Isometric elastic deformations

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Abstract. This paper deals with the problem of finding a class of isometric deformations of simple and closed curves, which decrease the total squared curvature $\int \kappa^2$, called also the elastic potential. The resulting curve is an elastic curve, which in the case of plane curves is a circle.

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1 Introduction

As it was first noticed by Daniel Bernoulli in 1743 and then continued by Euler, the potential energy of an elastic lamina, which is straight in its natural position, is given by $\int \frac{1}{2}\kappa^2$, where κ denotes its curvature. The general case of $\int \kappa^m$ was also studied, but only the case $m = 2$ has a special physical significance.

Definition 1.1. A regular curve $c : [0, L] \to \mathbb{R}^3$ with prescribed endpoints and given length, which minimizes the elastic potential energy $\int_0^L \frac{1}{2}\kappa^2(s)\, ds$ is called an elastic curve.

Finding the plane elastic curves of given length τ, which pass through two given points A and B, and have prescribed tangent lines at these points, was a problem first posed by Bernoulli. Euler found a solution in mid 1700's that involves the use of elliptic integrals and leads to a complete classification of the plane elastic curves into 9 distinct types, see Love [5]. Euler's idea was to minimize the action integral $\frac{1}{2}\int \frac{y''(x)^2}{[1+y'(x)^2]^3}\, dx$ subject to the length constraint $\int \sqrt{1 + y'(x)^2}\, dx = \tau$.

Elastic curves can be also studied using the Lagrangian formalism. The next approach shows the relation with the pendulum equation. First, we state a formula for the curvature, which is due to Euler. Consider a plane curve $c(s) = \big(x(s), y(s)\big)$ parameterized by the arc length s, with $s \in [0, \tau]$, τ being the length of the curve.

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Since $\dot{x}^2(s)+\dot{y}^2(s)=1$, there is a smooth function $\theta(s)$ such that $\dot{x}(s)=\cos\theta(s)$ and $\dot{y}(s)=\sin\theta(s)$. Then the curvature $\kappa(s)$ can be written as

$$\kappa^2(s)=\ddot{x}^2(s)+\ddot{y}^2(s)=\dot{\theta}^2(s). \tag{1.1}$$

Since $\tan\theta(s)=\dfrac{\dot{y}(s)}{\dot{x}(s)}=\dfrac{dy}{dx}$, then $\theta(s)$ is the angle made by the velocity $\dot{c}(s)$ with the x-axis. Consequently, the following constraints must hold

$$\dot{x}=\cos\theta, \qquad \dot{y}=\sin\theta. \tag{1.2}$$

The variational method involving non-holonomic constraints considers the unit speed curve $c(s)=\big(x(s),y(s)\big)$, $|\dot{c}(s)|=1$, and expresses its curvature by the formula (1.1) considering the velocity constraints (1.2). The resulting Lagrangian is

$$\mathcal{L}(\theta,x,y,\dot{\theta},\dot{x},\dot{y})=\frac{1}{2}\dot{\theta}^2+\lambda_1(\dot{x}-\cos\theta)+\lambda_2(\dot{y}-\sin\theta), \tag{1.3}$$

with λ_1,λ_2 Lagrange multipliers. The elastic curves satisfy the Euler-Lagrange equations

$$\ddot{\theta} = \lambda_1\sin\theta-\lambda_2\cos\theta \tag{1.4}$$
$$\dot{\lambda}_1 = 0 \tag{1.5}$$
$$\dot{\lambda}_2 = 0. \tag{1.6}$$

It is interesting to note the relationship between the elastic curves and the pendulum equation. This comes from the fact that the first equation can be transformed into the standard pendulum equation

$$\ddot{u}+a^2\sin u=0 \tag{1.7}$$

by letting $a^2=\sqrt{\lambda_1^2+\lambda_2^2}$ and $u=\theta+\pi-\alpha$, with $\alpha=\tan^{-1}(\lambda_2/\lambda_1)$. It is well know that the solution of the pendulum equation (1.7) can be represented in terms of elliptic functions (see Giaquinta and Hildebrand [4], vol. I, p. 142)

$$\sin\frac{u(s)}{2}=k\,\mathrm{sn}(as,k),$$

where $k=\sin\dfrac{\gamma}{2}$ is the elliptic modulus and $\gamma=\max u(s)$ is the maximal amplitude of the swing. The value of the curvature is given by

$$\kappa(s)=\Big|\frac{d\theta}{ds}\Big|=\Big|\frac{du}{ds}\Big|=2k|\mathrm{cn}(s,k)|,$$

and then the minimum value of the elastic potential becomes

$$\begin{aligned}\int_0^\tau\frac{1}{2}\kappa^2(s)\,ds &= 2k^2\int_0^\tau(1-\mathrm{sn}^2(s,k))\,ds\\ &= 2\tau(k^2-1)+2E(\tau,k)\\ &= 2E(\tau,k)-2\tau k'^2,\end{aligned} \tag{1.8}$$

where we used the formula

$$k^2 \int \mathrm{sn}^2(s,k)\,ds = s - E(s,k),$$

see Lawden [6], p. 62. Here $E(\cdot,k)$ denotes Jacobi's epsilon function defined by

$$E(s,k) = \int_0^s \mathrm{dn}^2(u,k)\,du, \qquad k \in (0,1),$$

and $k^2 + k'^2 = 1$. The plane elastic curve $c(s) = \big(x(s), y(s)\big)$ can be obtained by integrating in the constraints (1.2), see Love [5]

$$x(s) = s - 2E(am(s,k),k) \tag{1.9}$$

$$y(s) = -2k\mathrm{cn}(s,k), \tag{1.10}$$

where $am(s,k)$ is defined as the value of ϕ such that

$$s = \int_0^\phi \frac{dt}{\sqrt{1-k^2\sin^2 t}}.$$

This variational treatment of plane elastic curves can be also found in Giaquinta and Hildebrand [4], vol I, p. 142. The work of Laplace (1807) and J. C. Maxwell makes the relationship between the elastica and the the shape of the capillary, see Miller [7]. This led to the result that the cross section of the capillary surface in a cylindrical tube is an elastic curve.

In order to find an explicit formula for a deformation which transforms any simple, closed curve into a circle, the heat kernel on the unit sphere $\mathbb{S}^1$ is needed. For a geometric method of obtaining the heat kernel the reader is referred to Calin et. al. [3]. The formula for the heat kernel on $\mathbb{S}^1$ depends on the theta function

$$\theta_3(z|i\tau) = 1 + 2\sum_{n\geq 1} e^{-\pi n^2\tau}\cos(2nz),$$

and it is given by

$$K(s_0, s; t, 0) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(s-s_0)^2}{2t}} \theta_3\left(\frac{\pi(s-s_0)i}{t} \,\Big|\, \frac{2\pi i}{t}\right). \tag{1.11}$$

This paper deals with the study of existence and construction of isometric deformations of curves into elastic curves. The plan of the paper is as follows. Section 2 deals with a characterization of the elastic oval curves. In section 3 we construct heat-type deformations that decrease the elastic potential and in section 4 we deal with there existence.

2 Simple closed elastic curves

An *oval curve* is a smooth, plane, closed, convex, and simple curve. Its curvature is smooth and positive everywhere.

Theorem 2.1. *If an oval curve of given length τ is an elastic curve, then it is a circle of radius $r = \tau/(2\pi)$.*

Proof. Consider the curve defined by the one-to-one mapping $c : [0, \tau] \to \mathbb{R}^2$ and parametrized by the arc length s, with boundary conditions $c(0) = c(\tau)$, $\dot{c}(0) = \dot{c}(\tau)$, $\ddot{c}(0) = \ddot{c}(\tau)$. If κ denotes the curvature, by Fenchel's formula we have $\int_0^\tau \kappa = 2\pi$. Then the right side of the following Cauchy's integral inequality is constant

$$\int_0^\tau \frac{1}{2}\kappa^2 \geq \frac{1}{2\tau}\Big(\int_0^\tau \kappa\Big)^2 = \frac{2\pi^2}{\tau}.$$

The curve is elastic if and only if the left side is minimum. This occurs when the Cauchy's inequality becomes equality, i.e when $\kappa =$ constant. Since the curve is plane, simple and closed, it must be a circle of radius $r = \tau/(2\pi)$. □

This result makes physical sense: if an elastic circular lamina is slightly deformed, then after its release, it gets back to its circular shape, which is the shape with the least elastic potential.

We note that if the *smoothness* condition at the endpoints is dropped, considering perpendicular endpoint velocities $\dot{c}(0) \perp \dot{c}(\tau)$, the elastic curve becomes the *rectangular elastica* discovered by Bernoulli.

A counterexample for the case when the curve is not simple is given by the Bernoulli's lemniscate, which is a curve in shape of an ∞.

A straight line segment of length τ is a counterexample for the case when the *closeness* condition is removed.

3 Smooth isometric deformations

Any plane simple closed smooth curve of length $\tau = 2\pi$ can be considered as the image of an isometric immersion φ from the unit circle $\mathbb{S}^1$ to the plane $\mathbb{R}^2$

$$\varphi(s) = \big(\varphi^1(s), \varphi^2(s)\big),$$

where $s \in [0, 2\pi]$ is the arc length and the following periodicity holds

$$\varphi(0) = \varphi(2\pi), \qquad \dot{\varphi}(0) = \dot{\varphi}(2\pi).$$

Consider a smooth deformation φ_t of the immersion φ, such that

(i) $\varphi_0 = \varphi$.

(ii) φ_t is an isometric immersion for $t \geq 0$, satisfying $\varphi_t(0) = \varphi_t(2\pi)$, $\dot{\varphi}_t(0) = \dot{\varphi}_t(2\pi)$.

(iii) The deformation follows the following heat-type equation

$$\partial_t \varphi_t(s) - \frac{1}{2}\partial_s^2 \varphi_t(s) = F(t, \varphi_t(s)),$$

with the source function F specified later. This equation provides the evolution of the deformation φ_t.

Definition 3.1. Let $T > 0$ finite or infinite. An elastic deformation is a deformation satisfying the foregoing properties $(i) - (iii)$, such that $\lim_{t \to T} \varphi_t(\mathbb{S}^1)$ is an elastic curve.

Obviously, an elastic deformation deforms the curve $\varphi(\mathbb{S}^1)$ into an elastic curve, which under the hypothesis of Theorem 2.1 is a circle.

In this paper we shall deal with the case of plane curves only. Since φ_t is an immersion, the curve $\varphi_t(\mathbb{S}^1)$ is simple and since φ_t is isometric, then all curves $\varphi_t(\mathbb{S}^1)$ have the same length τ. We shall construct elastic deformations by choosing the source function F such that the elastic potential $\int \frac{1}{2}\kappa_t^2$ of the curve $\varphi_t(s)$ decreases during the deformation.

The next result states that heat-type deformations with the source function linear in φ_t leave the elastic potential invariant. In particular, the elastic curves are preserved by this deformation.

Theorem 3.1. *Let $\lambda \neq 0$ constant. If the isometric immersion φ_t satisfies the equation*

$$\partial_t \varphi_t(s) - \frac{1}{2}\partial_s^2 \varphi_t(s) = \lambda^2 \varphi_t(s), \tag{3.1}$$

then the elastic potential $\frac{1}{2}\int \kappa_t^2$ is conserved, i.e., is equal to the constant value $2\pi\lambda^2$, for any $t \geq 0$.

Proof. Because of the isometry, all curves in the family are unit speed

$$\|\dot{\varphi}_t(s)\|^2 = 1, \qquad \forall t \geq 0.$$

Then the curvature of the φ_t-curve satisfies $\kappa_t^2(s) = \|\ddot{\varphi}_t(s)\|^2$. We shall denote $\partial_s \varphi_t(s)$ by $\dot{\varphi}_t(s)$. Using the deformation equation (3.1), the elastic potential can be written as

$$\begin{aligned}
\frac{1}{2}\int_0^{2\pi} \kappa_t^2(s)\,ds &= \frac{1}{2}\int_0^{2\pi} \|\ddot{\varphi}_t(s)\|^2\,ds = \int_0^{2\pi} \langle \ddot{\varphi}_t(s), \frac{1}{2}\ddot{\varphi}_t(s)\rangle\,ds \\
&= \int_0^{2\pi} \langle \ddot{\varphi}_t(s), \partial_t \varphi_t(s)\rangle\,ds - \lambda^2 \int_0^{2\pi} \langle \ddot{\varphi}_t(s), \varphi_t(s)\rangle\,ds \\
&= -\int_0^{2\pi} \langle \dot{\varphi}_t(s), \partial_t \dot{\varphi}_t(s)\rangle\,ds + \lambda^2 \int_0^{2\pi} \langle \dot{\varphi}_t(s), \dot{\varphi}_t(s)\rangle\,ds \\
&= -\frac{1}{2}\partial_t \int_0^{2\pi} \langle \dot{\varphi}_t(s), \dot{\varphi}_t(s)\rangle\,ds + 2\pi\lambda^2 \\
&= 2\pi\lambda^2,
\end{aligned}$$

since the curves are unit speed and we used integration by parts and the periodicity property (ii). Therefore, the elastic potential is constant and independent of the parameter t. □

Corollary 3.2. *There is no isometric deformation φ_t of the circle $\mathbb{S}^1$ satisfying the heat equation*

$$\partial_t \varphi_t(s) - \frac{1}{2}\partial_s^2 \varphi_t(s) = 0.$$

Proof. Assume there is an isometric deformation φ_t satisfying the heat equation. Then making $\lambda = 0$ in the proof of Theorem 3.1 yields

$$\frac{1}{2}\int_0^{2\pi} \kappa_t^2(s)\, ds = -\frac{1}{2}\partial_t \int_0^{2\pi} \langle \dot\varphi_t(s), \dot\varphi_t(s)\rangle\, ds = 0.$$

Hence $\kappa_t(s) = 0$, which is a contradiction, since this relation cannot hold for a closed simple curve. $\square$

The next result deals with the construction of a family of deformations that decrease the elastic potential. This result is needed for the construction of simple and closed elastic curves of given length. It is worth noting that the initial curve $\varphi_0(s)$ is not necessarily convex.

Theorem 3.3. *Let $\sigma(t)$ be a smooth function defined on $[0,\infty)$ such that*

$$\sigma(t) > 0, \qquad \sigma'(t) < 0, \qquad \lim_{t\to\infty} \sigma(t) = \sigma = \frac{1}{2}.$$

If the immersion φ_t satisfies the initial value problem

$$\partial_t \varphi_t(s) - \frac{1}{2}\partial_s^2 \varphi_t(s) = \sigma(t)\, \varphi_t(s) \tag{3.2}$$

$$\varphi_{|t=0} = \varphi_0, \tag{3.3}$$

then the elastic potential $\frac{1}{2}\int \kappa_t^2$ is a decreasing function of t. The limit curve, obtained by taking $t \to \infty$, if exists, is a circle.

Proof. With the notations of Theorem 3.1 we have

$$\begin{aligned}
\frac{1}{2}\int_0^{2\pi} \kappa_t^2(s)\, ds &= \frac{1}{2}\int_0^{2\pi} \|\ddot\varphi_t(s)\|^2\, ds = \int_0^{2\pi} \langle \ddot\varphi_t(s), \frac{1}{2}\ddot\varphi_t(s)\rangle\, ds \\
&= \int_0^{2\pi} \langle \ddot\varphi_t(s), \partial_t\varphi_t(s)\rangle\, ds - \sigma(t)\int_0^{2\pi} \langle \ddot\varphi_t(s), \varphi_t(s)\rangle\, ds \\
&= -\int_0^{2\pi} \langle \dot\varphi_t(s), \partial_t\dot\varphi_t(s)\rangle\, ds + \sigma(t)\int_0^{2\pi} \langle \dot\varphi_t(s), \dot\varphi_t(s)\rangle\, ds \\
&= -\frac{1}{2}\partial_t \int_0^{2\pi} \langle \dot\varphi_t(s), \dot\varphi_t(s)\rangle\, ds + 2\pi\sigma(t) \\
&= 2\pi\sigma(t),
\end{aligned}$$

and hence the elastic potential is a decreasing function of t.

Assume the limit $\phi = \varphi_\infty = \lim_{t\to\infty} \varphi_t$ exists. Then ϕ satisfies the steady-state equation associated to (3.2)

$$\ddot\phi(s) = -2\sigma\phi(s).$$

Since the curve is unit speed, differentiating in $\langle \dot\phi, \dot\phi\rangle = 1$ yields

$$0 = \langle \ddot\phi, \dot\phi\rangle = -2\sigma\langle \phi, \dot\phi\rangle,$$

and then

$$\frac{d}{ds}\langle\phi(s),\phi(s)\rangle = 2\langle\dot{\phi}(s),\phi(s)\rangle = 0.$$

Therefore $\|\phi(s)\|$ is constant, and since the curve is closed, it follows that $\phi(s)$ is a circle. Since this is a unit circle, the curvature is $\kappa = 1$, and we have

$$2\pi\sigma = \lim_{t\to\infty} 2\pi\sigma(t) = \lim_{t\to\infty}\frac{1}{2}\int_0^{2\pi}\kappa_t^2 = \frac{1}{2}\int_0^{2\pi}\kappa^2 = \pi,$$

which is satisfied for $\sigma = 1/2$. □

With the same proof, we can construct a deformation which evolves into a unit circle in finite time $T > 0$. We shall deal with the existence of the limit $\lim_{t\to\infty}\varphi_t$ in Section 4, where additional conditions on the initial curve φ_0 are required.

4 Existence of elastic deformations

In section 3 we have constructed a smooth deformation of the immersion φ_0, which decreases the elastic energy and transforms the initial curve of length 2π into a unit circle. In this section we shall treat the existence and uniqueness of this type of deformation.

In order to find elastic deformations of the aforementioned type, it suffices to solve the initial value problem on $\mathbb{S}^1$

$$\begin{aligned}
&(4.1) & \partial_t\varphi_t(s) - \frac{1}{2}\partial_s^2\varphi_t(s) &= \sigma(t)\,\varphi_t(s), \qquad s\in[0,2\pi]\\
&(4.2) & \varphi_{|t=0}(s) &= \varphi_0(s),\\
&& \varphi_t(0) &= \varphi(2\pi),\\
&& \partial_s\varphi_t(0) &= \partial_s\varphi(2\pi),
\end{aligned}$$

where the function $\sigma(t)$ tends decreasingly to $\frac{1}{2}$.

This equation can be transformed into a homogeneous heat equation as follows. Let $\rho(t) = \int_0^t \sigma(x)\,dx$, so $\rho(0) = 0$. Considering the substitution $u_t(s) = e^{-\rho(t)}\varphi_t(s)$, we have

$$\partial_t u_t(s) - \frac{1}{2}\partial_s^2 u_t(s) = e^{-\rho(t)}\Big[\partial_t\varphi_t(s) - \frac{1}{2}\partial_s^2\varphi_t(s) - \sigma(t)\varphi_t(s)\Big] = 0,$$

i.e. $u_t(s)$ satisfies the following initial value heat equation on $\mathbb{S}^1$

$$\begin{aligned}
\partial_t u_t(s) - \frac{1}{2}\partial_s^2 u_t(s) &= 0\\
u_0(s) &= \varphi_0(s)\\
u_t(0) &= u_t(2\pi),\\
\partial_t u_t(0) &= \partial_t u_t(2\pi).
\end{aligned}$$

It is known that this problem has a unique solution which is given by

$$(4.3) \qquad u_t(s) = \int_0^{2\pi} K(s,v;t,0)\varphi_0(v)\,dv, \qquad t\geq 0,$$

where $K(s, v; t, \tau)$ is the heat kernel on $\mathbb{S}^1$ given by formula (1.11). Substituting in (4.3) yields the following result regarding elastic deformations.

Theorem 4.1. *For any simple and closed curve of length $\tau = 2\pi$, there is an elastic deformation which deforms it into a circle of radius* 1*. This deformation is the unique solution of the equation* $(4.1-4.2)$ *and is given by*

(4.4)
$$\varphi_t(s) = \frac{e^{\int_0^t \sigma}}{\sqrt{2\pi t}} \int_0^{2\pi} e^{-\frac{(s-v)^2}{2t}} \theta_3\Big(\frac{\pi(s-v)i}{t}\Big|\frac{2\pi i}{t}\Big)\varphi_0(v)\,dv, \qquad 0 < t,\ 0 \le s \le 2\pi.$$

Each of the above deformations depends on $\sigma(s)$. It is worth noting that this theorem does not exclude the existence of other elastic deformations different than the family (4.4).

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