Parallel displacements of directions on the Grassmann-like manifold of centered planes

Olga Belova

Abstract. The Grassmann-like manifold $Gr^*(m,n)$ of centered m-planes L_m^* (it has the same dimension as the space of m-planes) is considered in projective space. The analog of the strong Norden normalization of the Grassmann-like manifold is realized and induced connections are considered. Parallel displacements of directions on this manifold are studied.

M.S.C. 2010: 53A20, 53B15, 53B25.
Key words: projective space; Grassmann-like manifold; connection; covariant differential; parallel displacement.

1 Introduction

Extensive theory of Grassmann submanifolds has been developed by school of Baltic geometers, basically, works of Yu.G. Lumiste [1], V.I. Bliznikas [2], I.V. Bliznikene [3], etc., but here essentially new results are obtained and approaches to researches differ from earlier used.

The object of research of the present paper are connections in the fiberings associated with Grassmann-like manifold of centered planes. The work concerns to researches in the area of differential geometry. The research is based on an application of the G.F. Laptev's method [4] of defining a connection in a principal fiber bundle and his method of continuations and scopes, which generalizes the moving frame method and Cartan exterior forms method. The research depends on calculation of exterior differential forms. Grassmann manifold and the space of centered planes in the projective space were already studied by autor in this direction [5].

This paper is devoted to studies of parallel displacements of directions in connections along lines on Grassmann-like manifold of centered planes.

BSG Proceedings 21. The International Conference "Differential Geometry - Dynamical Systems" DGDS-2013, October 10-13, 2013, Bucharest-Romania, pp. 9-13.

2 Grassmann-like manifold and parallel displacements

In n-dimensional projective space P_n we consider the moving frame $\{A, A_I\}$ $(I, \dots = \overline{1, n})$ with derivation formulas

$$dA = \theta A + \omega^I A_I, \quad dA_I = \theta A_I + \omega_I^J A_J + \omega_I A,$$

where the form θ plays the role of a proportionality factor and the structure forms ω^I, ω_J^I, ω_I of the projective group $GP(n)$, which acts effectively on P_n, satisfy the Cartan equations

$$D\omega^I = \omega^J \wedge \omega_J^I,$$
$$D\omega_J^I = \omega_J^K \wedge \omega_K^I + \delta_J^I \omega_K \wedge \omega^K + \omega_J \wedge \omega^I,$$
$$D\omega_I = \omega_I^J \wedge \omega_J.$$

The Grassmann-like manifold $Gr^*(m, n)$ of the centered m-planes L_m^* is considered in P_n. Let's produce a specialization of the moving frame $\{A, A_a, A_\alpha\}$ $(a, \dots = \overline{1, m}; \quad \alpha, \dots = \overline{m+1, n})$ putting the tops A and A_a on the plane $L_m^* = [A, A_a]$ and fixing the center A.

The Grassmann-like manifold $Gr^*(m, n)$ [6] is given by the equations

$$\omega^a = \Lambda_\alpha^a \omega^\alpha + \Lambda_\alpha^{ab} \omega_b^\alpha,$$

where Λ_α^a, Λ_α^{ab} are functions; ω^α, ω_a^α are basic forms of this manifold,

$$dim Gr^*(m, n) = (n - m)(m + 1).$$

The components of the fundamental object $\Lambda = \{\Lambda_\alpha^a, \Lambda_\alpha^{ab}\}$ satisfy the differential comparisons modulo the basic forms $\omega^\alpha, \omega_a^\alpha$:

$$\Delta\Lambda_\alpha^a + \Lambda_\alpha^{ab}\omega_b + \omega_\alpha^a \equiv 0, \quad \Delta\Lambda_\alpha^{ab} \equiv 0.$$

The principal fiber bundle $G^*(Gr^*(m, n))$ is constructed over the manifold $Gr^*(m, n)$ and the stationarity subgroup G^* of the centered plane L_m^* is the typical fiber.

In the principal fiber bundle the fundamental-group connection is given in G. F. Laptev's method:

$$\tilde{\omega}_b^a = \omega_b^a - \Gamma_{b\alpha}^a \omega^\alpha - L_{b\alpha}^{ac} \omega_c^\alpha, \quad \tilde{\omega}_\beta^\alpha = \omega_\beta^\alpha - \Gamma_{\beta\gamma}^\alpha \omega^\gamma - L_{\beta\gamma}^{\alpha a} \omega_a^\gamma,$$
$$\tilde{\omega}_\alpha^a = \omega_\alpha^a - \Gamma_{\alpha\beta}^a \omega^\beta - L_{\alpha\beta}^{ab} \omega_b^\beta, \quad \tilde{\omega}_a = \omega_a - L_{a\alpha}\omega^\alpha - \Pi_{a\alpha}^b \omega_b^\alpha,$$
$$\tilde{\omega}_\alpha = \omega_\alpha - L_{\alpha\beta}\omega^\beta - \Pi_{\alpha\beta}^a \omega_a^\beta.$$

The connection in the associative fibering $G^*(Gr^*(m, n))$ is defined by the field of the connection object $\Gamma = \{\Gamma_{b\alpha}^a, L_{b\alpha}^{ac}, \Gamma_{\beta\gamma}^\alpha, L_{\beta\gamma}^{\alpha a}, L_{a\alpha}, \Pi_{a\alpha}^b, \Gamma_{\alpha\beta}^a, L_{\alpha\beta}^{ab}, L_{\alpha\beta}, \Pi_{\alpha\beta}^a\}$ on the base $Gr^*(m, n)$:

$$\Delta\Gamma_{b\alpha}^a + L_{b\alpha}^{ac}\omega_c - \omega_{b\alpha}^a \equiv 0, \ \Delta L_{b\alpha}^{ac} - \omega_{b\alpha}^{ac} \equiv 0,$$
$$\Delta\Gamma_{\beta\gamma}^\alpha + L_{\beta\gamma}^{\alpha a}\omega_a - \omega_{\beta\gamma}^\alpha \equiv 0, \quad \Delta L_{\beta\gamma}^{\alpha a} - \omega_{\beta\gamma}^{\alpha a} \equiv 0,$$
$$\Delta\Gamma_{\alpha\beta}^a + L_{\alpha\beta}^{ab}\omega_b - \Gamma_{b\beta}^a \omega_\alpha^b + \Gamma_{\alpha\beta}^\gamma \omega_\gamma^a - \omega_{\alpha\beta}^a \equiv 0, \quad \Delta L_{\alpha\beta}^{ab} + L_{\alpha\beta}^{\gamma b}\omega_\gamma^a - L_{c\beta}^{ab}\omega_\alpha^c - \omega_{\alpha\beta}^{ab} \equiv 0,$$
$$\Delta L_{a\alpha} + (\Pi_{a\alpha}^b + \Gamma_{a\alpha}^b)\omega_b \equiv 0, \quad \Delta\Pi_{a\alpha}^b + L_{a\alpha}^{cb}\omega_c + \delta_a^b \omega_\alpha \equiv 0$$

$$\Delta L_{\alpha\beta}+(\Pi^a_{\alpha\beta}+\Gamma^a_{\alpha\beta})\omega_a-L_{a\beta}\omega^a_\alpha+\Gamma^\gamma_{\alpha\beta}\omega_\gamma\equiv 0,\quad \Delta\Pi^a_{\alpha\beta}-\Pi^a_{b\beta}\omega^b_\alpha+L^{ba}_{\alpha\beta}\omega_b+L^{\gamma a}_{\alpha\beta}\omega_\gamma\equiv 0,$$

where

$$\omega^a_{b\alpha}=\Lambda^a_\alpha\omega_b+\delta^a_b\Lambda^c_\alpha\omega_c+\delta^a_b\omega_\alpha,\quad \omega^{ac}_{b\alpha}=\Lambda^{ac}_\alpha\omega_b+\delta^a_b\Lambda^{ec}_\alpha\omega_e-\delta^c_b\omega^a_\alpha,$$

$$\omega^\alpha_{\beta\gamma}=\delta^\alpha_\beta\Lambda^a_\gamma\omega_a+\delta^\alpha_\beta\omega_\gamma+\delta^\alpha_\gamma\omega_\beta,\quad \omega^{\alpha a}_{\beta\gamma}=\delta^\alpha_\gamma\omega^a_\beta+\delta^\alpha_\beta\Lambda^{ba}_\gamma\omega_b.$$

$$\omega^a_{\alpha\beta}=\Lambda^a_\beta\omega_\alpha,\quad \omega^{ab}_{\alpha\beta}=\Lambda^{ab}_\beta\omega_\alpha.$$

An analog of the strong Norden normalization [7] for this manifold is carried out. It consists of the fields of the planes C_{n-m-1} and N_{m-1}:

$$L^*_m\cap C_{n-m-1}=\varnothing\,,\quad A\notin N_{m-1}\subset L^*_m.$$

The planes C_{n-m-1} and N_{m-1} we shall set by the points $B_\alpha=A_\alpha+\lambda^a_\alpha A_a+\lambda_\alpha A$ and $B_a=A_a+\lambda_a A$, respectively.

The differential equations for the components of the clothing quasitensor $\lambda=\{\lambda^a_\alpha,\lambda_\alpha,\lambda_a\}$ are of the form [8]:

$$\Delta\lambda_a+\omega_a=\lambda_{a\alpha}\omega^\alpha+\lambda^b_{a\alpha}\omega^\alpha_b,$$

$$\Delta\lambda^a_\alpha+\omega^a_\alpha=\lambda^a_{\alpha\beta}\omega^\beta+\lambda^{ab}_{\alpha\beta}\omega^\beta_b,$$

$$\Delta\lambda_\alpha+\lambda^a_\alpha\omega_a+\omega_\alpha=\lambda_{\alpha\beta}\omega^\beta+\chi^a_{\alpha\beta}\omega^\beta_a.$$

We have (see [6])

$$\overset{0}{\Gamma}{}^a_{b\alpha}=-\delta^a_b\lambda_\alpha+\mu^a_\alpha\lambda_b+\delta^a_b\mu^c_\alpha\lambda_c,\quad \overset{0}{L}{}^{ac}_{b\alpha}=\delta^c_b\lambda^a_\alpha-(\delta^a_b\Lambda^{ec}_\alpha+\delta^e_b\Lambda^{ac}_\alpha)\lambda_e,$$

$$\overset{0}{\Gamma}{}^\alpha_{\beta\gamma}=-\delta^\alpha_\gamma\lambda_\beta-\delta^\alpha_\beta\lambda_\gamma+\delta^\alpha_\beta\mu^a_\gamma\lambda_a,\quad \overset{0}{L}{}^{\alpha a}_{\beta\gamma}=-\delta^\alpha_\gamma\lambda^a_\beta-\delta^\alpha_\beta\Lambda^{ba}_\gamma\lambda_b,$$

$$\overset{01}{\Gamma}{}^a_{\alpha\beta}=-\Lambda^a_\beta\lambda_\alpha-\mu^a_\beta\lambda_b\lambda^b_\alpha,\quad \overset{01}{L}{}^{ab}_{\alpha\beta}=-\Lambda^{ab}_\beta\lambda_\alpha+\Lambda^{ab}_\beta\lambda^c_\alpha\lambda_c-\lambda^a_\beta\lambda^b_\alpha,$$

$$\overset{01}{L}{}_{a\alpha}=\mu^b_\alpha\lambda_a\lambda_b,\quad \overset{01}{\Pi}{}^b_{a\alpha}=\delta^b_a\lambda_\alpha-\Lambda^{cb}_\alpha\lambda_a\lambda_c,$$

$$\overset{01}{L}{}_{\alpha\beta}=-\lambda_\alpha\lambda_\beta+\lambda_a\lambda_\alpha\mu^a_\beta-\lambda_a\lambda_b\lambda^b_\alpha\mu^a_\beta,\quad \overset{01}{\Pi}{}^a_{\alpha\beta}=-\Lambda^{ba}_\beta\lambda_b\lambda_\alpha-\lambda_\beta\lambda^a_\alpha+\Lambda^{ba}_\beta\lambda^c_\alpha\lambda_b\lambda_c.$$

Theorem 2.1. *The analog of the strong Norden normalization of the Grassmann-like manifold $Gr^*(m,n)$ induces a connection in the associated fibering.*

We'll consider the straight line in the plane L^*_m and passing through a point A. It crosses an analogue of a normal of the 2nd type N_{m-1} in the point $B=\mu^a B_a=\mu^a(A_a+\lambda_a A)$. We have

$$dB=[\theta+(-\lambda_\alpha+\mu^b_\alpha\lambda_b)\omega^\alpha-\Lambda^{cb}_\alpha\lambda_c\omega^\alpha_b]B+$$

$$+\nabla\mu^a B_a+\mu^a[\lambda_a\omega^\alpha+\omega^\alpha_a]B_\alpha+$$

$$+\mu^a[(\lambda_{a\alpha}+\lambda_a\lambda_b\mu^b_\alpha-\lambda_a\lambda_\alpha)\omega^\alpha+(\lambda^b_{a\alpha}-\lambda_a\lambda_c\Lambda^{cb}_\alpha-\delta^b_a\mu_\alpha)\omega^\alpha_b]A,$$

where $\mu^a_\alpha=\lambda^a_\alpha-\Lambda^a_\alpha,\quad \mu_\alpha=\lambda_\alpha-\lambda_a\lambda^a_\alpha$ and $\nabla\mu^a=d\mu^a+\mu^b\tilde{\omega}^a_b$.

If the covariant differential of the geometrical object μ vanishes we get parallel displacement and we have

Theorem 2.2. *The straight line* $AB \subset L^*_m$ *determined by a point* $B \in N_{m-1}$ *(*$A \notin N_{m-1} \subset L^*_m$*) is parallel displaced in the plane linear connection* $\Gamma_1 = \{\Gamma^a_{b\alpha}, L^{ac}_{b\alpha}\}$ *iff the point* B *is displaced in a plane* $P_{n-m+1} = N_{n-m} + B$*, where* N_{n-m} *is an analogue of a normal of the 1st type (*$N_{n-m} = A + C_{n-m-1}$*).*

We consider a normal straight line AC intersecting an analog of the Cartan plane $C_{n-m-1} \subset N_{n-m}$ in the point $C = \mu^\alpha B_\alpha = \mu^\alpha(A_\alpha + \lambda^a_\alpha A_a + \lambda_\alpha A)$. We find the differential of the point C

$$dC = [\theta - (\Lambda^a_\beta \lambda_a + \mu_\beta)\omega^\beta - \Lambda^{ba}_\beta \lambda_b \omega^\beta_a]C + \nabla\mu^\alpha B_\alpha +$$

$$+\mu^\alpha[(\lambda^a_{\alpha\beta} - \lambda_\alpha \mu^a_\beta)\omega^\beta + (\lambda^{ab}_{\alpha\beta} + \lambda_\alpha \Lambda^{ab}_\beta - \lambda^a_\beta \lambda^b_\alpha)\omega^\beta_b]B_a +$$

$$+\mu^\alpha[(\lambda_{\alpha\beta} - \lambda_a \lambda^a_{\alpha\beta} + \lambda_a \lambda_\alpha \mu^a_\beta - \lambda_\alpha \lambda_\beta)\omega^\beta + (\chi^a_{\alpha\beta} - \lambda_b \lambda^{ba}_{\alpha\beta} - \lambda_b \lambda_\alpha \Lambda^{ba}_\beta - \lambda^a_\alpha \mu_\beta)\omega^\beta_a]A,$$

where $\nabla\mu^\alpha = d\mu^\alpha + \mu^\beta \tilde{\omega}^\alpha_\beta$.

Theorem 2.3. *The straight line* $AC \subset N_{n-m}$ *determined by a point* $C \in C_{n-m-1}$ *(*C_{n-m-1} *is an analogue of the Cartan plane) is parallel displaced in the normal linear connection* $\Gamma_2 = \{\Gamma^\alpha_{\beta\gamma}, L^{\alpha a}_{\beta\gamma}\}$ *iff the point* C *is displaced in a plane* $P_{m+1} = L^*_m + C$*.*

We consider general parallel displacements. We consider the point $M \in L_{n-1} = N_{m-1} + C_{n-m-1}$.

$$M = \eta^a B_a + \eta^\alpha B_\alpha.$$

We find the differential of the point M

$$dM = \theta M + [\nabla\eta^a + (-\eta^a \lambda_\alpha + \eta^a \mu^b_\alpha \lambda_b + \eta^\beta \lambda^a_{\beta\alpha} - \eta^\beta \lambda_\beta \mu^a_\alpha)\omega^\alpha +$$

$$+(\eta^c \lambda_c \Lambda^{ab}_\alpha + \eta^\beta \lambda^{ab}_{\beta\alpha} + \eta^\beta \lambda_\beta \Lambda^{ab}_\alpha - \eta^\beta \lambda^a_\alpha \lambda^b_\beta - \eta^a \Lambda^{cb}_\alpha \lambda_c - \eta^c \Lambda^{ab}_\alpha \lambda_c)\omega^\alpha_b]B_a +$$

$$+[\nabla\eta^\alpha + (\delta^\alpha_\beta \eta^a \lambda_a - \eta^\alpha \lambda_\beta + \eta^\alpha \mu^a_\beta \lambda_a)\omega^\beta + (\delta^\alpha_\beta \eta^a - \eta^\alpha \Lambda^{ba}_\beta \lambda_b)\omega^\beta_a]B_\alpha +$$

$$+[(\eta^a \lambda_{a\alpha} + \eta^a \lambda_a \lambda_b \mu^b_\alpha - \eta^a \lambda_a \lambda_\alpha - \eta^\beta \lambda_a \lambda^a_{\beta\alpha} + \eta^\beta \lambda_{\beta\alpha} - \eta^\beta \lambda_a \lambda_\beta \Lambda^a_\alpha - \eta^\beta \lambda_\beta \mu_\alpha)\omega^\alpha +$$

$$+(\eta^a \lambda^b_{a\alpha} - \eta^a \lambda_a \lambda_c \Lambda^{cb}_\alpha - \eta^b \mu_\alpha - \eta^\beta \lambda_a \lambda^{ab}_{\beta\alpha} + \eta^\beta \chi^b_{\beta\alpha} - \eta^\beta \lambda_a \lambda_\beta \Lambda^{ab}_\alpha - \eta^\beta \lambda^b_\beta \mu_\alpha)\omega^\alpha_b]A,$$

where $\nabla\eta^a = d\eta^a + \eta^b \tilde{\omega}^a_b$, $\nabla\eta^\alpha = d\eta^\alpha + \eta^\beta \tilde{\omega}^\alpha_\beta$.

Theorem 2.4. *The straight line* AM *is parallely displaced in the compound connection* $\Gamma_1 \cup \Gamma_2$ *by any displacement of the point* M*, i.e. parallel displacement in this connection is degenerated.*

References

[1] Yu. G. Lumiste, *Induced connection in the embeded projective and affine fiberings* (in Russian), Acad. Proc. of Tartu University, Tartu, 177 (1965) 6–42.

[2] V. I. Bliznikas, *Some questions of geometry of hypercomplexes of lines* (in Russian), The Proc. of the Geom. Seminar, Moscow, 6 (1974) 43–111.

[3] I. V. Bliznikene, *About geometry of hemiholonomic congruence of the first kind*, The Proc. of the Geom. Seminar, Moscow, 3 (1971) 125–148.

[4] G. F. Laptev, *Differential geometry of the embeded manifolds* (in Russian), The Proc. of Moscow Math. Society, Moscow, 2 (1953) 275–383.

[5] O. O. Belova, *Connections in fiberings associated with the Grassman manifold and the space of centered planes*, J. Math. Sci, Springer, 162, 5 (2009), 605–632.

[6] O. O. Belova, *The connection in the fibering associated with Grassmann-like manifold of centered planes* (in Russian), Vestnik CHGPU Cheboksary 68 (2006), 18–20.

[7] A. P. Norden, *Spaces with an Affine Connection* (in Russian), Nauka, Moscow 1976.

[8] O. O. Belova, *Connections of the 2nd type in the fibering associated with Grassmann-like manifold of centered planes* (in Russian), Diff. Geom. of Figure Manifolds, Kaliningrad. 38 (2007), 6–12.

Author's address:

Olga Belova
Institute of Applied Mathematics and Information Technologies,
Immanuel Kant Baltic Federal University,
A. Nevsky Street, 236041, Kaliningrad, Russia.
E-mail: olgaobelova@mail.ru