# A geometrical overview of a mathematical model of malaria infection

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**Abstract.** In this paper we will describe some geometrical and dynamical properties of a mathematical model of malaria infection from the Poisson geometry point of view.

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**Key words**: Hamilton-Poisson system; Casimir function; nonlinear stability; Lax formulation; Kahan integrator; Runge-Kutta integrator.

## 1 Introduction

The model that we study was first proposed by Recker et al in [11] and studied in [8] or [12]. The model has the following form:

$$(1.1)\qquad \begin{cases} \dot{x_i} = x_i\left(\phi - \alpha y_i - \alpha' z_i\right), \\ \dot{y_i} = \beta x_i - \mu y_i, \\ \dot{z_i} = \beta' \sum\limits_{j=1}^{n} c_{ij} x_j - \mu' z_i, \end{cases}$$

where $i = 1, 2...n$. The variables $x_i, y_i$ and $z_i$ represent the abundance of the erythrocytes which are infected by the $i-$th parasite, and the magnitudes of the specific and cross-reactive immune response respectively. We assume that the immune responses are induced proportionally to the parasitic load at the rates $\beta$ and $\beta'$. The coefficients $\mu$ and $\mu'$ model the life-span of the corresponding immune responses. The efficiency of both responses are given by $\alpha$ and $\alpha'$. The coefficient $\phi$ represents the maximal growth rate of the parasite.

In all that follows we will consider the simplest case $n = 1$ so we rewrite the system (1.1) as:

$$(1.2)\qquad \begin{cases} \dot{x} = x\left(\phi - \alpha y - \alpha' z\right), \\ \dot{y} = \beta x - \mu y, \\ \dot{z} = \gamma x - \mu' z. \end{cases}$$

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The goal of our paper is to find the parameters' specific values for which the system (1.2) admits a Hamilton-Poisson realization. The Hamilton-Poisson realization offers us the tools to study the system (1.2) from mechanical geometry point of view.

The structure of this paper is as follows. In the second section of this work we prepare the framework of our study by writing the system as a Hamilton-Poisson one. The Poisson structure of the system and the Casimirs of our structure are presented here. The phase portrait of our dynamics is sketched in this section, too. The spectral stability and the nonlinear stability of the equilibrium states are presented in the third section. In the last section we give the Lax formulation of the system and we present numerical integration of our dynamics using two numerical integrators: Kahan's integrator and Runge-Kutta 4th steps one. Numerical simulations using MATHEMATICA 8.0 are presented, too.

To do this one needs first to find the constants of the motion of our system. Due to the numerous parameters of the system and trying to simplify the computation we shall focus to find only constants of motion being polynomials of degree at most three of the system (1.2).

**Proposition 1.1.** *The following smooth real functions $H$ are three degree polynomial constants of the motion defined by the system* (1.2)*:*

*(i) If $\alpha, \alpha', \beta, \beta', \gamma, \phi \in \mathbb{R}^*$, $\mu = \mu' = 0$, then the function*

$$H(x, y, z) = \beta z - \gamma y$$

*is the Hamiltonian of the system* (1.2)*.*

*(ii) If $\alpha', \beta, \beta', \gamma, \phi, \mu \in \mathbb{R}^*, \alpha = \mu' = 0$, then the function*

$$H(x, y, z) = \frac{\alpha'}{2\gamma} z^2 - \frac{\phi}{\gamma} z + x$$

*is the Hamiltonian of the system* (1.2)*.*

*(iii) If $\alpha, \beta, \beta', \gamma, \phi, \mu' \in \mathbb{R}^*, \alpha' = \mu = 0$, then the function*

$$H(x, y, z) = \frac{\alpha}{2\beta} z^2 - \frac{\phi}{\beta} z + x$$

*is the Hamiltonian of the system* (1.2)*.*

*Proof.* It is easy to see that $dH = 0$ for each case mentioned above. □

Let us focus now on the first case; if $\alpha, \alpha', \beta, \beta', \gamma, \phi \in \mathbb{R}^*$, $\mu = \mu' = 0$ the system (1.2) becomes:

$$\begin{cases} \dot{x} = x(\phi - \alpha y - \alpha' z), \\ \dot{y} = \beta x, \\ \dot{z} = \gamma x. \end{cases} \tag{1.3}$$

## 2 Hamilton-Poisson realizations

To find the Poisson structure in this case we shall use a method described by F. Haas and J. Goedert (see [4] for details ). Let us consider the skew-symmetric matrix given by:

$$\Pi := \begin{pmatrix} 0 & p_1(x,y,z) & p_2(x,y,z) \\ -p_1(x,y,z) & 0 & p_3(x,y,z) \\ -p_2(x,y,z) & -p_3(x,y,z) & 0 \end{pmatrix}.$$

We have to find the real smooth functions $p_1, p_2, p_3 : \mathbb{R}^3 \to \mathbb{R}$ such that:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \Pi \cdot \nabla H,$$

i.e. the following relations hold:

$$\begin{cases} -\gamma p_1(x,y,z) + \beta p_2(x,y,z) = x(\phi - \alpha y - \alpha' z), \\ \beta p_3(x,y,z) = \beta x, \\ \gamma p_3(x,y,z) = \gamma x. \end{cases}$$

It is easy to see that $p_3(x,y,z) = x$. Let us denote now $p_1(x,y,z) = p$.

Let us denote:

$$\begin{cases} v_1 := x(\phi - \alpha y - \alpha' z), \\ v_2 := \beta x, \\ v_3 := \gamma x. \end{cases}$$

The function $p$ is the solution of the following first order ODE (see [4] for details):

$$v_1 \frac{\partial p}{\partial x} + v_2 \frac{\partial p}{\partial y} + v_3 \frac{\partial p}{\partial z} = A \cdot p + B \tag{2.1}$$

where:

$$A = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} - \frac{\frac{\partial v_1}{\partial z}\frac{\partial H}{\partial x} + \frac{\partial v_2}{\partial z}\frac{\partial H}{\partial y} + \frac{\partial v_3}{\partial z}\frac{\partial H}{\partial z}}{\frac{\partial H}{\partial z}}$$

and

$$B = \frac{v_1 \frac{\partial v_2}{\partial z} - v_2 \frac{\partial v_1}{\partial z}}{\frac{\partial H}{\partial z}}.$$

The equation (2.1) becomes:

$$x(\phi - \alpha y - \alpha' z)\frac{\partial p}{\partial x} + \beta x \frac{\partial p}{\partial y} + \gamma x \frac{\partial p}{\partial z} = (\phi - \alpha y - \alpha' z)p + \alpha' x^2 \tag{2.2}$$

A solution of the equation (2.2) is $p = \frac{\alpha'}{\gamma} xz$. Now, one can reach the following result:

**Proposition 2.1.** *The system (1.3) has the Hamilton-Poisson realization:*

$$(\mathbb{R}^3, \Pi := \left[\Pi^{ij}\right], H),$$

*where*

$$\Pi = \begin{pmatrix} 0 & \dfrac{\alpha'}{\gamma}xz & -\dfrac{\alpha}{\beta}xy + \dfrac{\phi}{\beta}x \\ -\dfrac{\alpha'}{\gamma}xz & 0 & x \\ \dfrac{\alpha}{\beta}xy - \dfrac{\phi}{\beta}x & -x & 0 \end{pmatrix},$$

*and*

$$H(x, y, z) = \beta z - \gamma y.$$

**Remark 2.1.** There exists only one functionally independent Casimir of our Poisson configuration, given by $C : \mathbb{R}^3 \to \mathbb{R}$,

$$C(x, y, z) = \frac{\alpha}{2\beta}y^2 + \frac{\alpha'}{2\gamma}z^2 - \frac{\phi}{\beta}y + x.$$

**Proof:** Indeed, one can easily check that:

$$\Pi \cdot \nabla C = 0.$$

As the rank of $\Pi$ equals 2, it follows from the general theory of PDEs that $C$ is the only functionally independent Casimir function of the configuration (see e.g. reference [5] for details). The phase curves of the dynamics (1.3) are the intersections of the surfaces $H = const.$ and $C = const.$ (see Fig. 1).

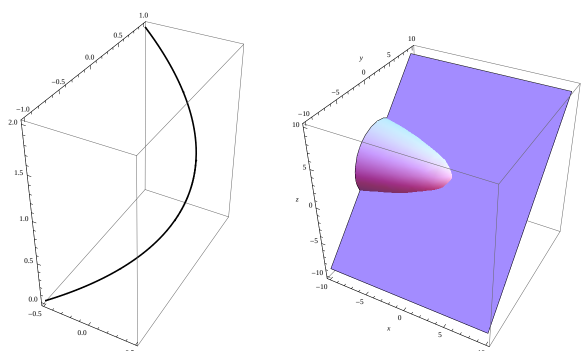

**Fig. 1. The phase curves of the dynamics** (1.3)**.**

The next result gives another Hamilton-Poisson realization of the system (1.3).

**Proposition 2.2.** *The system (1.3) may be modeled as an Hamilton-Poisson system in an infinite number of different ways, i.e. there are infinitely many different (in general non isomorphic) Poisson structures on* $\mathbb{R}^3$ *so that the system (1.3) is induced by an appropriate Hamiltonian.*

*Proof.* The triplets $\left(\mathbb{R}^3 \{\cdot\,,\,\cdot\}_{ab}\,, H_{cd}\right)$, where

$$\{f,g\}_{ab} = \nabla C_{ab} \cdot (\nabla f \times \nabla g)\,, \forall f,g \in C^\infty(\mathbb{R}^3,\mathbb{R}),$$

$$C_{ab} = aC + bH, \ \ H_{cd} = cC + dH, \ \ a,b,c,d \in \mathbb{R}, ad - bc = 1,$$

$$H = \beta z - \gamma y, \ \ C = x(\frac{\alpha}{2\beta}y^2 + \frac{\alpha'}{2\gamma}z^2 - \frac{\phi}{\beta}y + x)$$

define Hamilton-Poisson realizations of the dynamics (1.3). □

We further examine several geometrical and dynamical aspects of the system (1.3).

# 3 Stability problems

We discuss first the nonlinear stability of equilibrium states of our system (1.3) (see [6] for details).

It is obvious to see that the equilibrium points of our dynamics are given by:

$$e^{M,N} = (0, M, N), M, N \in \mathbb{R}.$$

Let $A$ be the matrix of the linear part of our system, i.e.

$$A = \begin{pmatrix} \phi - \alpha y - \alpha' z & -\alpha x & -\alpha' x \\ \beta & 0 & 0 \\ \gamma & 0 & 0 \end{pmatrix}.$$

Then the characteristic roots of $A(e^{M,N})$ are given by:

$$\lambda_{1,2} = 0, \lambda_3 = \phi - \alpha M - \alpha' N,$$

so we get the following result:

**Proposition 3.1.** *The equilibrium states $e^{M,N}, M, N \in \mathbb{R}$ are spectrally stable if $\phi - \alpha M - \alpha' N \leq 0$.*

Regarding the nonlinear stability we obtain

**Proposition 3.2.** *The equilibrium states $e^{M,N}$ are nonlinearly stable if $\phi - \alpha M - \alpha' N \leq 0$.*

*Proof.* We shall use the energy-Casimir method, see [1] for details. Let

$$H_{\varphi,\psi} = \frac{\gamma}{2\beta}y^2 + \frac{\beta}{2\gamma}z^2 - yz + \varphi(-\gamma y + \beta z) + \psi(\frac{\alpha}{2\beta}y^2 + \frac{\alpha'}{2\gamma}z^2 - \frac{\phi}{\beta}y + x)$$

be the energy-Casimir function, where $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ are two smooth real valued functions defined on $\mathbb{R}$.

The first variation of $H_{\varphi,\psi}$ equals zero at the equilibrium of interest if and only if

$$\dot{\varphi}(-\gamma M + \beta N) = \frac{M}{\beta} - \frac{N}{\gamma}$$

and

$$\dot{\psi}\left(-\frac{\alpha}{2\beta}M^2+\frac{\alpha'}{2\gamma}N^2-\frac{\phi}{\beta}M\right)=0.$$

Having chosen $\varphi$ and $\psi$ such that:

$$\begin{cases} \dot{\varphi}(-\gamma M+\beta N)=\frac{M}{\beta}-\frac{N}{\gamma} \\ \ddot{\varphi}(-\gamma M+\beta N)>0, \end{cases}$$

respectively

$$\begin{cases} \dot{\psi}(-\frac{\alpha}{2\beta}M^2+\frac{\alpha'}{2\gamma}N^2-\frac{\phi}{\beta}M)=0 \\ \ddot{\psi}(-\frac{\alpha}{2\beta}M^2+\frac{\alpha'}{2\gamma}N^2-\frac{\phi}{\beta}M)>0 \end{cases}$$

we can conclude that the second variation of $H_{\varphi,\psi}$ at the equilibrium of interest is positively defined and thus $e^{M,N}$ is nonlinearly stable. □

# 4 Lax formulation and numerical integration

**Proposition 4.1.** *The dynamics (1.3) allows a formulation in terms of Lax pairs.*

*Proof.* Let us consider:

$$L=\begin{pmatrix} 0 & u & v \\ -u & 0 & -i\sqrt{3}x+(-\sqrt{3}+i\alpha'\sqrt{\frac{3}{2}})z+\sqrt{3} \\ -v & i\sqrt{3}x-(-\sqrt{3}+i\alpha'\sqrt{\frac{3}{2}})z-\sqrt{3} & 0 \end{pmatrix},$$

and:

$$B=\begin{pmatrix} 0 & -\frac{2i}{\sqrt{3}}x & 2i\sqrt{\frac{2}{3}}x \\ \frac{2i}{\sqrt{3}}x & 0 & 2x \\ -2i\sqrt{\frac{2}{3}}x & -2x & 0 \end{pmatrix},$$

where

$$\begin{cases} u=-x-\frac{\alpha}{\sqrt{2}}y+iz+\frac{\phi}{\sqrt{2}}-\frac{\alpha\beta}{4}-i\frac{\gamma}{\sqrt{2}}-\frac{3\alpha'\gamma}{4}-i \\ v=\sqrt{2}x-\frac{\alpha}{2}y+\frac{1}{2}(-2i\sqrt{2}-3\alpha')z+i\sqrt{2}+\frac{\phi}{2}+\frac{\alpha\beta}{2\sqrt{2}}-i\frac{\gamma}{2}, \end{cases}$$

with $\phi,\alpha,\alpha',\beta,\gamma,\delta\in\mathbb{R}, i=\sqrt{-1}$. Then, using MATHEMATICA 8.0, we can put the system (1.3) in the equivalent form $\dot{L}=[L,B]$, as desired. □

We shall discuss now the numerical integration of the dynamics (1.3) via the Kahan integrator and also via the fourth-order Runge-Kutta integrator and we will point out some properties of Kahan's integrator. The Kahan's integrator [7] of the system (1.3) is given by:

$$(4.1)\qquad \begin{cases} x^{n+1}-x^n=\frac{h\phi}{2}(x^{n+1}+x^n)-\frac{h\alpha}{2}(x^{n+1}y^n+x^ny^{n+1}) \\ \qquad\qquad -\frac{h\alpha'}{2}(x^{n+1}z^n+z^ny^{n+1}) \\ y^{n+1}-y^n=\frac{h\beta}{2}(x^{n+1}+x^n) \\ z^{n+1}-z^n=\frac{h\gamma}{2}(x^{n+1}+x^n). \end{cases}$$

Using MATHEMATICA 8.0, we can prove the following proposition which shows the incompatibility of the Kahan integrator with the Poisson structure of the system (1.3).

**Proposition 4.2.** *Kahan's integrator (4.1) has the following properties:*

*(i) It is not Poisson preserving.*

*(ii) It does not preserve the Casimir $C$ of our Poisson configuration* $(\mathbb{R}^3, \Pi)$.

*(iii) It does not preserve the Hamiltonian $H$ of our system (1.3).*

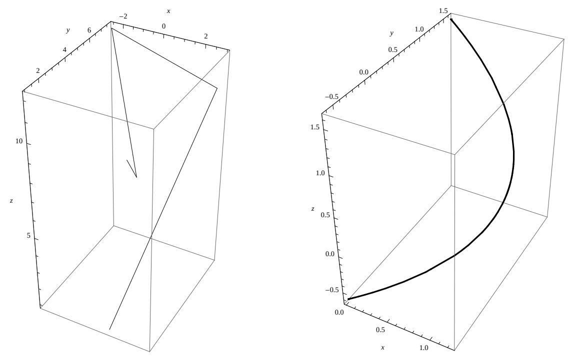

**Fig. 2. Kahan and Runge-Kutta 4 steps integrator, respectively ($x(0) = y(0) = z(0) = 1$).**

**Remark 4.1.** As we can see from the Figure 2 the two integrators give us almost the same results with the Figure 1, which is the exact solution of the system (1.3).

# 5 Conclusion

Trying to solve differential equations systems arising from different areas (like engineering, medicine, biology) we have to use numerical integrators. The exact solution of such a system often remains an open problem. The paper presents a geometrical overview of a mathematical model of malaria infection. The most important result is the possibility to obtain the exact solution of the system as the intersection between two surfaces: the first one given by the Hamiltonian of the system and the second one given by the Casimir of the Poisson configuration. Unfortunately, like other systems studied before - see for instance the Rikitake system ([13]), the Lotka-Volterra system ([9]), the Lorenz system ([3]) or the Lü system ([10]) - finding the corresponding Poisson structure implies the study of the system for particular values for its parameters.

More information about the application of geometrical methods in dynamical systems theory one can find in [2].

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