Generalized subbundles of vector bundles

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Abstract. The aim of this paper is to construct new examples concerning operations and specific properties of generalized smooth vector subbundles, many having different properties compared with the regular classical case. We show that using morphisms of vector bundles, smooth or cosmooth generalized vector subbundles can be obtained.

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1 Basic facts and examples

Some facts presented in this paper are largely discussed under various differentiability conditions in [1] and [4]. The various examples are intended to familiarize the reader with the specific behavior of generalized smooth vector subbundles (according to [2]).

According to [2], a *generalized vector subbundle* $\mathcal{D}$ of a vector bundle $E \to M$ is the assignment of a subspace $\mathcal{D}_x \subset E_x$ to every $x \in M$. We say that a vector $X_x \in \mathcal{D}_x$ is *allowed* if there is a smooth section Y of $\mathcal{D}_{|U_x}$ on an open neighborhood U_x of x, such that $Y_x = X_x$. We denote by $\mathcal{A}(\mathcal{D})_x \subset \mathcal{D}_x$ the set of allowed vectors in x. The null vector $\bar{0}_x \in E_x$ is obviously allowed since $\bar{0}_x \in \mathcal{A}(\mathcal{D})_x$, thus $\mathcal{A}(\mathcal{D})_x$ is non-void. It is easy to see that $\mathcal{A}(\mathcal{D}) = \underset{x\in M}{\cup} \mathcal{A}(\mathcal{D})_x$ is a generalized vector subbundle. According to [2] we say that $\mathcal{D}$ is *smooth* if $\mathcal{A}(\mathcal{D}) = \mathcal{D}$. According to its construction, $\mathcal{A}(\mathcal{D})$ is smooth.

The *orthogonal* of a vector subbundle $\mathcal{D}$ on E is the vector subbundle $\mathcal{D}^\perp \subset E^*$, where in every $x \in M$,

$$\mathcal{D}_x^\perp = \{\omega_x \in E_x^* : \omega(X_x) = 0, (\forall) X_x \in E_x\}.$$

According to the canonical isomorphism $E \cong E^{**}$, the orthogonal $\mathcal{E}^\perp$ of a vector subbundle $\mathcal{E}$ in E^* lies in E. A generalized vector subbundle $\mathcal{D}$ is *cosmooth* if $\mathcal{D}^\perp$ is smooth.

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Example 1.1. Let us consider the *smooth vector subbundle* $\mathcal{C}_r$ *on* $\mathbb{R}$ *generated by the vector field* $X_0 = \varphi_r \frac{d}{dt}$, where $r \geq 0$ and

$$\varphi_r(t) = \begin{cases} e^{-\frac{1}{(t-r)^2}} \text{ if } |t| > r, \\ 0 \text{ if } |t| \leq r \end{cases} . \tag{1.1}$$

Then $\left(\mathcal{C}_r^{\perp}\right)_t$ *is isomorphic with* $\mathbb{R}$ *for* $|t| \leq r$ *and is null for* $|t| > r$*. It is easy to see that* $C_r^{\perp}$ *is not smooth.*

The following simple result is useful in the following.

Lemma 1.1. *Let* $a > 0$ *and a smooth function* $f : (-a, a) \to \mathbb{R}$*,* $f(0) = 0$*. Then the function*

$$F(t) = \begin{cases} \frac{f(t)}{t} \text{ if } |t| > 0, \\ 0 \text{ if } t = 0 \end{cases} \tag{1.2}$$

is smooth on $(-a, a)$*.*

Example 1.2. *Let* $\mathcal{C}$ *be the smooth vector subbundle on* $\mathbb{R}^2$ *generated by the vector field* $C_{(x,y)} = y\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial y}$*. The generalized vector subbundle* $\mathcal{C}'$ *given by the orthogonal Euclidean orthogonal* $C'_{(x,y)} = (C_{(x,y)})^{\perp}$*,* $(\forall)(x, y) \in \mathbb{R}^2$ *is isomorphic to the cosmooth vector subbundle* $\mathcal{C}^{\perp} \subset T^*M$*. It is easy to see that* $\mathcal{C}'$*, thus also* $\mathcal{C}^{\perp}$*, are not smooth. Indeed, no non-null vector* $X_x \in T_{\bar{0}}\mathbb{R}^2 = C'_{\bar{0}}$ *can be extended to a vector field* X *tangent to* $\mathcal{C}$ *in a neighborhood of* $\bar{0}$*. The generalized vector subbundle* $\mathcal{A}(\mathcal{C}')$ *is* $\mathcal{C}^{\perp}_{(x,y)}$ *for* $(x, y) \neq \bar{0}$*, but* $\mathcal{A}(\mathcal{C}')_{\bar{0}=(0,0)} = \{\bar{0}\}$*. Using Lemma 1.1, it can be easily proved that the* $\mathcal{F}(\mathbb{R}^2)$ *modules* $\Gamma(\mathcal{C})$ *and* $\Gamma(\mathcal{A}(\mathcal{C}'))$ *are generated by* C *and the position vector field* $C'_{(x,y)} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ *respectively.*

In general, $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are simultaneously smooth generalized vector subbundle iff $\mathcal{D}$ has a constant rank. Indeed, considering the dimension function (of the fibers), this function is lower semi-continuos for a smooth vector subbundle and upper semi-continuos for a cosmooth one; thus it is continuos, thus constant on every connect component, iff the rank of $\mathcal{D}$ is constant.

Example 1.3. *For an* $r \geq 0$*, let us consider the smooth vector subbundle* $\mathcal{C}_{(r)}$ *on* $\mathbb{R}^2$ *generated by the vector field* $\mathcal{C}_{(r)(x,y)} = \varphi_r(x^2 + y^2)C_{(x,y)}$, where C is the vector field considered in Example 1.2 and φ_r is given by (1.1)*. Then* $\left(\mathcal{C}^{\perp}_{(r)}\right)_{(x,y)}$ *is isomorphic to* $\mathbb{R}^2$ *for* $\left|x^2 + y^2\right| \leq r$ *and is generated by* $\mathbb{C}_{(x,y)}$ *for* $\left|x^2 + y^2\right| > r$*. If* $r = 0$ we obtain Example 1.2.

Using the construction of $\mathcal{A}(\mathcal{D})$, the following property follows.

Proposition 1.2. *If* $\mathcal{D}_1 \subset \mathcal{D}_2$*, then* $\mathcal{A}(\mathcal{D}_1) \subset \mathcal{A}(\mathcal{D}_2)$*. If* $\mathcal{D}_1$ *or* $\mathcal{D}_2$ *are smooth, then* $\mathcal{D}_1 \subset \mathcal{A}(\mathcal{D}_2)$ *or* $\mathcal{A}(\mathcal{D}_1) \subset \mathcal{D}_2$ *respectively.*

Let $\mathcal{D} \subset E$ be a smooth generalized vector subbundle. Then $\mathcal{D}^{\perp} \subset E^*$ is cosmooth and $\mathcal{A}(\mathcal{D}^{\perp}) \subset E^*$ is smooth. We can consider the cosmooth vector subbundle $\mathcal{B}(\mathcal{D}) = \mathcal{A}(\mathcal{D}^{\perp})^{\perp} \subset E$; we say that $\mathcal{B}(\mathcal{D})$ is the cosmooth vector subbundle *generated* by the smooth vector subbundle $\mathcal{D}$.

Proposition 1.3. *We have* $(\mathcal{D}^{\perp})^{\perp} = \mathcal{D}$ *and* $\mathcal{D} \subset \mathcal{B}(\mathcal{D}) = \mathcal{A}(\mathcal{D}^{\perp})^{\perp}$.

Proposition 1.4. *Let* $\mathcal{D} \subset E$ *be a cosmooth generalized vector subbundle, i.e.* $\mathcal{D}^{\perp} \subset E^*$ *is smooth. We have* $\mathcal{A}(\mathcal{A}(\mathcal{D})^{\perp})^{\perp} \subset \mathcal{D}$.

Let $\Gamma_0 \subset \Gamma(E)$ be an $\mathcal{F}(M)$–submodule. For every $x \in M$ we denote by $\mathcal{D}(\Gamma_0)_x$ the linear span of $\{s_x | \ s \in \Gamma_0\}$. The generalized vector subbundle $\mathcal{D}(\Gamma_0)$ is smooth and we say that it is the canonical one associated with Γ_0. Notice that $\mathcal{D} = \mathcal{D}(\Gamma_0)$ can come from a different module $\Gamma_0' \neq \Gamma_0$, i.e. $\mathcal{D} = \mathcal{D}(\Gamma_0')$.

Let $\mathcal{D} \subset\Gamma(E)$ be a generalized smooth vector subbundle and $\mathcal{S} = \{s_1, \ldots, s_n\}$ be a finite set of sections such that $\mathcal{S}_x$ generate $\mathcal{D}_x$ for every $x \in M$. The *differential hull* of $\mathcal{S}$ is the $\mathcal{F}(M)$–hull $\Gamma(\mathcal{S})$ of $\mathcal{S}$. It is easy to see that for every finite set of generators $\mathcal{S}'$ of $\Gamma(\mathcal{S})$, we have $\Gamma(\mathcal{S}) = \Gamma(\mathcal{S}')$. An *anchor* of $\mathcal{D}$ is such an $\mathcal{F}(M)$–module $\Gamma(\mathcal{S})$; we denote an anchor by $(\mathcal{D}, \Gamma_0 = \Gamma(\mathcal{S}))$, but also by $\mathcal{D} = \mathcal{D}(\Gamma_0)$.

The existence of a finite generators of $\mathcal{D}$ depends on the module they belong. It happens that $\Gamma(\mathcal{D})$ is too large to be considered in the whole.

Example 1.4. *Consider the generalized vector subbundle* $\mathcal{D}_0$ *of* $\Theta^1(\mathbb{R})$, *generated by the vector field* $X_0 = \varphi_0 \frac{d}{dt}$, *where*

$$\varphi_0(t) = \left\{ \begin{array}{l} e^{-\frac{1}{t}} \text{ if } t > 0, \\ 0 \text{ if } t \leq 0 \end{array} \right. .$$

According to [2, Proposition 5.3], $\Gamma(\mathcal{D}_0)$ *is not finite dimensional.*

Example 1.5. *For an* $r \geq 0$, *let us consider the generalized vector subbundle considered in Example 1.1. For* $r = 0$, *the module* $\Gamma(\mathcal{C}_0)$ *is finitely generated and the vector field* $R_0 = t\frac{d}{dt}$ *is a generator. For* $r > 0$ *the module* $\Gamma(C_r)$ *is not finitely generated; these facts follow using similar arguments used in [2, Proposition 5.3.].*

Example 1.6. *Let us consider the generalized vector subbundle* $\mathcal{C}_{(r)}$ *from Example 1.3. In an analogous way as in Example 1.5 above, one can prove that the module* $\Gamma(\mathcal{C}_{(0)})$ *is generated by the the vector field* C *considered in Example 1.2. For* $r > 0$, *one can prove exactly like in Example 1.5, that the module* $\Gamma(\mathcal{C}_{(r)})$ *is not finitely generated.*

The sum and the intersection of generalized vector subbundles are discussed for example in [4].

If $\mathcal{D}'$ and $\mathcal{D}$ are generalized vector subbundles on a vector bundle over the base M, then $\mathcal{D}' \subset \mathcal{D}$ if $\mathcal{D}'_x \subset \mathcal{D}_x$, $(\forall) x \in M$.

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two generalized vector subbundles on a vector bundle. The *sum* $\mathcal{D}_1 + \mathcal{D}_2$ is the generalized vector subbundle defined in $x \in M$ by $(\mathcal{D}_1 + \mathcal{D}_2)_x = \mathcal{D}_{1,x} + \mathcal{D}_{2,x}$, as vector subspaces. If $(\mathcal{D}_1, \Gamma_1)$ and $(\mathcal{D}_2, \Gamma_2)$ are anchors, their *sum* is $(D_1 + \mathcal{D}_2, \Gamma_1 + \Gamma_2)$; it is easy to see that the is well defined. Notice that if $\mathcal{X}_1 = \{X_i\}_{i=\overline{1,k_1}} \subset \Gamma_1$ and $\mathcal{X}_2 = \{Y_\alpha\}_{\alpha=\overline{1,k_2}}$ are two sets of generators for the anchors (i.e. for modules and for the vector subbundle vectors), then their union $\mathcal{X}_{12} = \mathcal{X}_1 \cup \mathcal{X}_2$ is a set of generators for the sum anchor $(\Gamma_1 + \Gamma_2, D_1 + \mathcal{D}_2)$.

Obviously $\mathcal{D}_1 \subset \mathcal{D}_1 + \mathcal{D}_2$ and $\mathcal{D}_2 \subset \mathcal{D}_1 + \mathcal{D}_2$.

Example 1.7. Let $\mathcal{C}$, $\mathcal{A}(\mathcal{C}') \subset T\mathbb{R}^2$ be the generalized vector subbundles considered in Example 1.2. Then $(\mathcal{C} + \mathcal{A}(\mathcal{C}'))_{(x,y)} \subset \mathbb{R}^2$ is $\mathbb{R}^2$ for every $(x,y) \neq (0,0)$ and is $\{(0,0)\}$ in $(0,0)$. From Example 1.2, it follows that $\Gamma_0 = \Gamma(\mathcal{C}) + \Gamma(\mathcal{A}(\mathcal{C}'))$ is generated by the vector fields $\{C, C'\}$, but $\Gamma'_0 = \Gamma(\mathcal{C} + \mathcal{A}(\mathcal{C}'))$ is not generated by $\{C, C'\}$, in spite of the fact that C generates $\mathcal{C}$ and C' generates $\mathcal{A}(\mathcal{C}')$. Indeed, let us consider the vector field $C_0 = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \in \Gamma_0$. If would exist $f, g \in C^\infty(\mathbb{R})$ such that $C_0 = fC + gC'$, then for $(x,y) \neq (0,0)$ we must have $y = fy + gx$ and $x = -fx + gy$, thus $f = \frac{y^2-x^2}{x^2+y^2}$ and $g = \frac{2xy}{x^2+y^2}$, that are not differentiable in $(0,0)$. Thus $\{C, C'\}$ can not generate Γ_0. But using Lemma 1.1 one can prove that the $\mathcal{F}(\mathbb{R}^2)$–module Γ_0 is generated by the vector fields $\left\{x\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}, y\frac{\partial}{\partial y}\right\}$.

Example 1.8. Let $r \geq 0$ be given and $\mathcal{C}_{(r)}$, $\mathcal{A}(\mathcal{C}^{\perp}_{(r)}) \subset T\mathbb{R}^2$ *be the generalized vector subbundles considered in Example 1.3. Then* $(\mathcal{C}_{(r)} + \mathcal{A}(\mathcal{C}^{\perp}_{(r)}))_{(x,y)} \subset \mathbb{R}^2$ *is* $\mathbb{R}^2$ *for every* (x,y) where $x^2 + y^2 > r$ *and is* $\{(0,0)\}$ *if* $x^2 + y^2 \leq r$. *If* $r = 0$ *we obtain Example 1.7 above. The* $\mathcal{F}(\mathbb{R}^2)$*–module* $\Gamma(\mathcal{C}_{(r)} + \mathcal{A}(\mathcal{C}^{\perp}_{(r)}))$ *is not finitely generated, as well as its summands.*

Let $(\mathcal{D}_i, \Gamma_i)_{i\in I}$ be a set of anchors. Considering the $\mathcal{F}(M)$–module $\Gamma_0 = \bigcap\limits_{i\in I} \Gamma_i$ and $\mathcal{D}_0 = \mathcal{D}(\Gamma_0)$, we say that $(\mathcal{D}_0, \Gamma_0)$ is the *smooth intersection* of the anchors. Obviously $\mathcal{D}_0 \subset \bigcap\limits_{i\in I} \mathcal{D}_i$, but not equal, in general, as we see below.

In particular, if $(\mathcal{D}_1, \Gamma_1)$ and $(\mathcal{D}_2, \Gamma_2)$ are two anchored smooth vector subbundles, we can consider their differential intersection $(\mathcal{D}_0, \Gamma_0 = \Gamma_1 \cap \Gamma_2)$. We might think that $\mathcal{D}_0 = \mathcal{D}_1 \cap \mathcal{D}_2$, but it is not the case, as the following example shows. (See also [4, Example 1.3.2].)

Example 1.9. *On* $\mathbb{R}^3$ *we consider the regular vector subbundles* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *given by the* 2*-planes that contains the origin and have as normal vector fields (according to the canonical Euclidean metric in* $\mathbb{R}^3$*)* $\bar{N}_{1,(x,y,z)} = \dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + (1 + x^2 + y^2 + z^2)\dfrac{\partial}{\partial z}$ *and* $\bar{N}_{2,(x,y,z)} = 2\dfrac{\partial}{\partial x} + 2\dfrac{\partial}{\partial y} + (2 + x^2 + y^2 + z^2)\dfrac{\partial}{\partial z}$ *respectively. Then* $\mathcal{D}_{1,\bar{x}} \cap \mathcal{D}_{2,\bar{x}}$ *is* 1*–dimensional for* $\bar{x} = (x,y,z) \neq \bar{0} = (0,0,0)$*, generated by* $\bar{a}_{\bar{x}} = (x^2 + y^2 + z^2) \cdot (\dfrac{\partial}{\partial x} - \dfrac{\partial}{\partial y})$*, while* $\mathcal{D}_{1,\bar{0}} = \mathcal{D}_{2,\bar{0}} = \mathcal{D}_{1,\bar{0}} \cap \mathcal{D}_{2,\bar{0}}$ *is* 2*–dimensional. Since the dimension function is upper continuos in* $\bar{0}$*, it follows that* $\mathcal{D}_1 \cap \mathcal{D}_2$ *is not a generalized vector subbundle. In fact,* $\mathcal{D}_0 = \mathcal{D}(\Gamma_1 \cap \Gamma_2)$ *is regular,* 1*–dimensional and generated by the constant vector field* $\dfrac{\partial}{\partial x} - \dfrac{\partial}{\partial y}$.

Thus the intersection of two smooth generalized vector subbundles, even regular, *is not*, in general, a smooth generalized vector subbundle.

Example 1.10. *We give below an example where* $\mathcal{D}(\Gamma_1) = \mathcal{D}(\Gamma_2) \neq \mathcal{D}(\Gamma_1 \cap \Gamma_2)$*. For this, first consider the following smooth functions on* $\mathbb{R}$:

$$\varphi(t) = \begin{cases} e^{-\frac{1}{t^2}} \text{ if } t \neq 0, \\ 0 \text{ if } t = 0 \end{cases}, \psi(t) = \begin{cases} (2 + \sin\frac{1}{x})e^{-\frac{1}{t^2}} \text{ if } t \neq 0, \\ 0 \text{ if } t = 0 \end{cases},$$

and the following vector fields on $\mathbb{R}^2$:

$$\begin{aligned} Z_{1(x,y)} &= \varphi(x)\frac{\partial}{\partial x},\ Z_{2(x,y)} = \psi(x)\frac{\partial}{\partial y}, \\ X_1 &= Z_1 + \frac{\partial}{\partial y},\ X_2 = Z_2, \\ Y_{1(x,y)} &= \psi(x)\frac{\partial}{\partial x} + (1+\psi(x))\frac{\partial}{\partial y},\ Y_{2(x,y)} = \psi(x)\frac{\partial}{\partial x} + \psi(x)\frac{\partial}{\partial y}. \end{aligned}$$

Following a simple calculation, a relation of the type

$$\alpha X_1 + \beta X_2 = \alpha' Y_1 + \beta' Y_2 \tag{1.3}$$

yields for $x \neq 0$ *to*

$$\begin{aligned} \left(2+\sin\frac{1}{x}\right)(\alpha'+\beta') &= \alpha, \\ \alpha'(1+(2+\sin\frac{1}{x})e^{-\frac{1}{x^2}}) + \beta'(2+\sin\frac{1}{x})e^{-\frac{1}{x^2}} &= \alpha + \beta e^{-\frac{1}{x^2}}. \end{aligned} \tag{1.4}$$

It follows that the linear span of $\{X_{1(x,y)}, X_{2(x,y)}\}$ *is equal to the linear span of* $\{Y_{1(x,y)}, Y_{2(x,y)}\}$*; it is two dimensional for* $x \neq 0$ *and one dimensional for* $x = 0$*, generated by* $\frac{\partial}{\partial y}$*. If we suppose that the coefficients* α*,* β*,* α'*,* β' *from (1.3) are smooth, then the first relation (1.4) implies that* $\alpha(0,y) = \alpha'(0,y) + \beta'(0,y) = 0$*, while the second relation gives* $\alpha'(0,y) = \alpha(0,y)$*. Thus* $\alpha'(0,y) = \beta'(0,y) = 0$*, and the both sides in relation (1.3) vanish in the points* $(0,y)$*. It means that a smooth vector field* X *that is a smooth combination of* $\{X_1, X_2\}$ *as well as of* $\{Y_1, Y_2\}$ *vanishes in the points* $(0,y)$*, i.e.* $X_{(0,y)} = \bar{0}$*. In conclusion, consider* $\Gamma_1, \Gamma_2 \subset X(\mathbb{R}^2)$ *generated by* $\{X_1, X_2\}$ *and* $\{Y_1, Y_2\}$ *respectively; then* $\mathcal{D}(\Gamma_1) = \mathcal{D}(\Gamma_2)$ *and according to the above calculation* $\mathcal{D}_{(0,y)}(\Gamma_1 \cap \Gamma_2)$ *is zero dimensional, while* $\mathcal{D}_{(0,y)}(\Gamma_1) = \mathcal{D}_{(0,y)}(\Gamma_2)$ *is one dimensional and generated by* $\frac{\partial}{\partial y}$*.*

2 Generating generalized vector subbundles

In order to construct generalized vector subbundles that are smooth or cosmooth, one can use morphisms of vector bundles, as follows.

Proposition 2.1. *If* $f : E' \to E$ *is a morphism of vector bundles over the same base, then* $f(E') \subset E$ *is a smooth generalized subbundle and* $\ker f \subset E'$ *is a cosmooth generalized vector subbundle.*

Notice that if $\{s'_i\}_{i=\overline{1,r}} \subset \Gamma(E')$ is a set of generators, then $\{f(s'_i)\}_{i=\overline{1,r}} \subset \Gamma(E)$ are a set of generators for $f(E')$. Also if $\{\omega_\alpha\}_{\alpha=\overline{1,p}} \subset \Gamma(E^*)$ is a set of generators, then $\{f^*(\omega_\alpha)\}_{\alpha=\overline{1,p}} \subset \Gamma(E'^*)$ is a set of generators for $(\ker f)^\perp$.

Proposition 2.2. *Let us consider two vector bundles* E' *and* E *that have the bases* M' *and* M *respectively and* $f : E' \to E$ *an* f_0*-morphism of vector bundles, where* $f_0 : M' \to M$*. Then:*

1. *The vector generalized vector subbundle* $\ker f \subset E'$ *is cosmooth.*
2. *If* f_0 *is a submersion, then* $f(E') \subset E$ *is a smooth vector subbundle.*

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