

On a new unified geometric description of gravity and electromagnetism

N. Voicu

Abstract. In a previous paper, we have introduced a new unified description of the main equations of the gravitational and of the electromagnetic field, in terms of tidal tensors and connections on the tangent bundle TM of the space-time manifold. In the present work, we relate these equations to variational procedures on the tangent bundle. The Ricci scalar of the proposed connection is dynamically equivalent to the usual Einstein-Maxwell Lagrangian. Also, in order to be able to perform these variational procedures, we find an appropriate completion of the metric tensor (from the base manifold) up to a metric structure on TM .

M.S.C. 2010: 53Z05, 83C22, 83C50, 83E05.

Key words: tangent bundle; connection; tidal tensor; Einstein field equations.

1 Introduction

The main request of a unified, geometric description of gravity and electromagnetism is to find some geometric structures such that, in the Einstein field equations, the electromagnetic stress-energy tensor is enclosed in the left hand side, i.e., in the Einstein tensor. In the period between the two world wars, remarkable theories were built by: Einstein (teleparallelism, metric-affine theories), Eddington, Eisenhart, Schrödinger (affine geometry with torsion), Weyl (gauge theory), Kaluza and Klein (fifth space-time dimension). Afterwards, the interest for the classical approaches somehow waned and the focus shifted rather to quantum theories, [14].

More recently, interest for classical unified theories has grown again. Thus, Ferraris and Kijowski, [13], Chrusciel, [11], Poplawski, [22], developed the purely affine approach, in which the torsion of the affine connection (on the space-time manifold) accounts for electromagnetism.

A second path – which we follow here – uses the geometry of the tangent bundle TM of the space-time manifold. Here, we should mention the description due to R. Miron and collaborators, [16], [18], [17], [19], in which the metric tensor characterizes gravity, while electromagnetism is encoded in connections on the tangent bundle. In

the cited papers, they obtain a geometrization of the Lorentz equations of motion of charged particles and of Maxwell equations – but they do not solve the problem of enclosing the Maxwell stress-energy tensor into the Einstein tensor.

Other theories on the tangent bundle try to include information regarding electromagnetism in Finsler-type metrics (Randers, Beil or Weyl metrics, [9], [10]). Remarkable geometrizations of Maxwell's equations and of the equations of motion of charged particles are also provided in terms of Riemann-Jacobi and Finsler-Jacobi structures, [23], [24], [5], teleparallelism on TM , [27], complex Lagrange geometry, [20], or in the context of jet spaces, [6]-[7].

In [26], we proposed a new approach – based, on one side, on the notion of geodesic deviation (and subsequently, of tidal tensor, [12]) and on the other side, on the idea in [16], [18], [17], of encoding the information regarding gravity in the metric tensor on the base manifold and electromagnetism, in a 1-parameter family of connections $(\overset{\alpha}{N}, \overset{\alpha}{D})$ (where $\overset{\alpha}{N}$ are Ehresmann connections and $\overset{\alpha}{D}$, affine connections) on TM . We chose $(\overset{\alpha}{N}, \overset{\alpha}{D})$ such that:

- 1) worldlines of charged particles define autoparallel curves for both $\overset{\alpha}{N}, \overset{\alpha}{D}$;
- 2) worldline deviation equations are as simple as possible (their right hand side does not depend on the derivatives of the deviation vector field);
- 3) the Ricci tensor of a connection $\overset{\alpha}{D}$ can be obtained just by differentiating the trace of the tidal tensor with respect to the fiber coordinates on TM .

In [26], we wrote Maxwell equations directly in terms of tidal tensors attached to $\overset{\alpha}{N}$.

In the present paper, we build an analogue of the classical Hilbert action based on Ricci scalars of $\overset{\alpha}{D}$.

But, in order to perform variational procedures on TM , we also need a volume element on TM . With this aim, we propose a completion of the Lorentzian metric g_{ij} up to a metric on the total space TM and the subsequent Riemannian volume element, with two properties: a) there exists, for each $x \in M$, a canonical domain of integration $\tilde{\Delta} \subset T_x M$ with respect to the fiber coordinates such that, for functions $f = f(x)$ defined on M , the integral of f on a domain $\Delta \subset M$ and the integral of f on $\Delta \times \tilde{\Delta}$ coincide; b) the divergence of the horizontal lift to TM of a vector field on M coincides with the divergence of the original vector field.

This construction refines the one in [25]. With this, we get one more property of the connections $\overset{\alpha}{D}$:

- 4) for a conveniently chosen α , the Ricci scalar of $\overset{\alpha}{D}$ is dynamically equivalent to the usual Einstein-Maxwell Lagrangian on M . Einstein field equations (with the electromagnetic stress-energy tensor included in the Einstein tensor) and stress-energy conservation are then obtained in the usual way.

Property 4) is similar to the one in Kaluza-Klein theory, but it does not require additional space-time dimensions; moreover, our method has the advantage of providing geometrizations of the Lorentz equations of motion and of worldline deviation equations.

2 Preliminaries

2.1 Basic equations

Consider a 4-dimensional, C^∞ Lorentzian manifold (M, g) , with local coordinates $(x^i)_{i=\overline{0,3}}$, regarded as space-time manifold and ∇ , its Levi-Civita connection, with coefficients γ^i_{jk} and curvature tensor r^i_{jkl} ; we denote by ∂_i the elements of the natural basis for the module of vector fields on M .

In general relativity, the metric components g_{ij} describe the gravitational field. The electromagnetic field is described by the potential 1-form:

$$(2.1) \quad A = A_i(x)dx^i$$

and by the electromagnetic field tensor (Faraday 2-form) $F = dA$, or, locally,

$$(2.2) \quad F = \frac{1}{2}F_{ij}dx^i \wedge dx^j, \quad F_{ij} = \nabla_i A_j - \nabla_j A_i.$$

From the definition of F , it follows the identity: $dF = 0$, which is equivalent to homogeneous Maxwell equations: $\nabla_{\partial_i} F_{jk} + \nabla_{\partial_k} F_{ij} + \nabla_{\partial_j} F_{ki} = 0$.

The other basic equations of the two physical fields are obtained by variational methods. The total action attached to these, together with a system of particles with masses m_a , coordinates x^i_a and electric charges q_a is, [15]:

$$(2.3) \quad S = \underbrace{-\sum m_a c \int ds}_{S_m} - \underbrace{\sum \frac{q_a}{c} \int A_k(x) dx^k}_{S_i} - \underbrace{\frac{1}{16\pi c} \int F_{ij} F^{ij} \sqrt{-g} d^4x}_{S_f} - \underbrace{\frac{c^3}{16\pi k} \int r \sqrt{-g} d^4x}_{S_g};$$

where the sums are taken over the particles in the system, $g = \det(g_{ij})$, $d^4x = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$, r is the Ricci scalar of g and c, k are constants (the light speed in vacuum and the gravitational constant). The volume integrals are taken over a large enough compact domain $\Delta \subset M^1$. The first term S_m characterizes free particles, the third one S_f characterizes the electromagnetic field and the second one S_i , the interaction between the field and the particles. The fourth integral S_g is the Hilbert action for g_{ij} .

The line integrals S_m and S_i can be transformed, [2], into volume integrals, by

¹Traditionally, one integrates over a finite amount of time and over the whole spatial manifold, under the assumption that, far enough from sources, the considered fields vanish; loosely speaking, we can integrate over a large enough compact domain of M . Also, the variations of the fields at the limits of the region of integration are assumed to be zero, [15]. This way, we will be able to neglect boundary terms.

means of the Dirac delta function (involving the variables x^1, x^2, x^3):

$$(2.4) \quad S_m = - \int \mathcal{L}_m \sqrt{-g} d^4x, \quad \mathcal{L}_m = \sum_a \frac{\|dx_a\|}{dx_a^0} \frac{m_a c}{\sqrt{-g}} \delta^3(x - x_a),$$

$$(2.5) \quad S_i = - \frac{1}{c^2} \int A_i J^i \sqrt{-g} d^4x, \quad J^i = \sum_a \frac{dx_a^i}{dx_a^0} \frac{q_a c}{\sqrt{-g}} \delta^3(x - x_a).$$

Here, \mathcal{L}_m is a scalar, while J^i are components of the 4-current vector field, [15]. Thus, the total action S becomes:

$$(2.6) \quad S = - \int (\mathcal{L}_m + \frac{1}{c^2} A_i J^i + \frac{1}{16\pi c} F_{ij} F^{ij} + \frac{c^3}{16\pi k} r) \sqrt{-g} d^4x.$$

I. Variation of the 4-potential A in S yields the *inhomogeneous Maxwell equations*:

$$(2.7) \quad \nabla_{\partial_j} F^{ij} = - \frac{4\pi}{c} J^i.$$

II. Variation of S with respect to the metric components g^{ij} leads, [15], [2], to the *Einstein field equations*:

$$(2.8) \quad G_{ij} = \frac{8\pi k}{c^4} T_{ij};$$

here, the Einstein tensor

$$G_{ij} = r_{ij} - \frac{1}{2} r g_{ij}$$

is obtained from: $\delta S_g = - \frac{c^3}{16\pi k} \int G_{ij} \delta g^{ij} \sqrt{-g} d^4x = \frac{c^3}{16\pi k} \int G^{ij} \delta g_{ij} \sqrt{-g} d^4x.$

In the right hand side of (2.8), the stress-energy tensor T_{ij} is written as a sum:

$$T_{ij} = T_{ij}^f + T_{ij}^m,$$

where:

- T_{ij}^f is the stress-energy tensor of the electromagnetic field

$$T_{ij}^f = \frac{1}{4\pi} (-F_{il} F_j{}^l + \frac{1}{4} g_{ij} F^{lm} F_{lm}),$$

obtained from: $\delta S_f = \frac{1}{2c} \int T_{ij}^f \delta g^{ij} \sqrt{-g} d^4x = - \frac{1}{2c} \int T^{ij} \delta g_{ij} \sqrt{-g} d^4x;$

- T_{ij}^m is the stress-energy tensor of matter:

$$\delta S_m = \frac{1}{2c} \int T_{ij}^m \delta g^{ij} \sqrt{-g} d^4x = - \frac{1}{2c} \int T^{ij} \delta g_{ij} \sqrt{-g} d^4x.$$

III. In the case of a single particle (with mass m and charge q), variation of S with respect to its trajectory, i.e., the variation of:

$$(2.9) \quad S_m + S_i := mc \int ds + \frac{q}{c} \int A_k(x) dx^k$$

with respect to $x^i = x^i(s)$ (where s is the arc length), leads to the *Lorentz equations of motion of charged particles*:

$$(2.10) \quad \frac{\nabla \dot{x}^i}{ds} = \frac{q}{mc^2} F^i_j \dot{x}^j.$$

2.2 Conservation laws

Since action S is a scalar, it is invariant to diffeomorphisms, [2]. This invariance leads to the well-known energy-momentum conservation law.

Consider diffeomorphisms with pushforward $\tilde{x}^i = x^i + \varepsilon \xi^i(x)$ on M (where ξ^i are components of a vector field and $\varepsilon > 0$). Then, the variations of the field variables are given by their Lie derivatives:

$$(2.11) \quad \delta A_i = \mathcal{L}_\xi A_i = \xi^k \nabla_{\partial_k} A_i + A_k \nabla_{\partial_i} \xi^k, \quad \delta g_{ij} = \mathcal{L}_\xi g_{ij} = \nabla_{\partial_j} \xi_i + \nabla_{\partial_i} \xi_j.$$

The variation of the total action is $\delta S = \delta_A S + \delta_g S$; in detail:

$$\delta S = \int \left\{ -\frac{1}{4\pi c} (\nabla_{\partial_j} F^{ij} + \frac{4\pi}{c} J^i) \delta A_i + \frac{c^3}{16\pi k} (G^{ij} - \frac{8\pi k}{c^4} T^{ij}) \delta g_{ij} \right\} \sqrt{-g} d^4 x.$$

For solutions F of the inhomogeneous Maxwell equations (2.7), it remains

$$\delta S = \frac{c^3}{16\pi k} \int (G^{ij} - \frac{8\pi k}{c^4} T^{ij}) \delta g_{ij} \sqrt{-g} d^4 x.$$

Substituting δg_{ij} from (2.11) and integrating by parts, we get:

$$\delta S = \frac{c^3}{8\pi k} \int \left\{ \nabla_{\partial_j} [(G^{ij} - \frac{8\pi k}{c^4} T^{ij}) \xi_i] - \xi_i \nabla_{\partial_j} (G^{ij} - \frac{8\pi k}{c^4} T^{ij}) \right\} \sqrt{-g} d^4 x.$$

The first term becomes, by Stokes' theorem, a boundary one, hence it will not contribute to the integral; we get:

$$\delta S = -\frac{c^3}{8\pi k} \int \xi_i \nabla_{\partial_j} (G^{ij} - \frac{8\pi k}{c^4} T^{ij}) \sqrt{-g} d^4 x.$$

Since the variations ξ_i are independent, we are led to:

$$\nabla_{\partial_j} (G^{ij} - \frac{8\pi k}{c^4} T^{ij}) = 0.$$

Contracted Bianchi identities tell us that $\nabla_{\partial_j} G^{ij} = 0$. We thus get the *energy-momentum conservation law*

$$(2.12) \quad \nabla_{\partial_j} T^{ij} = 0.$$

In more detail, this is, [15]: $\nabla_{\partial_j} \overset{f}{T}{}^{ij} = -\frac{1}{c} F^i_j J^j = -\nabla_{\partial_j} \overset{m}{T}{}^{ij}$.

3 Geometric structures on TM

3.1 Ehresmann connections

Consider now the tangent bundle (TM, π, M) , with local coordinates $(x \circ \pi, y) =: (x^i, y^i)_{i=0,3}$; we denote by

$$(3.1) \quad l = \frac{y}{\|y\|}, \quad \|y\| = \sqrt{g_{ij}y^iy^j},$$

the *normalized supporting element* on TM , [8], and by $\cdot_{,i}$ and \cdot^i , partial differentiation with respect to x^i and y^i respectively. An Ehresmann connection N on TM , [16], [8], gives rise to the adapted basis

$$(3.2) \quad (\delta_i = \frac{\partial}{\partial x^i} - N^j_{\cdot i}(x, y) \frac{\partial}{\partial y^j}, \quad \delta^i = \frac{\partial}{\partial y^i}),$$

and to its dual $(dx^i, \delta y^i = dy^i + N^i_j dx^j)$.

Consider the following 1-parameter family of *Randers-type Lagrangians*, [8], [17] depending on a real parameter α :

$$(3.3) \quad \overset{\alpha}{L} = \sqrt{g_{ij}(x)\dot{x}^i\dot{x}^j} + \alpha A_i(x)\dot{x}^i.$$

The action $\int \overset{\alpha}{L} dt$ attached to $\overset{\alpha}{L}$ is formally similar to the action (2.9); though using the same notations as in the previous section, for the moment, we will not attribute any physical significance to A or α . Taking $t = \text{const} \cdot s$ as a parameter, extremal curves $x = x(t)$ are given by:

$$(3.4) \quad \frac{dy^i}{dt} + \gamma^i_{jk} y^j y^k - \alpha \|y\| F^i_j y^j = 0, \quad y^i = \dot{x}^i,$$

where

$$(3.5) \quad F^i_j := g^{ih}(\nabla_{\partial_h} A_j - \nabla_{\partial_j} A_h), \quad \|y\| = \sqrt{g_{ij}y^iy^j}.$$

We obtain a 1-parameter family of sprays, [1], [16], $G = \overset{\alpha}{G}$ on TM :

$$(3.6) \quad 2\overset{\alpha}{G}^i(x, y) = \gamma^i_{jk} y^j y^k + 2\overset{\alpha}{B}^i,$$

with

$$(3.7) \quad 2\overset{\alpha}{B}^i = -\alpha \|y\| F^i_j y^j =: -\alpha \|y\| F^i;$$

the corresponding spray connections, [1], $N = \overset{\alpha}{N}$ have the coefficients:

$$(3.8) \quad \overset{\alpha}{G}^i_j := \overset{\alpha}{G}^i_{\cdot j} = \gamma^i_{jk} y^k + B^i_j.$$

If there is no risk of confusion upon α , we will adopt the simpler notations: $G^i, B^i, \delta_i, G^i_j, B^i_j, \dots$ instead of: $\overset{\alpha}{G}^i, \overset{\alpha}{B}^i, \overset{\alpha}{\delta}_i, \overset{\alpha}{G}^i_j, \overset{\alpha}{B}^i_j$ etc.

Extremal curves of the action $\int \overset{\alpha}{L} dt$ are thus autoparallel curves (*geodesics*) for $N = \overset{\alpha}{N}$:

$$\frac{\delta y^i}{dt} = 0, \quad y^i = \dot{x}^i, \quad i = \overline{0,3}$$

and geodesic deviations are given, [26], by:

$$(3.9) \quad \frac{\delta^2 w^i}{dt^2} = E^i_j w^j, \quad E^i_j = R^i_{jk} y^k,$$

where $R^i_{jk} = \delta_k N^i_j - \delta_j N^i_k$ are the local coefficients of the curvature of N .

We will call the quantity

$$(3.10) \quad E = E^i_j \delta_i \otimes dx^j,$$

the *tidal tensor*² associated to N .

The functions B^i in (3.7) are the components of a horizontal vector field $B = B^i \delta_i$ on TM . Their derivatives with respect to the fiber coordinates are:

$$(3.11) \quad B^i_j = B^i_{.j} = -\frac{\alpha}{2}(F^i l_j + \|y\| F^i_j), \quad B^i_{jk} := B^i_{.jk} = -\frac{\alpha}{2}(l_{.jk} F^i + l_j F^i_k + l_k F^i_j).$$

Conversely, from the homogeneity of degree 2 of B in the fiber coordinates, it follows: $B^i_{.j} y^j = 2B^i$, $B^i_{jk} y^k = B^i_{.j}$ etc.

3.2 Affine connections on TM

Consider

$$(3.12) \quad G^i_{jk} := G^i_{.jk} = \gamma^i_{jk} + B^i_{jk};$$

and the affine connections $D = \overset{\alpha}{D}$ on TM which act on the $\overset{\alpha}{N}$ -adapted basis vectors as:

$$(3.13) \quad D_{\delta_k} \delta_j = G^i_{jk} \delta_i, \quad D_{\delta_k} \dot{\delta}_j = G^i_{jk} \dot{\delta}_i, \quad D_{\dot{\delta}_k} \delta_j = D_{\dot{\delta}_k} \dot{\delta}_j = 0.$$

Connections $\overset{\alpha}{D}$, $\alpha \in \mathbb{R}$, preserve by parallelism the distributions generated by $\overset{\alpha}{N}$ (hence, they are *distinguished linear connections*, [16], on TM), i.e., for any two vector fields X, Y on TM , we have: $D_X(hY) = hD_X Y$, $D_X(vY) = vD_X Y$. They are, generally, non-metrical.

$D = \overset{\alpha}{D}$ has generally nonvanishing torsion, given by:

$$(3.14) \quad \mathbb{T} = R^i_{jk} \dot{\delta}_i \otimes dx^k \otimes dx^j$$

and the curvature of D is:

$$(3.15) \quad \begin{aligned} \mathbb{R} = & R_j^i{}_{kl} \delta_i \otimes dx^l \otimes dx^k \otimes dx^j + R_j^i{}_{kl} \dot{\delta}_i \otimes \delta y^l \otimes dx^k \otimes dx^j + \\ & + B_j^i{}_{kl} \delta_i \otimes dx^l \otimes dx^k \otimes \delta y^j, \end{aligned}$$

²The tidal tensor is tightly related to the *Jacobi endomorphism* Φ , [4]: $\Phi = J \circ E$, where J is the almost tangent structure on TM .

where $B_j^i{}_{kl} = B^i{}_{.jkl}$ and $R_j^i{}_{kl}$ are obtained in terms of the tidal tensor as follows:

$$(3.16) \quad R_j^i{}_{kl} = \frac{1}{2}(E^i{}_{.k})_{.jl}.$$

In particular, the Ricci tensor of $\overset{\alpha}{D}$ is obtained from the trace $E^i{}_{.i}$:

$$(3.17) \quad R_{jl} = -\frac{1}{2}(E^i{}_{.i})_{.jl} = R_j^i{}_{li}.$$

Conversely, the tidal tensor E can be written³ in terms of \mathbb{R} as:

$$(3.18) \quad E^i{}_{.k} = R_j^i{}_{kl}y^jy^l, \quad E^i{}_{.i} = -R_{jl}y^jy^l.$$

Particular case: For $\alpha = 0$, we get: $2\overset{0}{G}^i = \gamma^i{}_{jk}y^jy^k$, i.e.,

$$\overset{0}{G}^i{}_{jk} = \gamma^i{}_{jk};$$

for vector fields X, Y on M , we have $l_h(\nabla_X Y) = \overset{0}{D}_{l_h(X)}l_h(Y)$ (where l_h denotes the horizontal lift to TM); thus, $\overset{0}{D}$ can be considered as the TM -equivalent of the Levi-Civita connection ∇ and each $\overset{\alpha}{D}$, as a "perturbation⁴" of $\overset{0}{D}$, with contortion tensor B . We obviously have: $\overset{0}{E}^i{}_{.k} = r_j^i{}_{kl}y^jy^l$, $\overset{0}{R}_j^i{}_{kl} = r_j^i{}_{kl}$ and the Ricci tensor of $\overset{0}{D}$ is $\overset{0}{R}_{jk} = r_{jk}$.

In [26], we have proved that the Euler-Lagrange equations for $\overset{\alpha}{L}$ are equivalent to:

$$(3.19) \quad D_V V = 0.$$

where V is the complete lift of the velocity vector field $\dot{x}^i\partial_i$:

$$(3.20) \quad V := y^i\delta_i + \frac{\delta y^i}{dt}\dot{\delta}_i, \quad y^i = \frac{dx^i}{dt}$$

and geodesic deviations can be also written as

$$(3.21) \quad \frac{D^2 w^i}{dt^2} = E^i{}_{.k}w^k;$$

here, all covariant derivatives are considered "with reference vector y ", [8], i.e., in their local expressions, $G^i{}_{.j} = G^i{}_{.j}(x, y)$, $G^i{}_{.jk} = G^i{}_{.jk}(x, y)$.

³Expression (3.18) points out an almost complete similarity between the tidal tensor and the notion of flag curvature in Finsler geometry. The difference consists in the metric tensor used in raising and lowering indices, which is here g_{ij} (not the Finslerian one corresponding to $\overset{\alpha}{L}$) and which leads to somehow different properties.

⁴Each of these perturbations gives rise to a notion of product for vector fields on TM ; thus, the module of vector fields on TM becomes an algebra – the so-called *deformation algebra*, [21].

3.3 Metric structure on TM

Fix a connection $\overset{\alpha}{N}$. The Lorentzian metric $g = (g_{ij})$ on M can be lifted into a metric

$$g = g_{ij}(x)dx^i \otimes dx^j$$

on the horizontal subbundle of TM , which we will extend up to a metric on the whole TM :

$$\mathcal{G} := g_{ij}(x)dx^i \otimes dx^j + v_{ij}(x)\delta y^i \otimes \delta y^j.$$

The functions $v_{ij} = v_{ij}(x)$ in (3.3) will be chosen as the components (lifted to TM) of a Riemannian – positive definite – metric tensor on M , with the property that the determinants of (g_{ij}) and (v_{ij}) have equal absolute values.

Such a choice is always possible. For instance, in Riemann normal coordinates $(x^{i'})$ for g at some $x_0 \in M$ (i.e., $g_{i'j'}(x_0) = \eta_{i'j'} = \text{diag}(-1, 1, 1, 1)$), we can set: $v_{i'j'}(x_0) = \delta_{i'j'}$, i.e., in the adapted basis,

$$\mathcal{G}(x'_0) := \text{diag}(-1, 1, 1, \dots, 1)$$

(in another coordinate system (x^k) , we will have $v_{kl} = \frac{\partial x^{i'}}{\partial x^k} \frac{\partial x^{j'}}{\partial x^l} v_{i'j'}$). With this choice, the dependence $x \mapsto \mathcal{G}(x)$ (accordingly, $x \mapsto v_{ij}(x)$) is a smooth one and

$$v := \det(v_{ij}) = \det\left(\frac{\partial x^{i'}}{\partial x^k}\right)^2 = -g.$$

As a consequence, the volume element on TM is:

$$(3.22) \quad d\Omega = \sqrt{-gv}d^4x \wedge \delta^4y,$$

where $\delta^4y = \delta y^0 \wedge \delta y^1 \wedge \delta y^2 \wedge \delta y^3$. Moreover, $d^4x \wedge \delta y^i = d^4x \wedge (dy^i + N^i_j dx^j) = d^4x \wedge dy^i$, i.e., we can actually write:

$$(3.23) \quad d\Omega = \sqrt{-gv}d^4x \wedge d^4y.$$

Since v is positive definite, the set:

$$(3.24) \quad \tilde{\Delta} = \{y \in T_x M \mid v_{ij}y^i y^j \leq \sqrt[4]{2/\pi^2}\}, \quad r > 0, \quad x \in M$$

is a compact subset of $T_x M$; in normal coordinates for v_{ij} , the domain $\tilde{\Delta} = \tilde{\Delta}(x)$ actually becomes a 3-sphere of volume equal to 1 in 4-dimensional Euclidean space.

For any $x \in M$, we will set $\tilde{\Delta}$ as a canonical integration domain with respect to $y \in T_x M$; thus, for any compact domain $\Delta \subset M$ and for any function $f : \Delta \rightarrow \mathbb{R}$, the integral of f over⁵ $\Delta \times \tilde{\Delta} \subset TM$ coincides with its integral over Δ :

$$(3.25) \quad \int_{\Delta \times \tilde{\Delta}} f(x)d\Omega = \int_{\Delta} f(x) \cdot \text{vol}(\tilde{\Delta})\sqrt{-g}d^4x = \int_{\Delta} f(x)\sqrt{-g}d^4x.$$

⁵For the sake of simplicity, we denote by the same letter f the composition $f \circ \pi$.

The divergence of a horizontal vector field $X = X^i(x, y)\delta_i$ on TM is, [25]:

$$\begin{aligned} \operatorname{div}(X) &= \frac{1}{\sqrt{-gv}}\delta_i(X^i\sqrt{-gv}) - \overset{0}{N}_{i,j}^j X^i = \\ &= \delta_i X^i + X^i \delta_i(\ln \sqrt{-g}) + X^i \delta_i(\ln \sqrt{v}) - \overset{0}{N}_{i,j}^j X^i. \end{aligned}$$

We have $\overset{0}{N}_{i,j}^j = \gamma_{ij}^j$. Taking into account that $\delta_i(\ln \sqrt{-g}) = \delta_i(\ln \sqrt{v}) = \gamma_{ij}^j$, we get:

$$(3.26) \quad \operatorname{div}(X) = \frac{1}{\sqrt{-g}}\delta_i(X^i\sqrt{-g}) = \overset{0}{D}_{\delta_i} X^i.$$

In particular, for a vector field $Y = Y^i(x)\partial_i$ on the base manifold, we have: $\operatorname{div}(Y) = \overset{0}{D}_{\delta_i} Y^i = \operatorname{div}(l_h Y)$.

4 Einstein field equations

4.1 In vacuum

Consider α as arbitrary and fixed. As analogue of the classical Hilbert action (this time, involving both g_{ij} and F_{ij}), we propose:

$$S_{fg} = -\frac{c^3}{16\pi k} \int_{\Delta \times \tilde{\Delta}} R d\Omega,$$

where R is the Ricci scalar of $\overset{\alpha}{D}$ with $\tilde{\Delta}$ as in (3.24) and Δ as in Section 2.

From (3.17), we get, by direct computation:

$$(4.1) \quad R = r + \overset{0}{D}_{\delta_i}(B^{ij}{}_j) - \frac{1}{2}g^{jk}(B^i{}_h B^h{}_i)_{.jk};$$

the term $\overset{0}{D}_{\delta_i}(B^{ij}{}_j)$ is a divergence, i.e., it will only produce a boundary term, which finally vanishes. A brief calculation leads to $-\frac{1}{2}g^{jk}(B^i{}_h B^h{}_i)_{.jk} = \frac{3\alpha^2}{2}F_{ij}F^{ij}$; thus, the two remaining terms in the integral S_{fg} do not depend on y any longer, i.e., S_{fg} can be written as an integral on the base manifold. We thus get:

Theorem 4.1. *The Ricci scalar R of $D = \overset{\alpha}{D}$ it is dynamically equivalent to the following Lagrangian on M :*

$$(4.2) \quad \tilde{R} = r + \frac{3\alpha^2}{2}F_{ij}F^{ij}.$$

In particular, for $\alpha = \alpha^*$ given by:

$$(4.3) \quad \frac{3(\alpha^*)^2}{2} = \frac{k}{c^4},$$

we get the usual Einstein-Maxwell action:

$$S_{fg}(\alpha^*) = -\frac{c^3}{16\pi k} \int \tilde{R} \sqrt{-g} d^4x = -\frac{c^3}{16\pi k} \int (r + \frac{k}{c^4} F_{ij} F^{ij}) \sqrt{-g} d^4x.$$

In terms of $\overset{\alpha}{D}$, we get, by a similar procedure to the one in [15]:

$$\delta_g S_{fg}(\alpha^*) = -\frac{c^3}{16\pi k} \int (\tilde{R}_{ij} - \frac{1}{2} \tilde{R} g_{ij} + \mathcal{B}_{.ij}) \delta g^{ij} \sqrt{-g} d^4x,$$

where: $\mathcal{B} := \frac{3}{2} \frac{B^l B_l}{\|y\|^2} + \frac{1}{2} B^i {}_h B^h{}_i$. Thus:

Proposition 4.2. *Einstein-Maxwell equations in vacuum are expressed in terms of $D = D(\alpha^*)$ as:*

$$(4.4) \quad \mathcal{G}_{ij} = 0,$$

where

$$(4.5) \quad \mathcal{G}_{ij} = \tilde{R}_{ij} - \frac{1}{2} \tilde{R} g_{ij} + \mathcal{B}_{.ij}.$$

are components of a symmetric horizontal tensor field $\mathcal{G} = \mathcal{G}_{ij} dx^i \otimes dx^j$ on TM .

4.2 In the presence of matter

Assuming that we also have some particles of masses m_a and electric charges q_a , the total action is:

$$(4.6) \quad S = S_{fg} + S_m + S_i.$$

Remark 4.1. The sum $S_{mi} := S_m + S_i$ can be also written in terms of the functions $\overset{\alpha}{L}$ as:

$$(4.7) \quad S_{mi} = - \int \sum_a \mu_a c \overset{\alpha}{L}(x_a, \frac{dx_a}{dx_a^0}) \sqrt{-g} d^4x, \quad \mu_a := \frac{m_a \delta^3(x - x_a)}{\sqrt{-g}},$$

where for each particle, we have a different value of α :

$$\alpha_a = \frac{q_a}{m_a c^2}.$$

The action S in (4.6) is nothing but the usual total action in Section 2.1, but here, the term S_f corresponding to the electromagnetic field is contained in the Ricci scalar R (equivalently, in \tilde{R}).

By varying the expression (4.6) of S with respect to g^{ij} , and using the fact that actually, S_i does not depend on g^{ij} , we get, similarly to Proposition 4.2:

Proposition 4.3. *Einstein field equations are written as:*

$$(4.8) \quad \mathcal{G}_{ij} = \frac{8\pi k^m}{c^4} T_{ij},$$

where \mathcal{G}_{jk} is the generalized Einstein tensor (4.5) (including the electromagnetic part of the stress-energy tensor) and $\overset{m}{T}_{ij}$ is the stress-energy tensor of matter.

5 Invariance to diffeomorphisms and conservation laws

Since we have proven that the total action $S = S_{fg} + S_m + S_i$ is equivalent to an action on the base manifold, it is enough to consider diffeomorphisms of M , as in Section 2.2. By a similar reasoning, we get that, as long as inhomogeneous Maxwell equations are satisfied by A , there hold the equalities:

$$\nabla_{\partial_j}(\mathcal{G}^{ij} - \frac{8\pi k^m}{c^4} T^{ij}) = 0,$$

which are equivalent to the following relations on TM :

$$(5.1) \quad \text{div}(\mathcal{G} - \frac{8\pi k^m}{c^4} T) = 0.$$

The above is just the usual energy-momentum conservation law, expressed in terms of the generalized Einstein tensor⁶.

Relation (5.1) is a consequence of the Bianchi identity for the horizontal component of the curvature and of Maxwell equations. The rest of the Bianchi identities for $\overset{\alpha}{D}$ do not yield any new information (the "perturbation" terms appearing from F cancel one another).

6 Equations of motion of charged particles

For a single charged particle, subject to the gravitational and the electromagnetic fields, we have:

$$S_{mi} = -mc \int \overset{\alpha}{L}(x, \dot{x}) dt,$$

with $\alpha = \frac{q}{mc^2}$. The equations of motion are:

$$(6.1) \quad \frac{\overset{\alpha}{D}y^i}{dt} = 0, \quad y^i = \dot{x}^i \quad (\alpha = \frac{q}{mc^2});$$

(where $t = \text{const} \cdot s$); for particles having the same ratio $\frac{q}{m}$, worldline deviation equations are given by

$$(6.2) \quad \frac{\overset{\alpha}{D}^2 w^i}{dt^2} = \overset{\alpha}{E}{}^i{}_j w^j, \quad \alpha = \frac{q}{mc^2}.$$

with E as in (3.10).

Acknowledgment. The work was supported by the Sectorial Operational Program Human Resources Development (SOP HRD), financed from the European Social Fund and by the Romanian Government under the Project number POSDRU/89/1.5/S/59323.

⁶Due to the presence of the stress-energy tensor of the electromagnetic field in the left hand side of (5.1), i.e., in the generalized Einstein tensor \mathcal{G} , the divergence $\text{div}(\mathcal{G})$ does not generally vanish; only the difference $\text{div}(\mathcal{G}) - \text{div}(\overset{m}{T})$ does. This is different from the classical setting, where both hand sides of the Einstein field equations represent tensors with zero divergence.

References

- [1] P.L. Antonelli, R.S. Ingarden, M. Matsumoto, *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology*, Kluwer Acad. Publ., Dordrecht, Boston, London, 1993.
- [2] E. Bertschinger, *Symmetry Transformations, the Einstein-Hilbert Action and Gauge Invariance*, MIT, 2002.
- [3] I. Bucataru, *Linear connections for systems of higher order differential equations*, Houston J. of Math. **31**(2) (2005), 315-331.
- [4] I. Bucataru, O. Constantinescu, M. Dahl, *A geometric setting for systems of ordinary differential equations*, Int. J. of Geom. Methods in Physics, 8(6) (2011), 1291-1327.
- [5] V. Balan, *Synge-Beil and Riemann-Jacobi jet structures with applications to physics*, Int. Journal of Math. and Math. Sci, (IJMMS), 27(2003) 1693-1702.
- [6] V. Balan, *Lagrangian structures in geometrized jet framework*, BSG Proc. 11(2004) 25-39.
- [7] Balan V., Stavrinou P.C., *Extended Maxwell equations with sources in geometrized jet framework*, Algebras, Groups and Geometries 21 (3) (2004) 263-276.
- [8] D. Bao, S.S. Chern, Z. Shen, *An Introduction to Riemann-Finsler Geometry*, Springer-Verlag, New York 2000.
- [9] R.G. Beil, *Finsler geometry and a unified field theory*, Contemporary Math., **196** (1996), 265-271.
- [10] R.G. Beil, *Comparison of unified field theories*, Tensor, N.S. **56** (1995), 175-184.
- [11] P.T. Chrusciel, *On the unified affine electromagnetism and gravitation theories*, Acta Physica Polonica, B15 (1) (1984), 35-51.
- [12] L.F.O. Costa, C.A.R. Herdeiro, *Gravitoelectromagnetic analogy based on tidal tensors*, Phys. Rev. D **78**(2) (2008), id: 024021.
- [13] M. Ferraris, J. Kijowski, *Unified geometric theory of electromagnetic and gravitational interactions*, Gen. Rel. and Gravitation 14 (1) (1982), 37-47.
- [14] H. Goenner, *On the History of Unified Field Theories*, Living Rev. Relativity, 7(2) (2004).
- [15] L.D. Landau, E.M. Lifschiz, *The Classical Theory of Fields*, 4th edn, Elsevier, 1975.
- [16] R. Miron, M. Anastasiei, *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Acad. Publ., Dordrecht, Boston, London, 1994.
- [17] R. Miron, *The geometry of Ingarden spaces*, Rep. on Math. Phys. **54**(2) (2004) 131-147.
- [18] R. Miron, R. Rosca, M. Anastasiei, K. Buchner, *New aspects in Lagrangian relativity*, Found. of Phys. Lett. **2**(5) (1992) 141-171.
- [19] R. Miron, H. Shimada, S.V. Sabau, M. Roman, *Geometrization of the scleronomic Riemannian mechanical systems*, Balkan J. of Geom. and Appl. 16(1) (2011) 78-97.
- [20] G. Munteanu, *A Yang-Mills electrodynamics theory on the holomorphic tangent bundle*, J. Nonlinear Math. Phys. **17**(2) (2010) 227-242.
- [21] L. Nicolescu, G.T. Pripoe, V. Damian, *On some families of linear connections*, Balkan J. of Geom. and Appl. 16(1) (2011) 98-110.

- [22] N.J. Poplawski, *The Maxwell Lagrangian in purely affine gravity*, Int. Journal of Modern Physics A, 23(3 & 4)(2008), 567-579.
- [23] C. Udriste, *Geometric dynamics*, Southeast Asian Bulletin of Math. 24(2000) 313322.
- [24] C. Udriste, A. Udriste, *Electromagnetic Dynamical Systems*, Balkan J. of Geom. and Appl. 2(1)(1997) 129-140.
- [25] N. Voicu, *On the fundamental equations of electromagnetism in Finslerian spacetimes*, Progress In Electromagnetics Research 113(2011) 83-102.
- [26] N. Voicu, *Tidal tensors in the description of gravity and electromagnetism*, arXiv:1111.1435v2 [math-ph], 2011; to appear in Journal of Nonlinear Mathematical Physics, DOI 10.1142/S1402925112500180.
- [27] M.I. Wanas, N.L. Youssef, A.M. Sid-Ahmed, *Teleparallel Lagrange Geometry and a Unified Field Theory*, Classical and Quantum Gravity, **27**(4) (2010), 045005 .

Author's address:

Nicoleta Voicu
Faculty of Mathematics and Computer Science,
50, Iuliu Maniu str., RO-500091 Brasov, Romania.
E-mail: nico.brinzei@unitbv.ro