

On Pfaff systems

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Abstract. The aim of the paper is to extend the dynamics of first order Lagrangians to that of Pfaff forms, i.e. differential forms on tangent bundles. The action on curves of a Pfaff form is natural associated with that of a second order Lagrangian linear in accelerations, while the converse association is not unique. An equivalence relation of Pfaff form is considered, that is compatible with gauge equivalent Lagrangians. In order to express the Euler-Lagrange equation as a second order Lagrange derivative of a Pfaff form, then controlled and higher order Pfaff forms are considered.

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Key words: Pfaff form; Lagrangian; Euler-Lagrange equation.

1 Introduction

The second order Lagrangians are considered, for example, in [4], [13], [18] etc. The second order Lagrangians that are affine in acceleration are involved in some special problems and they are studied for example in [1], [2], [3], [4], [7], [11], [14] etc. These Lagrangians, are the most singular possible - their vertical Hessian vanishes and according to [4, Sect. 6.3], some special regularity conditions can be considered. Third order Lagrangians that are affine in the third order derivatives, possessing an acceleration-extended Galilean symmetry, are studied in [8]; they extend the second order case considered previously by the authors and can be a model for a future development of the present paper. Other geometric aspects concerning Pfaff forms can be found in [15].

In this paper we study of some aspects of Pfaff forms, i.e. differentiable one forms on $\mathbb{R} \times TM$. Considering an action of a Pfaff form ω on differentiable curves on the differentiable manifold M , it is the same as the action of a second order Lagrangian affine in accelerations that corresponds canonically to ω (Proposition 2.1). Conversely, the action of a second order Lagrangian affine in accelerations can correspond to at least a Pfaff form (Proposition 2.2). We prove that one can consider a certain equivalence relation on Pfaff forms such that an equivalence class corresponds to gauge equivalent Lagrangians that give the actions (Proposition 3.1).

Considering controlled Pfaff forms (Proposition 3.3), higher order Pfaff forms, top Pfaff forms and Lagrange derivatives of Pfaff forms, then the Euler Lagrange equation of a Pfaff form can be obtained by (two) successive Lagrange derivatives of Pfaff forms (Proposition 4.1), considering an Ostrogradski Pfaff form, closed related to Ostrogradski momenta. The Euler-Lagrange equation contains the second derivatives and we prove that in the case of a regular Lagrangian, its solutions are integral curves of a global second order differential equation (Proposition 4.2).

2 Pfaff forms, Lagrangians and actions on curves

A Pfaff form is a differentiable form $\omega \in \mathcal{X}^*(\mathbb{R} \times TM)$. Using these local bases, a Pfaff form ω has the local form

$$(2.1) \quad \omega = \omega_0(t, x^i, y^i)dt + (\omega_i(t, x^j, y^j)dx^i + \bar{\omega}_i(t, x^j, y^j)dy^i) = \omega_0 + \omega',$$

where

$\omega' = \omega_i(t, x^j, y^j)dx^i + \bar{\omega}_i(t, x^j, y^j)dy^i$ is a new (global) Pfaff form that we call the *essential component* of ω ;

$\omega_0 : \mathbb{R} \times TM \rightarrow \mathbb{R}$ is a (global) Lagrangian that we call the *Lagrangian component* of ω .

We say that the Pfaff form $\omega = \omega_0 + \omega'$ is *pure* if $\omega_0 = 0$, *Lagrangian* if $\omega' = 0$ and *mixed* if ω is neither pure nor Lagrangian.

It is easy to see that a first order Lagrangian $L : TM \rightarrow \mathbb{R}$ is the same as the Lagrangian Pfaff form $\omega_0 = Ldt$.

Let $\eta : \mathbb{R} \times TM \rightarrow \pi^*T^*M$, having local the form $\eta = \eta_i(t, x^j, y^j)dx^i$ and coming from a section of the induced vector bundle $\mathbb{R} \times p_2^*T^*M \rightarrow \mathbb{R} \times TM$. It can be regarded as well as a Pfaff form, and is called a *top Pfaff form*. A Pfaff form ω having the form (2.1) defines a top Pfaff form $\tilde{\omega} = \tilde{\omega}_i(t, x^j, y^j)dx^i$. According to a definition given below, the top Pfaff form $\tilde{\omega}$ is degenerated when it is regarded as a Pfaff form.

A Pfaff form can be related to a second order dynamic form considered in [4]. According to [4, Section 2], a *first order dynamic form* on the bundle $Y = \mathbb{R} \times M \rightarrow M$ is a one contact and horizontal two form ν on $J^1(Y)$, having the local form $\nu = \nu_i(t, x^j, y^j)dx^i \wedge dt + \bar{\nu}_i(t, x^j, y^j)dy^i \wedge dt$. Obviously a first order dynamic form is equivalent to give a pure Pfaff form. An advantage to use Pfaff forms is having the Lagrangian forms in the same setting. An other motivation to use Pfaff forms is given by their action on curves, that relates them to the second order Lagrangians that are affine in accelerations, i.e. the vertical Hessian vanishes.

If $\gamma : [a, b] \rightarrow M$ is a curve on M , then for every $t \in [a, b]$ one can consider in $\tilde{\gamma}(t) = (t, \frac{d\gamma}{dt}(t)) \in \mathbb{R} \times TM$ the scalar $\omega_{\tilde{\gamma}(t)}\left(\frac{d^2\gamma}{dt^2}(t)\right)$. The *action* of the Pfaff form ω on γ is given by the formula

$$(2.2) \quad I_\omega(\gamma) = \int_a^b \omega_{\tilde{\gamma}(t)}\left(\frac{d^2\gamma}{dt^2}(t)\right) dt.$$

Let us relate the action of Pfaff forms on curves to the actions of Lagrangians on curves. First, we define the *action* of a first order Lagrangian $L^{(1)} : \mathbb{R} \times TM \rightarrow \mathbb{R}$ on a curve $\gamma : [a, b] \rightarrow M$, by the formula:

$$I_L(\gamma) = \int_a^b L^{(1)} \left(t, \gamma(t), \frac{d\gamma}{dt}(t) \right) dt.$$

If $\gamma : [a, b] \rightarrow M$ is a curve on M , then the curves $\frac{d\gamma}{dt} : [a, b] \rightarrow TM$ (the *velocity curve*) and $\frac{d^2\gamma}{dt^2} : [a, b] \rightarrow T^2M \subset TTM$ (the *acceleration curve*) are called here the *first order lift* and the *second order lift* respectively, of the curve γ . A *second order Lagrangian* on M is a differentiable map $L^{(2)} : \mathbb{R} \times T^2M \rightarrow \mathbb{R}$, where T^2M is the second order tangent space of M . Then $L^{(2)}$ defines also an *action* on γ by formula:

$$I_\omega(\gamma) = \int_a^b L^{(2)} \left(t, \gamma(t), \frac{d\gamma}{dt}(t), \frac{d^2\gamma}{dt^2}(t) \right) dt.$$

A second order Lagrangian L is *affine in accelerations* if its vertical Hessian vanishes. Using local coordinates, $L(t, x^i, y^i, z^i) = f_0(t, x^i, y^i) + z^i g_i(t, x^i, y^i)$. Notice that if $g_i = 0$, then $L = f_0$ is a first order Lagrangian. In this case f_0 is obtained projecting $L : T^2M \rightarrow \mathbb{R}$ on $f_0 : TM \rightarrow \mathbb{R}$, by the natural projection $T^2M \rightarrow TM$.

It is easy to see that the Lagrangian action I_{L_0} of a Lagrangian L_0 is the same as the Pfaff action I_{ω_0} of the Lagrangian Pfaff form $\omega_0 = L_0 dt \in \mathcal{X}^*(\mathbb{R} \times TM)$. It worth to remark that ω_0 is a closed form only if $L_0 = L_0(t)$. Let us remark also that the formula (2.2) is free of coordinates and gives an easy tool to obtain an action involving accelerations, velocities, position and time. More precisely, the accelerations are involved affinely, as follows.

Proposition 2.1. *If $\omega \in \mathcal{X}^*(\mathbb{R} \times TM)$ is a Pfaff form, then there is a second order Lagrangian $L^{(2)} : T^2M \rightarrow \mathbb{R}$, affine in accelerations, such that $I_\omega = I_L$.*

Notice that using coordinates (x^i, y^i, z^i) on T^2M , the Pfaff form given by the above Proposition has the form:

$$(2.3) \quad L_\omega^{(2)} : T^2M \rightarrow \mathbb{R}, \quad L^{(2)}(t, x^i, y^i, z^i) = \omega_0(t, x^i, y^i) + \omega_i(t, x^j, y^j) y^i + \bar{\omega}_i(t, x^j, y^j) z^i.$$

The following result shows that the action of every second order Lagrangian, affine in accelerations, can be represented as well as an action of a suitable Pfaff form.

Proposition 2.2. *Let $L^{(2)} : \mathbb{R} \times T^2M \rightarrow \mathbb{R}$ be a second order Lagrangian affine in accelerations. Then there is a Pfaff form $\omega \in \mathcal{X}^*(\mathbb{R} \times TM)$ such that $I_\omega = I_L$.*

The actions of Pfaff forms on curves are related to the well-known actions of the first and the second order Lagrangians on curves. Let us consider two points $x, y \in M$ and $\gamma_0 = (x_0^i(t))$ be a curve joining x and y , i.e. $x_0^i(0) = x$ and $x_0^i(1) = y$. Let us consider variations of γ_0 , as curves joining x and y , having the local form $\gamma_\varepsilon = (x_\varepsilon^i(t))$, where $x_\varepsilon^i(t) = x_0^i(t) + \varepsilon h^i(t)$.

In the case of the actions of second order Lagrangians on curves, the specific variational conditions, impose:

$$(2.4) \quad h^i(a) = h^i(b) = 0,$$

$$(2.5) \quad \frac{dh^i}{dt}(a) = \frac{dh^i}{dt}(b) = 0.$$

For a second order Lagrangian $L^{(2)} : \mathbb{R} \times T^2M \rightarrow \mathbb{R}$, the extreme curves of the action $I_{L^{(2)}}$ are given by the well-known Euler-Lagrange equations

$$(2.6) \quad \frac{\partial L^{(2)}}{\partial x^i} - \frac{d}{dt} \frac{\partial L^{(2)}}{\partial y^i} + \frac{d^2}{dt^2} \frac{\partial L^{(2)}}{\partial z^i} = 0.$$

In the particular case of a Lagrangian (2.3), the Euler-Lagrange equations have the form

$$(2.7) \quad \frac{\partial \omega_0}{\partial x^i} + \frac{\partial \omega_j}{\partial x^i} \frac{dx_0^j}{dt} + \frac{\partial \bar{\omega}_j}{\partial x^i} \frac{d^2 x_0^j}{dt^2} - \frac{d}{dt} \left(\frac{\partial \omega_0}{\partial y^i} + \frac{\partial \omega_j}{\partial y^i} \frac{dx_0^j}{dt} + \omega_i + \frac{\partial \bar{\omega}_j}{\partial y^i} \frac{d^2 x_0^j}{dt^2} \right) + \frac{d^2}{dt^2} \bar{\omega}_i = 0.$$

Let us consider \mathbb{R}^2 with coordinates x and y . The canonical symplectic form $\alpha = dx \wedge dy$ gives the Pfaff form $\omega^{(1)} = \dot{x}dy - \dot{y}dx$ and the second order Lagrangian $L_0(t, \dot{x}, \dot{y}, \ddot{x}, \ddot{y}) = \dot{x}\ddot{y} - \dot{y}\ddot{x}$ on \mathbb{R}^2 ; here $(x, y) := (x^1, x^2)$; $(\dot{x}, \dot{y}) := (y^1, y^2)$, in the previous notations. This Lagrangian was involved in [7], concerning its invariance to the $(2 + 1)$ -Galilean symmetry; the authors prove in the Appendix that *the general form of a one-particle Lagrangian which is at most linearly dependent on \ddot{x} , and \ddot{y} leading to Euler-Lagrange equations of motion which are covariant with respect to the $D = 2$ Galilei group, is given, up to gauge transformations, by $L(t, x, y, \dot{x}, \dot{y}, \ddot{x}, \ddot{y}) = -k(\dot{x}\ddot{y} - \dot{y}\ddot{x}) + \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$. This Lagrangian is affine in velocities, but it can come from two Pfaff forms:*

$$(2.8) \quad \begin{aligned} \omega_1 &= -k(\dot{x}d\dot{y} - \dot{y}d\dot{x}) + \frac{m}{2}(\dot{x}dx + \dot{y}dy), \\ \omega_2 &= -k(\dot{x}d\dot{y} - \dot{y}d\dot{x}) + \frac{m}{2}(\dot{x}^2 + \dot{y}^2)dt. \end{aligned}$$

In order to put together these two Pfaff forms, we define below an equivalence relation, ruled by their action and implicitly by their second order Lagrangians, affine in velocities.

3 Equivalence of Pfaff forms

A first order Lagrangian $F : \mathbb{R} \times TM \rightarrow \mathbb{R}$ and a Pfaff form $\omega \in \mathcal{X}^*(\mathbb{R} \times TM)$ gives the Pfaff form $\omega' = \omega + dF$. Then

$$I_{\omega'}(\gamma) = I_{\omega}(\gamma) + \int_a^b \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x^i} \frac{dx^i}{dt} + \frac{\partial F}{\partial y^i} \frac{dy^i}{dt} \right) dt = I_{\omega}(\gamma) + F(x^i(b), \frac{dx^i}{dt}(b)) - F(x^i(a), \frac{dx^i}{dt}(a)).$$

According to the variation conditions (2.4) and (2.5), it is easy to see that I_{ω} and $I_{\omega'}$ have the same extreme curves.

Analogous considerations as made in [6] for the gauge equivalence of first order Lagrangians can be transposed for second order Lagrangians (see for example [12,

Section 4.4). It reads that the second order Lagrangians L and $L' = L + \frac{d}{dt}F$, $F : \mathbb{R} \times TM \rightarrow \mathbb{R}$, are gauge equivalent, i.e. they have the same extreme curves. Here $\frac{d}{dt}F$ stands for L_{dF} , the second order Lagrangian associated with the Pfaff form dF . The analogous gauge form for actions of the corresponding Pfaff forms, reads that the Pfaff forms ω and $\omega' = \omega + dF$ have the same extreme curves.

We notice that the Lagrangians given by [3, formula (11)] or [2, formula (34)] are gauge equivalent, but they are studied without using this fact.

Let us consider the submodule $\mathcal{G} \subset \mathcal{X}^*(\mathbb{R} \times TM)$ generated by the local differential forms $\{\delta x^i = dx^i - y^i dt\}_{i=1, \dots, m}$. A form $\eta \in \mathcal{G}$ iff it has the local form $\eta = a_i(t, x^j, y^j)\delta x^i$. It is easy to see that any form $\eta \in \mathcal{G}$ vanishes along the (second order) lift of a curve on M . Thus for any Pfaff form $\omega \in \mathcal{X}^*(\mathbb{R} \times TM)$, the Pfaff forms ω and $\omega' = \omega + \eta$ have the same extreme curves (see [11] for other implications concerning the module \mathcal{G}).

We say that:

- two Pfaff forms $\omega, \omega' \in \mathcal{X}^*(\mathbb{R} \times M)$ are *equivalent* if there is an $F \in \mathcal{X}^*(\mathbb{R} \times M)$ such that $\omega' - \omega - dF \in \mathcal{G}$;
- two second order Lagrangians L' and L are *gauge equivalent* if there is an $F \in \mathcal{X}^*(\mathbb{R} \times TM)$ such that $L' - L = \frac{d}{dt}F$.

It is easy to see that two equivalent Pfaff forms have the same extreme curves. Analogously, two second order Lagrangians L' and L that are gauge equivalents have the same extreme curves.

Proposition 3.1. *Two Pfaff forms ω' and ω are equivalent iff their second order Lagrangians $L_{\omega'}$ and L_{ω} are gauge equivalent.*

It follows that the property of the above Proposition can be used as a definition of equivalent Pfaff forms.

Corollary 3.2. *If two Pfaff forms correspond to the same second order Lagrangian affine in accelerations, then they are equivalent.*

The Poincaré-Cartan form $\theta_L = Ldt + \frac{\partial L}{\partial y^i}\delta x^i$ of a first order Lagrangian L , where $\delta x^i = dx^i - y^i dt$, is obviously equivalent to the canonical Lagrangian form Ldt and both correspond to the same Lagrangian L , seen of second order by $T^2M \rightarrow TM \rightarrow \mathbb{R}$.

There are two possibilities to associate a Pfaff form to a pointed Lagrangian $L(t, x^i, y^i) = y^i \nu_i$: $\omega_1 = \nu_i dx^i$ and $\omega_2 = Ldt$ respectively. The first is pure and the second is a Lagrangian one, but they have the same action on curves, that given by the action of the same Lagrangian. It is easy to see that $d\omega_1 - d\omega_2 = 0$ iff $\nu^i = 0$; thus ω_1 and ω_2 are not differential equivalent (i.e. $\omega_1 - \omega_2 = dF$) for $L \neq 0$. Thus there are Pfaff forms that are not differential equivalent (i.e. their difference is not a exact differential), but equivalent.

Every Pfaff form ω that has the local form $\omega = \omega_0 dt + \omega_i dx^i + \bar{\omega}_i dy^i$ is locally equivalent to the local Pfaff form $\omega' = (\omega_0 + y^i \omega_i)dt + \bar{\omega}_i dy^i$, since $\omega - \omega' = \omega_i \delta x^i$. But, in general ω' is not a global Pfaff form.

The Pfaff forms $\omega_1 = -k(\dot{x}d\dot{y} - \dot{y}d\dot{x}) + \frac{m}{2}(\dot{x}dx + \dot{y}dy)$ and $\omega_2 = -k(\dot{x}d\dot{y} - \dot{y}d\dot{x}) + \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$ considered previously are equivalent, since $\omega_1 - \omega_2 = \frac{m}{2}(\dot{x}(dx - \dot{x}dt) + \dot{y}(dy - \dot{y}dt))$. Let us consider below two other situations.

1) Considering the canonical symplectic form in \mathbb{R}^2 given by $(\varepsilon_{ij}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we obtain the pure Pfaff form $\omega_1 = -k\varepsilon_{ij}y^i dy^j$, or $\omega_1 = -k\varepsilon_{ij}\dot{x}^i d\dot{x}^j$ where k is a non-null constant, that corresponds to the second order Lagrangian $L_0(x^i, y^i, z^i) = -k\varepsilon_{ij}y^i z^j$, or $L_0(x^i, \dot{x}^i, \ddot{x}^j) = -k\varepsilon_{ij}\dot{x}^i \ddot{x}^j$.

The Lagrangian $L(x^i, y^i, z^i) = \frac{my^i y^j \delta_{ij}}{2} - k\varepsilon_{ij}y^i z^j$, or $L(x^i, \dot{x}^i, \ddot{x}^j) = \frac{m\dot{x}^i \ddot{x}^j \delta_{ij}}{2} - k\varepsilon_{ij}\dot{x}^i \ddot{x}^j$ was considered in [7, 8, 1].

In order to obtain a Pfaff form we have two possibilities: $\omega = my^i \delta_{ij} dx^j - k\varepsilon_{ij}y^i dy^j$ and $\omega' = \frac{my^i y^j \delta_{ij}}{2} dt - k\varepsilon_{ij}y^i dy^j$; the first is pure and the second is a mixed one.

2) The Lagrangian $L(x^i, y^i, z^i) = -m \|y^{(1)}\| + \frac{\varepsilon_{ij}y^i z^j}{\|y^{(1)}\|^3}$, where $\|y^{(1)}\| = \sqrt{\frac{y^i y^j \delta_{ij}}{2}}$ or $L(x^i, \dot{x}^i, \ddot{x}^j) = -m \|\dot{x}\| + \frac{\varepsilon_{ij}\dot{x}^i \ddot{x}^j}{\|\dot{x}\|^3}$, where $\|\dot{x}\| = \sqrt{\frac{m\dot{x}^i \ddot{x}^j \delta_{ij}}{2}}$ was considered in [10]. The two Pfaff forms, one pure and one mixed, can also be considered: $\omega = -\frac{my^i \delta_{ij}}{\|y^{(1)}\|} dx^j + \frac{\varepsilon_{ij}y^i}{\|y^{(1)}\|^3} dy^j$, and $\omega' = -m \|y^{(1)}\| dt + \frac{\varepsilon_{ij}y^i}{\|y^{(1)}\|^3} dy^j$.

Unlike the first example, in the second example the Pfaff form ω' is not differentiable in the points where $(y^i = 0)$.

3.1 Controlled and higher order Pfaff forms

Let us extend the top Lagrange derivative to a broader class of Pfaff forms.

Let $\pi_E : E \rightarrow TM$ a weak fiber manifold over the tangent bundle (i.e. a submersion) such that the composed projection $\pi_M : E \rightarrow M$ is a fiber manifold (i.e. a surjective submersion). A *bundle map* of two fibered manifolds over the base M is a map that send fibers to fibers ($E_x = \pi_M^{-1}(x)$ is the fiber of $x \in M$). We denote by $\pi_{TM} : TM \rightarrow M$ the canonical projection. Using local coordinates (x^i) on M , (x^i, u^α) on E and (x^i, y^j) on TM that are adapted to the fibered manifold structures, then we have the local forms $(x^i, y^j) \xrightarrow{\pi_{TM}} (x^i)$, $(x^i, u^\alpha) \xrightarrow{\pi_E} (x^i)$, $(x^i, y^j, u^\alpha) \xrightarrow{\pi_E} (x^i, y^j)$, $(x^i, u^\alpha, y^j, v^\beta) \xrightarrow{\pi_E} (x^i, u^\alpha)$ and $(x^i, y^j, u^\alpha, v^\beta) \xrightarrow{\pi_E} (\rho^i(x^i, y^j, u^\alpha, v^\beta))$. If coordinates change, then one follow the rules: $x^{i'} = x^i(x^i)$, $y^{j'} = \frac{\partial x^{j'}}{\partial x^i} y^i$, $u^{\alpha'} = u^\alpha(x^i, u^\alpha)$, $v^{\beta'} = \frac{\partial u^{\beta'}}{\partial x^i} y^i + \frac{\partial u^{\beta'}}{\partial u^\alpha} v^\alpha$.

A *controlled Pfaff form* on E is bundle map $\omega : \mathbb{R} \times E \rightarrow T^*TM$ over the base M . A *controlled top Pfaff form* on E is a bundle map $\bar{\omega} : \mathbb{R} \times E \rightarrow T^*M$ over the base M . Using local coordinates a controlled Pfaff form ω has the local form, $\omega = \omega_i(t, x^i, u^\alpha) dx^i + \bar{\omega}_i(t, x^i, u^\alpha) dy^i$; the local functions change according to the rules $\bar{\omega}_i = \frac{\partial x^{i'}}{\partial x^i} \bar{\omega}_{i'}$ and $\omega_i = \frac{\partial y^{i'}}{\partial x^i} \bar{\omega}_{i'} + \frac{\partial x^{i'}}{\partial x^i} \omega_{i'}$ respectively. Analogously, a controlled top Pfaff form $\bar{\omega}$ has the form $\bar{\omega} = \bar{\omega}_i dx^i$ and $\bar{\omega}_i = \frac{\partial x^{i'}}{\partial x^i} \bar{\omega}_{i'}$. Obviously a controlled Pfaff form ω as above gives rise to a top Pfaff form $\bar{\omega}$.

Let us define the Lagrange controlled *top derivative* of ω as

$$(3.1) \quad \mathcal{E}'_\omega(t, x^i, y^i, u^\alpha, v^\alpha) = [\omega_i - (\frac{\partial \bar{\omega}_i}{\partial x^j} y^j + \frac{\partial \bar{\omega}_i}{\partial u^\alpha} v^\alpha + \frac{\partial \bar{\omega}_i}{\partial t})] dx^i.$$

Proposition 3.3. *The Lagrange controlled top derivative of ω is a global map $\mathcal{E}'_\omega : \mathbb{R} \times TE \rightarrow T^*M$, over M .*

Considering the (local) operator $\frac{d}{dt} = \frac{\partial}{\partial t} + y^i \frac{\partial}{\partial x^i} + v^\alpha \frac{\partial}{\partial u^\alpha}$, then $\mathcal{E}'_\omega = (\omega_i - \frac{d}{dt}\bar{\omega}_i)dx^i$.

If $E = TM$, we obtain the definition of a controlled Pfaff form ω or top Pfaff form, as above. The Lagrange controlled derivative of $\omega : \mathcal{E}'_\omega : \mathbb{R} \times TTM \rightarrow T^*M$ restricts to the Lagrange derivative of $\omega : \mathcal{E}'_\omega : \mathbb{R} \times T^2M \rightarrow T^*M$ given above by formula (3.1), since $T^2M \subset TTM$; using local coordinates, this inclusion looks as $(x^i, y^i, z^i) \rightarrow (x^i, y^i, y^i, z^i)$.

We call:

a *k-order Pfaff form* any controlled Pfaff form $\omega : \mathbb{R} \times T^kM \rightarrow T^*TM$ and

a *k-order top Pfaff form* any bundle map $\bar{\omega} : \mathbb{R} \times T^kM \rightarrow T^*M$.

It is easy to see that any k -order Pfaff form $\omega = \omega_i dx^i + \bar{\omega}_i dy^i$ gives a k -order top Pfaff form $\bar{\omega} = \bar{\omega}_i dx^i$.

Let us define now the *Lagrange top derivative* $\mathcal{E}_\omega^{(k+1)}$ of a k -order Pfaff form $\omega : \mathbb{R} \times T^kM \rightarrow T^*TM$. The Lagrange controlled top derivative of ω is $\mathcal{E}_\omega^{(k+1)} : \mathbb{R} \times TT^kM \rightarrow T^*M$ restricts to the Lagrange derivative of $\omega : \mathcal{E}_\omega^{(k+1)} : \mathbb{R} \times T^{k+1}M \rightarrow T^*M$, according to the inclusion $T^{k+1}M \subset TT^kM$. Using local coordinates, the inclusion has the form $(x^i, y^i, \dots, w^i, \tilde{w}^i) \rightarrow (x^i, y^i, \dots, w^i, y^i, \dots, w^i, \tilde{w}^i)$, $\mathcal{E}_\omega^{(k+1)} : \mathbb{R} \times TT^kM \rightarrow T^*M$ has the local form

$$\mathcal{E}'_\omega(t, x^i, y^i, \dots, w^i, X^i, Y^i, \dots, W^i) = [\omega_i - (\frac{\partial \bar{\omega}_i}{\partial t} + \frac{\partial \bar{\omega}_i}{\partial x^j} X^j + \frac{\partial \bar{\omega}_i}{\partial y^j} Y^j + \dots + \frac{\partial \bar{\omega}_i}{\partial w^j} W^j)] dx^i$$

and the restriction to $\mathbb{R} \times T^{k+1}M$ is

$$\begin{aligned} \mathcal{E}_\omega^{(k+1)} : \mathbb{R} \times T^{k+1}M \rightarrow T^*M, \mathcal{E}_\omega^{(k+1)}(t, x^i, y^i, \dots, z^i, w^i, \tilde{w}^i) = \\ [\omega_i - (\frac{\partial \bar{\omega}_i}{\partial t} + \frac{\partial \bar{\omega}_i}{\partial x^j} y^j + \dots + \frac{\partial \bar{\omega}_i}{\partial z^i} w^i + \frac{\partial \bar{\omega}_i}{\partial w^i} \tilde{w}^i)] dx^i, \end{aligned}$$

or

$$(3.2) \quad \mathcal{E}_\omega^{(k+1)} = [\omega_i - \frac{d}{dt}\bar{\omega}_i] dx^i,$$

where $\frac{d}{dt}$ is the local operator given by $\frac{d}{dt} = \frac{\partial}{\partial t} + y^j \frac{\partial}{\partial x^j} + \dots + w^i \frac{\partial}{\partial z^i} + \tilde{w}^i \frac{\partial}{\partial w^i}$.

The first order Lagrange top derivative of a Pfaff form $\omega = \omega_i dx^i + \bar{\omega}_i dy^i$ is the second order top Pfaff form $\mathcal{E}_\omega^{(2)} = \mathcal{E}'_\omega : \mathbb{R} \times T^2M \rightarrow T^*M$ given using (3.2).

In the case $k = 3$, the local operator $\frac{d}{dt}$ is given by $\frac{d}{dt} = \frac{\partial}{\partial t} + y^j \frac{\partial}{\partial x^j} + z^i \frac{\partial}{\partial y^i} + w^i \frac{\partial}{\partial z^i} + \tilde{w}^i \frac{\partial}{\partial w^i}$. In the case when $\bar{\omega} = \bar{\omega}_i dx^i$ has the order 2, then $\frac{\partial \bar{\omega}_i}{\partial w^i} = 0$ and the third order Lagrange top derivative $\mathcal{E}_\omega^{(3)}$ has the order at most 3, as ω . This is the case below when the Euler-Lagrange top form of a Pfaff form has the third order.

We say that a k -order Pfaff form $\omega : \mathbb{R} \times T^kM \rightarrow T^*TM$ is *effectively of order k* (or the order k is *effective*) if ω can not be induced by a $(k-1)$ -Pfaff form $\omega' : \mathbb{R} \times T^{k-1}M \rightarrow T^*TM$ (using the canonical projection $T^kM \rightarrow T^{k-1}M$). Analogously, a k -order top Pfaff form $\omega : \mathbb{R} \times T^kM \rightarrow T^*M$ is *effectively of order k* if ω can not be induced by a $(k-1)$ -Pfaff form $\bar{\omega}' : \mathbb{R} \times T^{k-1}M \rightarrow T^*M$.

The Lagrange top derivative of an effective k -order Pfaff form $\omega : \mathbb{R} \times T^kM \rightarrow T^*TM$ is at most effective $(k+1)$ -order top Pfaff form $\mathcal{E}_\omega^{(k+1)} : \mathbb{R} \times T^{k+1}M \rightarrow T^*M$. In the case when an effective k -order Pfaff form $\omega = \omega_i dx^i + \bar{\omega}_i dy^i$ has the associated

top Pfaff form $\bar{\omega} = \bar{\omega}_i dx^i$ of an effective s -order, $s \leq k - 1$, then the Lagrange top derivative $\mathcal{E}_\omega^{(k+1)}$ has an effective k -order, as ω .

It is easy to see that if $\bar{\omega} : \mathbb{R} \times T^k M \rightarrow T^*M$, $\bar{\omega}(t, x^i, y^i, \dots, z^i) = \bar{\omega}_i(t, x^i, y^i, \dots, z^i) dx^i$ is a k -order top Pfaff form, then $(E_{ij} = \frac{\partial \bar{\omega}_i}{\partial z^j})$ and $(E_{ijk} = \frac{\partial^2 \bar{\omega}_i}{\partial z^j \partial z^k})$ are covariant tensors: a bilinear form E'_ω and a trilinear form E''_ω respectively, on the fibers of T^*M . Then $\bar{\omega}$ is effective of order k iff $E'_\omega \neq 0$. We say the k -order top Pfaff form $\bar{\omega}$ is *affine in k -accelerations* if $E''_\omega = 0$.

We say also that a k -order Pfaff form ω , with $\bar{\omega}$ the associated Pfaff form, is *affine in k -accelerations* if its Lagrange top derivative $\mathcal{E}_\omega^{(k+1)}$ is affine in k -accelerations (if the effective order of $\bar{\omega}$ is less than the effective order of ω) or in $(k + 1)$ -accelerations (if the effective orders of ω and $\bar{\omega}$ are the same).

Using relation (3.2) it is easy to see that if a k -order Pfaff form ω and its associated top Pfaff form $\bar{\omega}$ have effectively the orders k , then the Lagrange top derivative of ω is affine in the $(k + 1)$ -accelerations, i.e. denoting $\bar{\theta} = \mathcal{E}_\omega^{(k+1)}$, then $E''_{\bar{\theta}} = 0$ and $E'_{\bar{\theta}} = \mathcal{E}_\omega^{(k+1)}$, specifically $E_{ij} = \frac{\partial \bar{\omega}_i}{\partial z^j}$.

4 The Euler-Lagrange equation as a top Pfaff form

Any second order Lagrangian $L : \mathbb{R} \times T^2 M \rightarrow \mathbb{R}$ gives rise to at most fourth order top Pfaff form $\mathcal{E}_i dx^i$, that we call the *Euler-Lagrange top Pfaff form* of L , where

$$(4.1) \quad \mathcal{E}_i = \frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial y^i} + \frac{d^2}{dt^2} \frac{\partial L}{\partial z^i},$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + y^j \frac{\partial}{\partial x^j} + z^j \frac{\partial}{\partial y^j} + w^j \frac{\partial}{\partial z^j} + \tilde{w}^j \frac{\partial}{\partial w^j}$ and $(x^i, y^i, z^i, w^i, \tilde{w}^i)$ are the canonical local coordinates on $T^4 M$ induced by the local coordinates (x^i) on M . In the case when a second order Lagrangian L_ω is affine in accelerations and it is associated with a Pfaff form ω , its local formula is given as in formula (2.3) with $L^{(2)} = L_\omega$. We say that the Euler-Lagrange top Pfaff form $\mathcal{E} = \mathcal{E}_\omega$ of L_ω is the *Euler-Lagrange top Pfaff form* of ω . Specifically, if ω is a Pfaff form given by formula (2.1), then $L_\omega(t, x^i, y^i, z^i) = \omega_0 + y^i \omega_i + z^i \bar{\omega}_i$, thus

$$(4.2) \quad \mathcal{E}_i = \frac{\partial \omega_0}{\partial x^i} + \frac{\partial \omega_j}{\partial x^i} y^j + \frac{\partial \bar{\omega}_j}{\partial x^i} z^j - \frac{d}{dt} \left(\frac{\partial \omega_0}{\partial y^i} + \frac{\partial \omega_j}{\partial y^i} y^j + \omega_i + \frac{\partial \bar{\omega}_j}{\partial y^i} z^j \right) + \frac{d^2}{dt^2} \bar{\omega}_i,$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + y^j \frac{\partial}{\partial x^j} + z^j \frac{\partial}{\partial y^j} + w^j \frac{\partial}{\partial z^j}$, since $\frac{\partial L}{\partial z^i} = \bar{\omega}_i(t, x^i, y^i, z^i)$ and the fourth order coordinates (\tilde{w}^i) are not involved. Thus the top Pfaff form \mathcal{E} is at most third order in this case.

We prove below that the Euler-Lagrange top Pfaff form can be obtained using two second order Pfaff forms.

Proposition 4.1. *Let ω be a (first order) Pfaff form such that the Euler-Lagrange top Pfaff form \mathcal{E}_ω is of third order. Then the following assertions holds true.*

1. *If Ω is a first or a second order Pfaff form such that $\bar{\Omega} = \bar{\omega}$, then there is a second or a third order Pfaff form Φ , uniquely determined by the conditions that the Lagrange top derivative of Ω is $\bar{\Phi}$ and the Lagrange top derivative of Φ is the Euler-Lagrange top Pfaff form \mathcal{E}_ω .*

2. There are two second order Pfaff forms Ω and Φ such that $\bar{\Omega} = \bar{\omega}$, the Lagrange top derivative of Ω is $\bar{\Phi}$ and the Lagrange top derivative of Φ is \mathcal{E}_ω .

We call the second order Pfaff form Ω constructed in 2. of Proposition 4.1 as an *Ostrogradski Pfaff form* of ω .

The above construction is related to a general approach, related to the classical Ostrogradski theory.

Let $L : \mathbb{R} \times T^2M \rightarrow \mathbb{R}$ be a second order Lagrangian. There is a top Pfaff form $\bar{\omega} = \frac{\partial L}{\partial z^i} dx^i$ associated with this Lagrangian, that is of order at most 2. Let us suppose that ω is a first or second order Pfaff form ω such that its top Pfaff form is $\bar{\omega}$, i.e. $\omega = \omega_i dx^i + \frac{\partial L}{\partial z^i} dy^i$. One can consider for example $\omega = \frac{1}{2} \frac{\partial L}{\partial y^i} dx^i + \frac{\partial L}{\partial z^i} dy^i$. Then the formula $\Omega = \Omega_i dx^i + \bar{\Omega}_i dx^i = \left(\frac{\partial L}{\partial y^i} - \omega_i \right) dx^i + \frac{\partial L}{\partial z^i} dx^i$ defines a Pfaff form of order at most 2.

Then $\eta : \mathbb{R} \times T^2M \rightarrow T^*TM$, $\eta(t, x^i, y^i, z^i) = \left(\frac{\partial L}{\partial y^i} - \omega_i \right) dx^i + \frac{\partial L}{\partial z^i} dy^i$ is a second order Pfaff form. The Lagrange top derivative of η is $\mathcal{E}_\eta^{(3)} : \mathbb{R} \times T^2M \rightarrow T^*M$, $\mathcal{E}_\eta'' = \left(\frac{\partial L}{\partial y^i} - \omega_i - \frac{d}{dt} \frac{\partial L}{\partial z^i} \right) dx^i$. Usually $\left(\frac{\partial L}{\partial y^i} - \omega_i - \frac{d}{dt} \frac{\partial L}{\partial z^i} \right)$ is denoted by p_i .

Since $\mathcal{E} = \frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial y^i} + \frac{d^2}{dt^2} \frac{\partial L}{\partial z^i} = \frac{\partial L}{\partial x^i} - \frac{d}{dt} \omega_i - \frac{d}{dt} \left(\frac{\partial L}{\partial y^i} - \omega_i - \frac{d}{dt} \frac{\partial L}{\partial z^i} \right)$, it follows that $\mu = \left(\frac{\partial L}{\partial x^i} - \frac{d}{dt} \omega_i \right) dx^i + \left(\frac{\partial L}{\partial y^i} - \omega_i - \frac{d}{dt} \frac{\partial L}{\partial z^i} \right) dy^i$ is at most third order Pfaff form and its Lagrange top derivative is $\mathcal{E}_\mu^{(4)} = \mathcal{E}_\omega$, the Euler-Lagrange top Pfaff form of ω . This algorithm can produce Pfaff forms for a second order Lagrangian, taking a suitable Pfaff form ω .

A k -order semi-spray $S : \mathbb{R} \times T^k M \rightarrow T^{k+1} M$ is a section $(t, x^{(k)}) \xrightarrow{\bar{S}} (t, S(t, x^{(k)}))$ of the affine bundle $\mathbb{R} \times T^{k+1} M \rightarrow \mathbb{R} \times T^k M$, obtained as a product of the affine bundle $T^{k+1} M \rightarrow T^k M$ and the identity $\mathbb{R} \rightarrow \mathbb{R}$.

Let $\mathcal{E}_\omega : \mathbb{R} \times T^3 M \rightarrow T^* M$ be the Euler-Lagrange top Pfaff form of a Pfaff form ω . We say that a second order semi-spray $\bar{S} : \mathbb{R} \times T^2 M \rightarrow \mathbb{R} \times T^3 M$ is adapted to the Pfaff form ω if $\mathcal{E}_\omega \circ \bar{S} = 0$. The local form of \bar{S} and \mathcal{E}_ω are $(t, x^i, y^i, z^i) \xrightarrow{\bar{S}} (t, x^i, y^i, z^i, S^i(t, x^i, y^i, z^i))$ and $\mathcal{E}_\omega = \mathcal{E}_i dx^i$ with \mathcal{E}_i given by (4.2) respectively. Denoting $h_{ij} = \frac{\partial \bar{\omega}_i}{\partial y^j} - \frac{\partial \bar{\omega}_j}{\partial y^i}$, then there are local functions $f_i(t, x^i, y^i, z^i)$ such that

$$\mathcal{E}_i(t, x^i, y^i, z^i, w^i) = h_{ij} w^j + f_i(t, x^i, y^i, z^i).$$

More precisely:

$$\begin{aligned} \mathcal{E}_i = & \frac{\partial \omega_0}{\partial x^i} - \frac{\partial^2 \omega_0}{\partial t \partial y^i} - \frac{\partial^2 \omega_0}{\partial x^j \partial y^i} y^j + \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial^2 \omega_j}{\partial t \partial y^i} - \frac{\partial \omega_j}{\partial x^j \partial y^i} - \right. \\ & \left. \frac{\partial \omega_j}{\partial y^j \partial y^i} \right) y^j - \frac{\partial \omega_i}{\partial t} - \frac{\partial \omega_i}{\partial x^j} y^j + \frac{\partial^2 \bar{\omega}_i}{\partial t^2} + \frac{\partial^2 \bar{\omega}_i}{\partial x^j \partial t} y^j + \left(\frac{\partial^2 \bar{\omega}_i}{\partial t \partial x^j} + \frac{\partial^2 \bar{\omega}_i}{\partial x^k \partial x^j} y^k \right) y^j - \\ & - \frac{\partial^2 \omega_0}{\partial y^j \partial y^i} z^j - \frac{\partial \omega_j}{\partial y^i} z^j - \frac{\partial \omega_j}{\partial y^j} z^j + \frac{\partial \bar{\omega}_j}{\partial x^i} z^j + \left(2 \frac{\partial^2 \bar{\omega}_i}{\partial x^k \partial y^j} - \frac{\partial \bar{\omega}_j}{\partial x^k \partial y^i} \right) y^k z^j + \\ & + \left(2 \frac{\partial^2 \bar{\omega}_i}{\partial y^j \partial t} - \frac{\partial^2 \bar{\omega}_j}{\partial t \partial y^i} \right) z^j + \frac{\partial \bar{\omega}_i}{\partial x^j} z^j + \left(\frac{\partial^2 \bar{\omega}_i}{\partial y^k \partial y^j} - \frac{\partial \bar{\omega}_j}{\partial y^k \partial y^i} \right) z^k z^j + \left(\frac{\partial \bar{\omega}_i}{\partial y^j} - \frac{\partial \bar{\omega}_j}{\partial y^i} \right) w^j. \end{aligned}$$

The condition that \bar{S} is adapted to the Pfaff form ω reads

$$(4.3) \quad h_{ij} S^j + f_i(t, x^i, y^i, z^i) = 0.$$

Notice that the third order semi-spray S gives a system of third order equations, having the form

$$\frac{d^3 x^i}{dt^3} + S^i(t, x^i, \frac{dx^i}{dt}, \frac{d^2 x^i}{dt^2}) = 0;$$

its solutions are the integral curves of the vector field S .

A Pfaff form ω is *regular* if the local matrices $(h_{ij} = \frac{\partial \bar{\omega}_i}{\partial y^j} - \frac{\partial \bar{\omega}_j}{\partial y^i})$ are non-singular.

Proposition 4.2. *If the Pfaff form ω is regular, then the solutions of the generalized Euler-Lagrange equation $\mathcal{E} = 0$, where \mathcal{E} is given by (4.2) has the same solutions as of a second order equation given by a global second order semi-spray $S : \mathbb{R} \times T^2 M \rightarrow T^3 M$.*

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