]

$$\begin{aligned}[X_1 \wedge ... \wedge X_p, Y_1 \wedge ... \wedge Y_q] &= (-1)^{p+1} \sum_{i=1}^{p} \sum_{j=1}^{q} (-1)^{i+j} [X_i, Y_j]_E \wedge X_1 \wedge ... \wedge \\ &\qquad \wedge \hat{X}_i \wedge ... \wedge X_p \wedge Y_1 \wedge ... \wedge \hat{Y}_j \wedge ... \wedge Y_q.\end{aligned}$$

Let us consider the bivector $\Pi \in \Gamma(\wedge^2 E)$ (contravariant, skew-symmetric, 2-section) given by

$$\Pi = \frac{1}{2} \pi^{\alpha\beta}(x) s_\alpha \wedge s_\beta. \tag{3.1}$$

Definition 3.1. The bivector Π is a Poisson bivector on E if and only if $[\Pi, \Pi] = 0$, where $[\cdot, \cdot]$ is the Schouten-Nijenhuis bracket.

Locally, the condition $[\Pi, \Pi] = 0$ implies that

$$\sum_{(\alpha,\varepsilon,\delta)} \left(\pi^{\alpha\beta} \sigma^i_\beta \frac{\partial \pi^{\varepsilon\delta}}{\partial x^i} + \pi^{\alpha\beta} \pi^{\gamma\delta} L^\varepsilon_{\beta\gamma} \right) = 0. \tag{3.2}$$

If Π is a Poisson bivector on a Lie algebroid E, then the pair (E, Π) is called the *Poisson-Lie algebroid*, which generalize the Poisson manifolds. The Poisson bracket on E is given by

$$\{f_1, f_2\} = \Pi(d^E f_1, d^E f_2), \quad f_1, f_2 \in C^\infty(E).$$

We also have the bundle map $\pi^\# : E^* \to E$ defined by

$$\pi^\# \rho = i_\rho \Pi, \quad \rho \in \Gamma(E^*).$$

Let us consider the bracket [8]

$$[\rho, \theta]_\pi = \mathcal{L}_{\pi^\# \rho} \theta - \mathcal{L}_{\pi^\# \theta} \rho - d^E(\Pi(\rho, \theta)),$$

where $\mathcal{L}$ is Lie derivative and $\rho, \theta \in \Gamma(E^*)$. With respect to this bracket and the usual Lie bracket on the vector fields $\mathcal{X}(M)$, the map $\widetilde{\sigma} : E^* \to TM$ given by $\widetilde{\sigma} = \sigma \circ \pi^\#$ is a Lie algebra homomorphism

$$\widetilde{\sigma}[\rho, \theta]_\pi = [\widetilde{\sigma}\rho, \widetilde{\sigma}\theta].$$

The bracket $[\cdot, \cdot]_\pi$ satisfies also the Leibniz rule

$$[\rho, f\theta]_\pi = f[\rho, \theta]_\pi + \widetilde{\sigma}(\rho)(f)\theta,$$

and it results that $(E^*, [\cdot, \cdot]_\pi, \widetilde{\sigma})$ is a Lie algebroid. Next, we can define the contravariant exterior differential $d^\pi : \bigwedge^k(E^*) \to \bigwedge^{k+1}(E^*)$ by

$$\begin{aligned} d^\pi \omega(s_1, ..., s_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i+1} \widetilde{\sigma}(s_i) \omega(s_1, ..., \hat{s}_i, ..., s_{k+1}) + \\ &\qquad + \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega([s_i, s_j]_\pi, s_1, ..., \hat{s}_i, ..., \hat{s}_j, ... s_{k+1}).\end{aligned}$$

In fact, is obtained the cohomology of the Lie algebroid E^* with the anchor $\widetilde{\sigma}$ and the bracket $[\cdot,\cdot]_\pi$ which generalize the Poisson cohomology of Lichnerowicz for Poisson manifolds [10].
If $E=\mathfrak{g}$ is a Lie algebra and $\Pi = r \in \wedge^2\mathfrak{g}$, then the condition $[\Pi,\Pi]=0$ become $[r,r]_{\mathfrak{g}}=0$ and is obtained the classical *Yang-Baxter equation* (see [8]).

3.1 Linear contravariant connections on Poisson-Lie algebroids

The particular case of the contravariant connection on a Poisson manifold can be found in [16], [2].

Definition 3.2. If $\rho,\theta\in\Gamma(E^*)$ and $\Phi,\Psi\in\Gamma(E)$ then the linear contravariant connection on a Lie algebroid is an application $D:\Gamma(E^*)\times\Gamma(E)\to\Gamma(E)$ which satisfies the relations:
i) $D_{\rho+\theta}\Phi = D_\rho\Phi + D_\theta\Phi$,
ii) $D_\rho(\Phi+\Psi) = D_\rho\Phi + D_\rho\Psi$,
iii) $D_{f\rho}\Phi = fD_\rho\Phi$,
iv) $D_\rho(f\Phi) = fD_\rho\Phi + \widetilde{\sigma}(\rho)(f)\Phi, \quad f\in C^\infty(M)$.

The contravariant connection induces a contravariant derivative $D_\alpha:\Gamma(E)\to\Gamma(E)$ such that the following equalities are fulfilled

$$D_{f_1\alpha_1+f_2\alpha_2} = f_1D_{\alpha_1} + f_2D_{\alpha_2}, \quad f_i\in C^\infty(M), \quad \alpha_i\in\Gamma(E^*)$$

$$D_\rho(f\theta) = fD_\rho\theta + \widetilde{\sigma}(\rho)(f)\theta, \quad f\in C^\infty(M), \quad \rho,\theta\in\Gamma(E^*)$$

In the case where the contravariant connection D is induced by a covariant connection ∇ on a Lie algebroid E (see [1]) we have $D_\rho = \nabla_{\pi\#\rho}$.

Definition 3.3. The torsion and curvature of the linear contravariant connection D are given by the formulas

$$T(\rho,\theta) = D_\rho\theta - D_\theta\rho - [\rho,\theta]_\pi,$$

$$R(\rho,\theta)\mu = D_\rho D_\theta\mu - D_\theta D_\rho\mu - D_{[\rho,\theta]_\pi}\mu, \quad \rho,\theta,\mu\in\Gamma(E^*).$$

The curvature tensor satisfies the equalities $R(\rho,\theta)=-R(\theta,\rho)$, $R(f\rho,\theta)=fR(\rho,\theta)$ and the Bianchi identities have the following form

$$\sum_{(\rho,\theta,\mu)} (D_\rho R(\theta,\mu) + R(T(\rho,\theta),\mu)) = 0,$$

$$\sum_{(\rho,\theta,\mu)} (R(\rho,\theta)\mu - T(T(\rho,\theta),\mu) - D_\rho T(\theta,\mu)) = 0.$$

In local coordinates we define the Christoffel symbols $\Gamma^{\alpha\beta}_\gamma$ by the following formula $D_{s^\alpha}s^\beta = \Gamma^{\alpha\beta}_\gamma s^\gamma$, and under a change of coordinates

$$\left\{ \begin{array}{ll} x^{i'} = x^{i'}(x^i), & i,i' = \overline{1,n} \quad on \quad M \\ y^{\alpha'} = A^{\alpha'}_\alpha y^\alpha, & \alpha,\alpha' = \overline{1,m} \quad on \quad E, \end{array} \right. \tag{3.3}$$

corresponding to a new base $s^{\alpha'} = A^{\alpha'}_{\alpha} s^{\alpha}$, these symbols transform according to

$$\Gamma^{\alpha'\beta'}_{\gamma'} = A^{\alpha'}_{\alpha} A^{\beta'}_{\beta} A^{\gamma}_{\gamma'} \Gamma^{\alpha\beta}_{\gamma} + A^{\alpha'}_{\alpha} A^{\gamma}_{\gamma'} \sigma^{i}_{\varepsilon} \frac{\partial A^{\beta'}_{\gamma}}{\partial x^{i}} \pi^{\alpha\varepsilon}. \tag{3.4}$$

If we denote $T(s^{\alpha}, s^{\beta}) = T^{\alpha\beta}_{\gamma} s^{\gamma}$ and $R(s^{\alpha}, s^{\beta}) s^{\gamma} = R^{\alpha\beta\gamma}_{\delta} s^{\delta}$ then, under a change of coordinates (3.3), we obtain that

$$T^{\alpha'\beta\prime}_{\gamma'} = A^{\alpha'}_{\alpha} A^{\beta'}_{\beta} A^{\gamma}_{\gamma'} T^{\alpha\beta}_{\gamma}, \qquad R^{\alpha'\beta'\gamma'}_{\delta'} = A^{\alpha'}_{\alpha} A^{\beta'}_{\beta} A^{\gamma'}_{\gamma} A^{\delta}_{\delta'} R^{\alpha\beta\gamma}_{\delta}.$$

Proposition 3.1. *The local components of the torsion and curvature tensors of the linear contravariant connection are given by*

$$T^{\alpha\beta}_{\varepsilon} = \Gamma^{\alpha\beta}_{\varepsilon} - \Gamma^{\beta\alpha}_{\varepsilon} - \pi^{\alpha\gamma} L^{\beta}_{\gamma\varepsilon} + \pi^{\beta\gamma} L^{\alpha}_{\gamma\varepsilon} - \sigma^{i}_{\varepsilon} \frac{\partial \pi^{\alpha\beta}}{\partial x^{i}},$$

$$R^{\alpha\beta\gamma}_{\delta} = \Gamma^{\alpha\varepsilon}_{\delta} \Gamma^{\beta\gamma}_{\varepsilon} - \Gamma^{\beta\varepsilon}_{\delta} \Gamma^{\alpha\gamma}_{\varepsilon} + \pi^{\alpha\varepsilon} \sigma^{i}_{\varepsilon} \frac{\partial \Gamma^{\beta\gamma}_{\delta}}{\partial x^{i}} - \pi^{\beta\varepsilon} \sigma^{i}_{\varepsilon} \frac{\partial \Gamma^{\alpha\gamma}_{\delta}}{\partial x^{i}} + (\pi^{\beta\nu} L^{\alpha}_{\nu\varepsilon} - \pi^{\alpha\nu} L^{\beta}_{\nu\varepsilon} - \sigma^{i}_{\varepsilon} \frac{\partial \pi^{\alpha\beta}}{\partial x^{i}}) \Gamma^{\varepsilon\gamma}_{\delta}.$$

Definition 3.4. A tensor field T on E is called *parallel* if and only if $DT = 0$.

Definition 3.5. A contravariant connection D is called a Poisson connection if the Poisson bivector Π is parallel with respect to D.

Remark 3.6. If the Poisson connection D is induced by a covariant connection ∇ (i.e. $D\Pi = 0$, $D_{\rho} = \nabla_{\pi^{\#}\rho}$, $\pi^{\#} D_{\rho} \phi = \nabla_{\pi^{\#}\rho} \pi^{\#} \phi$) then the torsion and curvature tensors of the both connections are related by the following equalities

$$T^{\nabla}(\pi^{\#}\rho, \pi^{\#}\theta) = \pi^{\#} T^{D}(\rho, \theta), \quad R^{\nabla}(\pi^{\#}\rho, \pi^{\#}\theta) \pi^{\#} \mu = \pi^{\#} R^{D}(\rho, \theta) \mu, \ \forall \rho, \theta, \mu \in \Gamma(E^{*}).$$

Let T be a tensor of type (r, s) with the components $T^{i_1 \dots i_r}_{j_1 \dots j_s}$ and $\theta = \theta_{\alpha} s^{\alpha}$ a section of E^{*}. The local coordinates expression of the contravariant derivative is given by

$$D_{\theta} T = \theta_{\alpha} T^{i_1 \dots i_r}_{j_1 \dots j_s} /^{\alpha} s_{i_1} \otimes \dots \otimes s_{i_r} \otimes s^{j_1} \otimes \dots \otimes s^{j_s},$$

$$T^{i_1 \dots i_r}_{j_1 \dots j_s} /^{\alpha} = \pi^{\alpha\varepsilon} \sigma^{i}_{\varepsilon} \frac{\partial T^{i_1 \dots i_r}_{j_1 \dots j_s}}{\partial x^{i}} + \sum_{a=1}^{r} \left(\Gamma^{i_a \alpha}_{\varepsilon} T^{i_1 \dots \varepsilon \dots i_r}_{j_1 \dots j_s} \right) - \sum_{b=1}^{s} \left(\Gamma^{\varepsilon\alpha}_{j_b} T^{i_1 \dots i_r}_{j_1 \dots \varepsilon \dots j_s} \right),$$

and $/$ denote the *contravariat derivative operator.*

Proposition 3.2. *a) The functions*

$$\Gamma^{\alpha\beta}_{\gamma} = \sigma^{i}_{\gamma} \frac{\partial \pi^{\alpha\beta}}{\partial x^{i}}, \tag{3.5}$$

are the coefficients of a contravariant connection.
b) The contravariant connection with the coefficients given by (3.5) is a Poisson connection if and only if

$$\sum_{(\alpha,\varepsilon,\delta)} \pi^{\alpha\beta} \pi^{\gamma\delta} L^{\varepsilon}_{\beta\gamma} = 0.$$

Proof. a) Using the change of coordinates (3.3) and the fact that the structure function σ_α^i change by the rule [12]

$$\sigma_{\alpha'}^{i'} A_\alpha^{\alpha'} = \frac{\partial x^{i'}}{\partial x^i} \sigma_\alpha^i,$$

we obtain that the coefficients (3.5) satisfy the transformation law (3.4).
b) Using the equality (3.5) we obtain that

$$\pi^{\beta\gamma}/^{\alpha} = \pi^{\alpha\varepsilon}\sigma_\varepsilon^i \frac{\partial \pi^{\beta\gamma}}{\partial x^i} - \Gamma_\varepsilon^{\beta\alpha}\pi^{\varepsilon\gamma} - \Gamma_\varepsilon^{\gamma\alpha}\pi^{\beta\varepsilon} = \sum_{(\alpha,\beta,\gamma)} \pi^{\alpha\varepsilon}\sigma_\varepsilon^i \frac{\partial \pi^{\beta\gamma}}{\partial x^i}.$$

From the condition $[\Pi, \Pi] = 0$, locally given by the equation (3.2), it results that $\pi^{\beta\gamma}/^{\alpha} = 0$ if and only if the required relation is fulfilled. □

The image of the anchor map $\sigma(E) \subseteq TM$ defines an integrable smooth distribution on M. Therefore, the manifold M is foliated by the integral leaves of $\sigma(E)$, which are called the leaves of the Lie algebroid. A curve $u : [t_0, t_1] \to E$ is called admissible if $\sigma(u(t)) = \dot{c}(t)$, where $c(t) = \pi(u(t))$ is the base curve on M. It follows that $u(t)$ is admissible if and only if the base curve $c(t)$ lies on a leaf of the Lie algebroid whereas two points can be joint by an admissible curve if and only if they are situated on the same leaf. We can choose a smooth family $t \to \theta(t) \in E^*$ of 1-form such that $\widetilde{\sigma}\theta(t) = \dot{c}(t)$. We shall call the pair $(u(t), \theta(t))$ the *dual curve.*

Definition 3.7. Let $(u(t), \theta(t))$ a *dual curve* on E. We say that $(u(t), \theta(t))$ is a geodesic if

$$(D_\theta \theta)_{u(t)} = 0.$$

In local coordinates, a dual curve $(u(t), \theta(t)) = (x^1(t), ..., x^n(t), \theta_1(t), ..., \theta_m(t))$ is geodesic if and only if it satisfies the following system

$$\begin{cases} \frac{dx^i(t)}{dt} = \sigma_\gamma^i \pi^{\alpha\gamma}(x^1(t), ..., x^n(t))\theta_\alpha(t) \\ \\ \frac{d\theta_\alpha(t)}{dt} = -\Gamma_\alpha^{\beta\gamma}(x^1(t), ..., x^n(t))\theta_\beta\theta_\gamma \end{cases}$$

4 The prolongation of a Lie algebroid to its dual bundle

Let $\tau : E^* \to M$ be the dual bundle of $\pi : E \to M$ and $(E, [\cdot, \cdot]_E, \sigma)$ a Lie algebroid structure over M. One can construct a Lie algebroid structure over E^*, by taking the prolongation over $\tau : E^* \to M$ (see [6], [9], [7]). The associated vector bundle is $(\mathcal{T}E^*, \tau_1, E^*)$ where $\mathcal{T}E^* = \underset{u^* \in E^*}{\cup} \mathcal{T}_{u^*}E^*$,

$$\mathcal{T}_{u^*}E^* = \{(u_x, v_{u^*}) \in E_x \times T_{u^*}E^* | \sigma(u_x) = T_{u^*}\tau(v_{u^*}), \tau(u^*) = x \in M\}$$

and the projection $\tau_1 : \mathcal{T}E^* \to E^*$, $\tau_1(u_x, v_{u^*}) = u^*$. The anchor is the projection $\sigma^1 : \mathcal{T}E^* \to TE^*$, $\sigma^1(u, v) = v$. Notice that if $\mathcal{T}\tau : \mathcal{T}E^* \to E$, $\mathcal{T}\tau(u, v) = u$ then $(V\mathcal{T}E^*, \tau_{1|V\mathcal{T}E^*}, E^*)$ with $V\mathcal{T}E^* = Ker\mathcal{T}\tau$ is a subbundle of $(\mathcal{T}E^*, \tau_1, E^*)$, called the *vertical subbundle.* If (q^i, μ_α) are local coordinates on E^* at u^* and $\{s_\alpha\}$ is a local basis of sections of $\pi : E \to M$, then a local basis of $\Gamma(\mathcal{T}E^*)$ is $\{\mathcal{Q}_\alpha, \mathcal{P}^\alpha\}$ where [9]

$$\mathcal{Q}_\alpha(u^*) = \left(s_\alpha(\tau(u^*)), \sigma^i_\alpha \frac{\partial}{\partial q^i}|_{u^*}\right), \quad \mathcal{P}^\alpha(u^*) = \left(0, \frac{\partial}{\partial \mu_\alpha}|_{u^*}\right).$$

The structure functions on $\mathcal{T}E^*$ are given by the following formulas

$$\sigma^1(\mathcal{Q}_\alpha) = \sigma^i_\alpha \frac{\partial}{\partial q^i}, \quad \sigma^1(\mathcal{P}^\alpha) = \frac{\partial}{\partial \mu_\alpha},$$

$$[\mathcal{Q}_\alpha, \mathcal{Q}_\beta]_{\mathcal{T}E^*} = L^\gamma_{\alpha\beta}\mathcal{Q}_\gamma, \quad [\mathcal{Q}_\alpha, \mathcal{P}^\alpha]_{\mathcal{T}E^*} = 0, \quad [\mathcal{P}^\alpha, \mathcal{P}^\beta]_{\mathcal{T}E^*} = 0,$$

$$d^E\mathcal{Q}^\gamma = -\frac{1}{2}L^\gamma_{\alpha\beta}\mathcal{Q}^\alpha \wedge \mathcal{Q}^\beta, \quad d^E\mathcal{P}_\alpha = 0, \quad d^E q^i = \sigma^i_\alpha \mathcal{Q}^\alpha, \quad d^E\mu_\alpha = \mathcal{P}_\alpha,$$

where $\{\mathcal{Q}^\alpha, \mathcal{P}_\alpha\}$ is the dual basis of $\{\mathcal{Q}_\alpha, \mathcal{P}^\alpha\}$. In local coordinates the *Liouville section* is given by $\theta_E = \mu_\alpha \mathcal{Q}^\alpha$. The *canonical symplectic structure* ω_E is defined by $\omega_E = -d^E\theta_E$. It follows that ω_E is a non degenerate 2-section, $d^E\omega_E = 0$ and

$$\omega_E = \mathcal{Q}^\alpha \wedge \mathcal{P}_\alpha + \frac{1}{2}\mu_\alpha L^\alpha_{\beta\gamma}\mathcal{Q}^\beta \wedge \mathcal{Q}^\gamma. \tag{4.1}$$

A nonlinear connection on $\mathcal{T}E^*$ is an almost product structure $\mathcal{N}$ on $\tau_1 : \mathcal{T}E^* \to E^*$ (i.e. a bundle morphism $\mathcal{N} : \mathcal{T}E^* \to \mathcal{T}E^*$, such that $\mathcal{N}^2 = id$) smooth on $\mathcal{T}E^*\backslash\{0\}$ such that $V\mathcal{T}E^* = \ker(id + \mathcal{N})$. If $\mathcal{N}$ is a connection on $\mathcal{T}E^*$ then $H\mathcal{T}E^* = \ker(id - \mathcal{N})$ is the horizontal distribution associated to $\mathcal{N}$ and $\mathcal{T}E^* = V\mathcal{T}E^* \oplus H\mathcal{T}E^*$. A connection $\mathcal{N}$ on $\mathcal{T}E^*$ induces two projectors $\mathrm{h}, \mathrm{v} : \mathcal{T}E^* \to \mathcal{T}E^*$ such that $\mathrm{h}(\rho) = \rho^{\mathrm{h}}$ and $\mathrm{v}(\rho) = \rho^{\mathrm{v}}$ for every $\rho \in \Gamma(\mathcal{T}E^*)$, given by $\mathrm{h} = \frac{1}{2}(id + \mathcal{N})$ and $\mathrm{v} = \frac{1}{2}(id - \mathcal{N})$. The local sections $\{\mathcal{P}^\alpha\}_{\alpha=\overline{1,m}}$ define a local frame of $V\mathcal{T}E^*$ and the sections $\delta^*_\alpha = (\mathcal{Q}_\alpha)^{\mathrm{h}} = \mathcal{Q}_\alpha + \mathcal{N}_{\alpha\beta}\mathcal{P}^\beta$ generate a local frame of $H\mathcal{T}E^*$. The frame $\{\delta^*_\alpha, \mathcal{P}^\alpha\}$ is a local basis of $\mathcal{T}E^*$ called *Berwald basis.* The dual basis is $\{\mathcal{Q}^\alpha, \delta\mathcal{P}_\alpha\}$ where $\delta\mathcal{P}_\alpha = \mathcal{P}_\alpha - \mathcal{N}_{\alpha\beta}\mathcal{Q}^\beta$. A connection $\mathcal{N}$ is called symmetric if $\omega_E(\mathrm{h}\rho_1, \mathrm{h}\rho_2) = 0$, $\forall \rho_1, \rho_2 \in \Gamma(\mathcal{T}E^*)$ and it follows that $\mathcal{N}$ is symmetric if and only if

$$\mathcal{N}_{\alpha\beta} - \mathcal{N}_{\beta\alpha} = \mu_\gamma L^\gamma_{\alpha\beta}. \tag{4.2}$$

The Lie brackets of Berwald basis $\{\delta^*_\alpha, \mathcal{P}^\alpha\}$ are [7]

$$[\delta^*_\alpha, \delta^*_\beta]_{\mathcal{T}E^*} = L^\gamma_{\alpha\beta}\delta^*_\gamma + \mathcal{R}_{\alpha\beta\gamma}\mathcal{P}^\gamma, \quad [\delta^*_\alpha, \mathcal{P}^\beta]_{\mathcal{T}E^*} = -\frac{\partial \mathcal{N}_{\alpha\gamma}}{\partial \mu_\beta}\mathcal{P}^\gamma, \quad [\mathcal{P}^\alpha, \mathcal{P}^\beta]_{\mathcal{T}E^*} = 0,$$

$$\mathcal{R}_{\alpha\beta\gamma} = \delta^*_\alpha(\mathcal{N}_{\beta\gamma}) - \delta^*_\beta(\mathcal{N}_{\alpha\gamma}) - L^\varepsilon_{\alpha\beta}\mathcal{N}_{\varepsilon\gamma}. \tag{4.3}$$

The curvature of a connection $\mathcal{N}$ on $\mathcal{T}E^*$ is given by $\mathcal{R} = -\mathrm{N}_{\mathrm{h}}$, where h is horizontal projector and N_{h} is the Nijenhuis tensor of h. In local coordinates we get

$$\mathcal{R} = -\frac{1}{2}\mathcal{R}_{\alpha\beta\gamma}\mathcal{Q}^\alpha \wedge \mathcal{Q}^\beta \otimes \mathcal{P}^\gamma,$$

where $\mathcal{R}_{\alpha\beta\gamma}$ is given by (4.3) and is called the *curvature tensor* of $\mathcal{N}$. The curvature is an obstruction to the integrability of $H\mathcal{T}E^*$, understanding that a vanishing curvature entails that horizontal sections are closed under the Lie algebroid bracket of $\mathcal{T}E^*$. On the Lie algebroid $(\mathcal{T}E^*, [\cdot,\cdot]_{\mathcal{T}E^*}, \sigma^1)$ we have the canonical symplectic section ω_E given by (4.1) which induces a vector bundle isomorphism

$$\natural_{\omega_E} : E^* \to E, \quad i_\zeta \omega_E \in E^* \to \zeta \in E.$$

Definition 4.1. The canonical Poisson bivector is given by $\Lambda = \natural_{\omega_E} \omega_E$.

It follows that

$$\Lambda(dF, dG) = -\omega_E(\natural(dF), \natural(dG)), \quad F, G \in C^\infty(E^*),$$

and in local coordinates we get

$$\Lambda = \mathcal{P}^\alpha \wedge \mathcal{Q}_\alpha + \frac{1}{2}\mu_\alpha L^\alpha_{\beta\gamma} \mathcal{P}^\beta \wedge \mathcal{P}^\gamma.$$

Remark 4.2. The Schouten-Nijenhuis bracket $[\Lambda, \Lambda]$ leads, locally, to the expression

$$[\Lambda, \Lambda] = \frac{1}{3} \sum_{(\alpha,\beta,\gamma)} \left(\sigma^i_\alpha \frac{\partial L^\varepsilon_{\beta\gamma}}{\partial x^i} + L^\varepsilon_{\alpha\delta} L^\delta_{\beta\gamma} \right) \mu_\varepsilon \mathcal{P}^\beta \wedge \mathcal{P}^\alpha \wedge \mathcal{P}^\gamma$$

and using the structure equations on the Lie algebroid (2.1) it result $[\Lambda, \Lambda] = 0$.

Definition 4.3. Let us consider a Poisson bivector on E given by (3.1), then the horizontal lift of Π to $\mathcal{T}E^*$ is the bivector defined by

$$\Pi^H = \frac{1}{2} \pi^{\alpha\beta}(x) \delta^*_\alpha \wedge \delta^*_\beta.$$

Proposition 4.1. *The horizontal lift* Π^H *is a Poisson bivector if and only if* Π *is a Poisson bivector on* E *and*

$$\mathcal{R}\left((\pi^\# \rho)^h, (\pi^\# \theta)^h \right) = 0, \quad \forall \rho, \theta \in \Gamma(E^*) \tag{4.4}$$

Proof. The Poisson condition $[\Pi, \Pi] = 0$ leads to the equation

$$\sum_{(\alpha,\varepsilon,\delta)} \left(\pi^{\alpha\beta} \pi^{\gamma\delta} L^\varepsilon_{\beta\gamma} + \pi^{\alpha\beta} \sigma^i_\beta \frac{\partial \pi^{\varepsilon\delta}}{\partial x^i} \right) = 0,$$

and $[\Pi^H, \Pi^H] = 0$ yields

$$\sum_{(\varepsilon,\delta,\alpha)} \left(\pi^{\alpha\beta} \pi^{\gamma\delta} L^\varepsilon_{\beta\gamma} + \pi^{\alpha\beta} \sigma^i_\beta \frac{\partial \pi^{\varepsilon\delta}}{\partial x^i} \right) \delta^*_\varepsilon \wedge \delta^*_\alpha \wedge \delta^*_\delta + \pi^{\alpha\beta} \pi^{\gamma\delta} \mathcal{R}_{\beta\gamma\varepsilon} \mathcal{P}^\varepsilon \wedge \delta^*_\alpha \wedge \delta^*_\gamma = 0,$$

and it results $\pi^{\alpha\beta} \pi^{\gamma\delta} \mathcal{R}_{\beta\gamma\varepsilon} = 0$, which is the locally expression of (4.4). □

Proposition 4.2. *If the connection* $\mathcal{N}$ *on* $\mathcal{T}E^*$ *is defined by a linear connection* ∇ *with the coefficients* $\Gamma^\gamma_{\alpha\beta}$ *on the Lie algebroid* E *then the bivector* Π^H *has the following form*

$$\Pi^H = \frac{1}{2} \pi^{\alpha\beta} \mathcal{Q}_\alpha \wedge \mathcal{Q}_\beta + \frac{1}{2} \pi^{\alpha\beta} \mu_\gamma \mu_\theta \Gamma^\gamma_{\alpha\varepsilon} \Gamma^\theta_{\beta\delta} \mathcal{P}^\varepsilon \wedge \mathcal{P}^\delta + \pi^{\alpha\beta} \mu_\gamma \Gamma^\gamma_{\beta\varepsilon} \mathcal{Q}_\alpha \wedge \mathcal{P}^\varepsilon. \tag{4.5}$$

Proof. The coefficients of the nonlinear connection are $\mathcal{N}_{\alpha\beta} = \mu_\gamma \Gamma^\gamma_{\alpha\beta}$ and introducing the relation $\delta^*_\alpha = \mathcal{Q}_\alpha + \mathcal{N}_{\alpha\beta}\mathcal{P}^\beta$ into the expression of Π^H it results (4.5). □

Proposition 4.3. *If $\mathcal{N}$ is a symmetric nonlinear connection then the canonical Poisson bivector has the form*

$$\Lambda = \mathcal{P}^\alpha \wedge \delta^*_\alpha.$$

Proof. By direct computations we obtain

$$\begin{aligned}
\Lambda &= \mathcal{P}^\alpha \wedge \mathcal{Q}_\alpha + \tfrac{1}{2}\mu_\alpha L^\alpha_{\beta\gamma}\mathcal{P}^\beta \wedge \mathcal{P}^\gamma = \mathcal{P}^\alpha \wedge (\delta^*_\alpha - \mathcal{N}_{\alpha\beta}\mathcal{P}^\beta) + \tfrac{1}{2}\mu_\gamma L^\gamma_{\beta\alpha}\mathcal{P}^\beta \wedge \mathcal{P}^\alpha \\
&= \mathcal{P}^\alpha \wedge \delta^*_\alpha - \tfrac{1}{2}\mathcal{N}_{\alpha\beta}\mathcal{P}^\alpha \wedge \mathcal{P}^\beta - \tfrac{1}{2}\left(\mathcal{N}_{\alpha\beta} + \mu_\gamma L^\gamma_{\beta\alpha}\right)\mathcal{P}^\alpha \wedge \mathcal{P}^\beta \\
&= \mathcal{P}^\alpha \wedge \delta^*_\alpha - \tfrac{1}{2}\mathcal{N}_{\alpha\beta}\mathcal{P}^\alpha \wedge \mathcal{P}^\beta - \tfrac{1}{2}\mathcal{N}_{\beta\alpha}\mathcal{P}^\alpha \wedge \mathcal{P}^\beta \\
&= \mathcal{P}^\alpha \wedge \delta^*_\alpha - \tfrac{1}{2}\mathcal{N}_{\alpha\beta}\mathcal{P}^\alpha \wedge \mathcal{P}^\beta - \tfrac{1}{2}\mathcal{N}_{\alpha\beta}\mathcal{P}^\beta \wedge \mathcal{P}^\alpha \\
&= \mathcal{P}^\alpha \wedge \delta^*_\alpha - \tfrac{1}{2}\mathcal{N}_{\alpha\beta}\mathcal{P}^\alpha \wedge \mathcal{P}^\beta + \tfrac{1}{2}\mathcal{N}_{\alpha\beta}\mathcal{P}^\alpha \wedge \mathcal{P}^\beta = \mathcal{P}^\alpha \wedge \delta^*_\alpha.
\end{aligned}$$

□

Recall that two Poisson structures are said to be *compatible* if the bivectors Π_1 and Π_2 satisfy the condition

$$[\Pi_1, \Pi_2] = 0.$$

Proposition 4.4. *If Π^H is a Poisson bivector and $\mathcal{N}$ is a symmetric nonlinear connection, then Π^H is compatible with the canonical Poisson structure Λ if and only if the following relations fulfilled*

$$\sigma^i_\gamma \frac{\partial \pi^{\alpha\beta}}{\partial x^i} + \pi^{\varepsilon\alpha}\left(\frac{\partial \mathcal{N}_{\varepsilon\gamma}}{\partial \mu_\beta} - L^\beta_{\varepsilon\gamma}\right) - \pi^{\varepsilon\beta}\left(\frac{\partial \mathcal{N}_{\varepsilon\gamma}}{\partial \mu_\alpha} - L^\alpha_{\varepsilon\gamma}\right) = 0, \tag{4.6}$$

$$\pi^{\alpha\beta}\mathcal{R}_{\alpha\gamma\varepsilon} = 0. \tag{4.7}$$

Proof. If $\mathcal{N}$ is symmetric then $\mathcal{N}_{\alpha\beta} - \mathcal{N}_{\beta\alpha} = \mu_\gamma L^\gamma_{\alpha\beta}$ and with respect with the basis $\{\delta^*_\alpha, \mathcal{P}^\alpha\}$ it results $\Lambda = \mathcal{P}^\alpha \wedge \delta^*_\alpha$. By a straightforward computation we obtain

$$\begin{aligned}
[\Pi^H, \Lambda] &= -\frac{1}{2}\left(\sigma^i_\gamma \frac{\partial \pi^{\alpha\beta}}{\partial x^i} + \pi^{\varepsilon\alpha}\left(\frac{\partial \mathcal{N}_{\varepsilon\gamma}}{\partial \mu_\beta} - L^\beta_{\varepsilon\gamma}\right) - \pi^{\varepsilon\beta}\left(\frac{\partial \mathcal{N}_{\varepsilon\gamma}}{\partial \mu_\alpha} - L^\alpha_{\varepsilon\gamma}\right)\right)\delta^*_\alpha \wedge \delta^*_\beta \wedge \mathcal{P}^\gamma \\
&\quad + \pi^{\alpha\beta}\mathcal{R}_{\alpha\gamma\varepsilon}\mathcal{P}^\varepsilon \wedge \delta^*_\beta \wedge \mathcal{P}^\gamma.
\end{aligned}$$

and $[\Pi^H, \Lambda] = 0$ is equivalent with the relations (4.6), (4.7). □

Remark 4.4. If the nonlinear connection $\mathcal{N}$ is defined by a linear connection ∇ with the coefficients $\Gamma^\gamma_{\alpha\beta}$ on the Lie algebroid E, then the equations (4.6), (4.7) have the form

$$\sigma^i_\gamma \frac{\partial \pi^{\alpha\beta}}{\partial x^i} + \pi^{\varepsilon\alpha}\left(\Gamma^\beta_{\varepsilon\gamma} - L^\beta_{\varepsilon\gamma}\right) - \pi^{\varepsilon\beta}\left(\Gamma^\alpha_{\varepsilon\gamma} - L^\alpha_{\varepsilon\gamma}\right) = 0,$$

$$\pi^{\alpha\beta}\mu_\varepsilon\left(\sigma^i_\alpha \frac{\partial \Gamma^\varepsilon_{\beta\gamma}}{\partial q^i} - \sigma^i_\beta \frac{\partial \Gamma^\varepsilon_{\alpha\gamma}}{\partial q^i} + \Gamma^\varepsilon_{\alpha\theta}\Gamma^\theta_{\beta\gamma} - \Gamma^\varepsilon_{\beta\theta}\Gamma^\theta_{\alpha\gamma} - L^\theta_{\alpha\beta}\Gamma^\varepsilon_{\theta\gamma}\right) = 0.$$

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References

[1] J. Cortes, E. Martinez, *Mechanical control systems on Lie algebroids*, IMA J. Math. Control Inf. 21, no. 4, (2004) 457–492.

[2] R. Fernandes, *Connections in Poisson geometry I: holonomy and invariants*, J. Differ. Geom. 54 (2000) 303-365.

[3] R. Fernandes, *Lie algebroids, holonomy and characteristic classes*, Adv. Math. 170 (2002) 119-179.

[4] D. Fish, S. Preston, *On the integrability of orthogonal distributions in Poisson manifolds*, Balkan J. Geom. Appl. 15, no. 2 (2010) 57-68.

[5] J. Grabowski, P. Urbanski, *Lie algebroids and Poisson-Nijenhuis structures*, Rep. Math. Phys. 40 (1997) 195-208.

[6] P. Higgins, K. Mackenzie, *Algebraic constructions in the category of Lie algebroids*, J. Algebra, 129 (1990) 194-230.

[7] D. Hrimiuc, L. Popescu, *Nonlinear connections on dual Lie algebroids*, Balkan J. Geom. Appl. 11 no.1 (2006) 73-80.

[8] Y. Kosmann-Schwarzbach, *Poisson manifolds, Lie algebroids, modular classes: a survey*, SIGMA, Symmetry, Integrability Geom. Methods Appl. 4 (2008) 005.

[9] M. de Leon, J. Marrero, E. Martinez, *Lagrangian submanifolds and dynamics on Lie algebroids*, J. Phys. A: Math. Gen. 38 (2005) 241-308.

[10] A. Lichnerowicz, *Les variétés de Poisson et leurs algèbres de Lie associées*, J. Differ. Geom. 12 (1977) 253-300.

[11] K. Mackenzie, *Lie groupoids and Lie algebroids in differential geometry*, London Math. Society Lecture Note Series, Cambridge 124, 1987.

[12] E. Martinez, *Lagrangian mechanics on Lie algebroids*, Acta Appl. Math. 67 (2001) 295-320.

[13] G. Mitric, *Poisson structures on cotangent bundles*, IJMMS 29 (2003) 1833-1855.

[14] L. Popescu, *A note on Poisson-Lie algebroids (I)*, Balkan J. Geom. Appl. 14, no. 2 (2009) 79-89.

[15] L. Popescu, *A note on Poisson-Lie algebroids,* J. Geom. Symmetry Phys. 12 (2008) 63-73.

[16] I. Vaisman, *Lectures on the geometry of Poisson manifolds*, Progress in Math. 118, Birkhäuser, Berlin, 1994.

[17] I. Vaisman, *Complementary 2-forms of Poisson structures*, Compos. Math. 101, no.1 (1996) 55-75.

[18] A. Weinstein A., *The local structure of Poisson manifolds*, J. Differ. Geom. 18 (1983) 523-557.

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