An overview for nonlinear A-Dirac equations with variable growth

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Abstract. In this survey we first briefly give some related definitions and notations that will be involved. Then we are confined to present current advances for nonlinear A-Dirac equations with variable growth. Furthermore, some unresolved problems to the above-mentioned topic are proposed.

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1 Introduction

The qualitative analysis of nonlinear partial differential equations involving differential operators with variable exponent is motivated by wide applications to various fields. Materials requiring such more advanced theory have been studied experimentally since the middle of last century. The first major discovery on electrorheological fluids is due to Willis Winslow in 1949. These fluids have the interesting property that their effective viscosity depends on the electric field in the fluid. The dramatical increase of the effective viscosity (or shear stress) is due to the existence of special particle structures that appear in the presence of an electric field hindering the flow. Winslow noticed that in such fluids (for instance Lithium polymetachrylate) the effective viscosity in an electrical field is inversely proportional to the strength of the field. The field induces string-like formations in the fluid, which are parallel to the field. They can raise the effective viscosity by as much as five orders of magnitude. This phenomenon is known as the *Winslow effect*. For a general account of the underlying physics we refer to Halsey [27]. For overviews of microscopic models in relationship with applications to electrorheology we refer the reader to Parthasarathy and Klingenberg [39] and Růžička [40]. Electrorheological fluids have been used in robotics and space technology. The experimental research has been done mainly in the USA, for instance in NASA laboratories. We also point out an interesting recent mathematical model developed by Rajagopal and Růžička [41]. The model takes into

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account the delicate interaction between the electromagnetic fields and the moving fluids. Particularly, in the context of continuum mechanics, these fluids are seen as non-Newtonian fluids. Other relevant applications of nonlinear equations involving differential operators with variable exponent include obstacle problems [42], porous medium equation [1], and Kirchhoff problems [2, 3].

Most materials can be modeled with sufficient accuracy using classical Lebesgue and Sobolev spaces, L^p and $W^{1,p}$, where p is a fixed constant. With the emergence of nonlinear problems in applied sciences, Lebesgue and Sobolev spaces L^p and $W^{1,p}$ have demonstrated their limitations in applications. A class of nonlinear problems with variable exponent is a new research field and reflects a new kind of physical phenomena. For instance, for some materials with inhomogeneities, e.g., the electrorheological fluids (sometimes referred to as "smart fluids"), this is not adequate, but rather the exponent p should be able to vary.

To the best of our knowledge, variable exponent Lebesgue spaces appeared in the literature for the first time already in a 1931 article by Orlicz [38]. In the 1950's this study was carried on by Nakano [34], who made the first systematic study of spaces with variable exponent (called *modular spaces*). Nakano mentioned explicitly variable exponent Lebesgue spaces as an example of more general spaces he considered, see Nakano [34]. Later, the Polish mathematicians investigated the modular function spaces (see e.g. Musielak [33]). Since Kováčik and Rákosník discussed $L^{p(x)}$ space and $W^{k,p(x)}$ space in [31], many results have been obtained concerning this kind of variable exponent spaces, see for instance [8–10, 10–12, 16] for basic properties of variable exponent spaces and [13–15, 17–19, 44] for the applications of variable exponent spaces on elliptic equations. For an overview of variable exponent spaces with various applications to differential equations we refer to [28, 43]. In particular, one of the reasons that forced the rapid expansion of the theory of variable exponent function spaces has been the models of electrorheological fluids introduced by Rajagopal and Růžička [41], which can be described by the boundary-value problem for the generalized Navier-Stokes equations.

As a powerful tool for solving elliptic boundary value problems in the plane, the methods of complex function theory play an important role. One way to extend these ideas to higher dimension is to start with a generalization of algebraic and geometrical properties of the complex numbers. In this way, Hamilton studied the algebra of quaternion in 1843. Further generalizations were introduced by Clifford in 1878. He initiated the so-called geometric algebras or Clifford algebras, which are generalizations of the complex numbers, the quaternions, and the exterior algebras, see [26]. Clifford analysis is usually the study of Dirac equation or of a generalized Cauchy-Riemann system, in which solutions are defined on domains in the Euclidean space and take values in Clifford algebras, see [4, 24, 45]. In particular, Gürlebeck and Sprößig [25, 35] developed an operator calculus, which is analogous to the known complex analytic approach in the plane and based on three operators: a Cauchy-Riemann-type operator, a Teodorescu transform, and a generalized Cauchy-type integral operator, to investigate elliptic boundary value problems of fluid dynamics over bounded and unbounded domains, especially the Navier-Stokes equations and related equations.

The purpose of this survey is to summarize advances of nonlinear A-Dirac equations. We are confined to the above topic for our personal interests, here we make no attempt at completeness because there has so far been many comprehensive surveys of differential equations with variable growth, for example, see [8, 28] and references therein. We hope that this article can give some basic ideas for further investigations in this field.

2 Definitions and notations

We first recall some related notions and results concerning Clifford algebra. For a detailed discussion we refer to [4, 24–26, 35–37, 45].

Let $C\ell_n$ be the real universal Clifford algebra over $\mathbb{R}^n$. Denote $C\ell_n$ by

$$C\ell_n = \operatorname{span}\{e_0, e_1, e_2, \ldots, e_n, e_1e_2, \ldots, e_{n-1}e_n, \ldots, e_1e_2\ldots e_n\},$$

where $e_0 = 1$(the identity element in $\mathbb{R}^n$), $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of $\mathbb{R}^n$ with the relation $e_ie_j + e_je_i = -2\delta_{ij}e_0$, where δ_{ij} is a Kronecker symbol. Thus the dimension of $C\ell_n$ is 2^n. We have an increasing tower $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset C\ell_3 \subset \cdots$, where $\mathbb{H}$ is the algebra of real quaternions. For $I = \{i_1, \ldots, i_r\} \subset \{1, \ldots, n\}$ with $1 \le i_1 < i_2 < \ldots < i_n \le n$, put $e_I = e_{i_1}e_{i_2}\ldots e_{i_r}$, while for $I = \varnothing$, $e_{\varnothing} = e_0$. For $0 \le r \le n$ fixed, the space $C\ell_n^r$ is defined by

$$C\ell_n^r = \operatorname{span}\{e_I : |I| := \operatorname{card}(I) = r\}.$$

The Clifford algebra $C\ell_n$ is a graded algebra as

$$C\ell_n = \bigoplus_{0 \le r \le n} C\ell_n^r.$$

Any element $a \in C\ell_n$ may thus be written in a unique way as

$$a = [a]_0 + [a]_1 + \ldots + [a]_n,$$

where $[\ ]_r : C\ell_n \to C\ell_n^r$ denotes the projection of $C\ell_n$ onto $C\ell_n^r$.

It is customary to identify $\mathbb{R}$ with $C\ell_n^0$ and identify $\mathbb{R}^n$ with $C\ell_n^1$ respectively. This means that each element x of $\mathbb{R}^n$ may be represented by $x = \sum_{i=1}^n x_ie_i$. From an analysis viewpoint, an extremely important property of the universal Clifford algebra is that every non-zero vector $x \in \mathbb{R}^n$ has a multiplicative inverse given by $-x/|x|^2$. Up to a sign this inverse corresponds to the kelvin inverse of a vector in Euclidean space.

Notice that there is an isometric isomorphism between the Grassman algebra and the Clifford algebra. Here the Grassman algebra is denoted by Λ^* with grading $\bigoplus_l \Lambda_l^*(\Omega)$. More precisely, the isometric isomorphism can be established by the linear extension of the map $\lambda : \Lambda^*(\Omega) \to C\ell_n(\Omega)$ defined on reduced multivectors:

$$\lambda : e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_l} \to e_{\alpha_1} \cdots e_{\alpha_l},$$

where $\{e_1, e_2, \ldots, e_n\}$ is independent of the choice of basis, see [36].

For $u \in C\ell_n$, we denotes by $[u]_0$ the scalar part of u, that is the coefficient of the element e_0. We define the Clifford conjugation as follows:

$$\overline{e_{i_1}e_{i_2}\ldots e_{i_r}} = (-1)^{\frac{r(r+1)}{2}} e_{i_1}e_{i_2}\ldots e_{i_r}.$$

For $A \in C\ell_n$, $B \in C\ell_n$, we have

$$\overline{AB} = \overline{B}\ \overline{A}, \quad \overline{\overline{A}} = A.$$

We denote

$$(A, B) = [\overline{A}B]_0.$$

Then an inner product is thus obtained, leading to the norm $|\cdot|$ on $C\ell_n$ given by

$$|A|^2 = [\overline{A}A]_0.$$

For all what follows let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a sufficiently smooth boundary $\partial\Omega$. A Clifford-valued function $u : \Omega \to C\ell_n$ can be written as $u = \Sigma_I u_I e_I$, where the coefficients $u_I : \Omega \to \mathbb{R}$ are real valued functions.

The Dirac operator on Euclidean space used here is as follows:

$$D = \sum_{j=1}^{n} \mathrm{e}_j \frac{\partial}{\partial x_j} = \sum_{j=1}^{n} \mathrm{e}_j \partial_j.$$

If u is C^1 real-valued function defined on a domain Ω in $\mathbb{R}^n$, then $Du = \nabla u = (\partial_1 u, \partial_2 u, \cdots, \partial_n u)$, where ∇ is the distributional gradient. Further $D^2 = -\Delta$, where Δ is the Laplace operator over $\mathbb{R}^n$ which operates only on coefficients. A function is left monogenic if it satisfies the equation $Du(x) = 0$ for each $x \in \Omega$. A similar definition can be given for right monogenic function. An important example of a left monogenic function is the generalized Cauchy kernel

$$G(x) = \frac{1}{\omega_n} \frac{\overline{x}}{|x|^n},$$

where ω_n denotes the surface area of the unit ball in $\mathbb{R}^n$. This function is a fundamental solution of the Dirac operator. Basic properties of left monogenic functions one can refer to [4, 24–26, 45].

Next we recall some basic properties of variable exponent spaces of Clifford-valued functions, for a detailed treat we refer to [20, 21].

Let $\mathcal{P}(\Omega)$ be the set of all Lebesgue measurable functions $p : \Omega \to (1, \infty)$. Given $p \in \mathcal{P}(\Omega)$ we define the conjugate function $p'(x) \in \mathcal{P}(\Omega)$ by

$$p'(x) = \frac{p(x)}{p(x) - 1}, \quad x \in \Omega.$$

We denote by w the Radon measure canonically associated with the weight $w(x)$ in the following way:

$$\mu(E) = \int_E \omega(x) dx.$$

Let $\mathcal{M}(\Omega)$ be the set of all measurable real functions defined on Ω. Note that two measurable functions are considered as the same element of $\mathcal{M}(\Omega)$ when they are equal almost everywhere. The variable exponent Lebesgue space $L^{p(x)}(\Omega, \omega)$ is defined by

$$L^{p(x)}(\Omega, \omega) = \Big\{ u \in \mathcal{M}(\Omega) : \int_\Omega |u(x)|^{p(x)} d\mu < \infty \Big\}.$$

with the norm

$$\|u\|_{L^{p(x)}(\Omega,\omega)} = \inf\Big\{t > 0 : \int_\Omega \Big|\frac{u(x)}{t}\Big|^{p(x)} d\mu \le 1\Big\},$$

and the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega,\omega) = \Big\{u \in L^{p(x)}(\Omega,\omega) : |\nabla u| \in L^{p(x)}(\Omega,\omega)\Big\},$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega,\omega)} = \|\nabla u\|_{L^{p(x)}(\Omega,\omega)} + \|u\|_{L^{p(x)}(\Omega,\omega)}. \tag{2.1}$$

Denote $W_0^{1,p(x)}(\Omega,\omega)$ by the completion of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega,\omega)$ with respect to the norm (2.1). The space $W^{-1,p(x)}(\Omega,\omega)$ is defined as the dual of the space $W_0^{1,p'(x)}(\Omega,\omega)$. For more details we refer to [16] and references therein.

In the sequel, we say that $u \in L^{p(x)}(\Omega, \mathrm{C}\ell_n,\omega)$ can be understood coordinatewisely. For instance, $u \in L^{p(x)}(\Omega, \mathrm{C}\ell_n,\omega)$ means that $\{u_I\} \subset L^{p(x)}(\Omega,\omega)$ for $u = \Sigma_I u_I e_I \in \mathrm{C}\ell_n$ with the norm $||u||_{L^{p(x)}(\Omega,\mathrm{C}\ell_n,\omega)} = \sum_I ||u_I||_{L^{p(x)}(\Omega,\omega)}$. Throughout this way, spaces $W^{1,p(x)}(\Omega, \mathrm{C}\ell_n,\omega)$, $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n,\omega)$, $C_0^\infty(\Omega, \mathrm{C}\ell_n,\omega)$, etc., can be understood similarly. In particular, if $\omega(x) \equiv 1$, then the space $L^2(\Omega, \mathrm{C}\ell_n)$ can be converted into a right Hilbert $\mathrm{C}\ell_n$-module by defining the following Clifford-valued inner product (see [24, Definition 3.74])

$$\big(f, g\big)_{\mathrm{C}\ell_n} = \int_\Omega \overline{f(x)} g(x) dx. \tag{2.2}$$

Definition 2.1.(see [39]) A function $a : \Omega \to \mathbb{R}$ is globally log-Hölder continuous in Ω if there exist $L_i > 0$ $(i = 1, 2)$ and $a_\infty \in \mathbb{R}^n$ such that

$$|a(x) - a(y)| \le \frac{L_1}{\log(\mathrm{e} + |x-y|^{-1})},\ |a(x) - a_\infty| \le \frac{L_2}{\log(\mathrm{e} + |x|)},$$

hold for all $x, y \in \Omega$. We define the following class of variable exponents

$$\mathcal{P}^{\log}(\Omega) = \Big\{p \in \mathcal{P}(\Omega) : \frac{1}{p} \text{ is globally log} - \text{Hölder continuous}\Big\}.$$

3 Advance in nonlinear A-Dirac equations

In this section, we will focus our attention on advance of nonlinear A-Dirac equations.

In [35, 36], Nolder first introduced A-Dirac equations $DA(x, Du) = 0$ and developed some tools for the study of weak solutions to nonlinear A-Dirac equations in the space $W_0^{1,p}(\Omega, \mathrm{C}\ell_n)$. Note that If $A(x, \xi) = \xi$, then A-Dirac equations $DA(x, Du) = 0$ becomes $-\triangle u = 0$, i.e., Clifford Laplacian equation. If $A(x, \xi) = |\xi|^{p-2}\xi$, then A-Dirac equations becomes $D(|Du|^{p-2}Du) = 0$, that is, p-Dirac equation, see [37]. Moreover, if u is real-valued function, then the scalar part of A-Dirac equations correspond to A-harmonic equations $-\mathrm{div}(A(x, \nabla u)) = 0$. These equations have been extensively studied with many applications, see [29]. It is worth pointing out that

Nolder [35, 36] showed a Caccioppoli-type estimate and the removability result for solutions to scalar part of A-Dirac equations.

However, the existence of weak solutions to the homogeneous A-Dirac equations was not proved in [35, 36]. Inspired by his works, Fu and Zhang were interested in the the existence of weak solutions for A-Dirac equations with variable growth. For this purpose, it is natural to consider variable exponent spaces as the corresponding working spaces. In [20], the authors introduced weighted variable exponent Lebesgue spaces $L^{p(x)}(\Omega, \mathrm{C}\ell_n, \omega)$ and weighted variable exponent Dirac Sobolev spaces $W^{D,p(x)}(\Omega, \mathrm{C}\ell_n, \omega)$ as follows.

Definition 3.1. The space $L^{p(x)}(\Omega, \mathrm{C}\ell_n, \omega)$ is defined by

$$L^{p(x)}(\Omega, \mathrm{C}\ell_n, \omega) = \Big\{ u \in \mathrm{C}\ell_n : u = \sum_I u_I e_I, u_I \in L^{p(x)}(\Omega, \omega) \Big\},$$

with the norm

$$\|u\|_{L^{p(x)}(\Omega, \mathrm{C}\ell_n, \omega)} = \Big\| \sum_I u_I e_I \Big\|_{L^{p(x)}(\Omega, \mathrm{C}\ell_n, \omega)} = \sum_I \|u_I\|_{L^{p(x)}(\Omega, \omega)},$$

and the Dirac Sobolev Space $W^{D,p(x)}(\Omega, \mathrm{C}\ell_n, \omega)$ is defined by

$$W^{D,p(x)}(\Omega, \mathrm{C}\ell_n, \omega) = \Big\{ u \in L^{p(x)}(\Omega, \mathrm{C}\ell_n, \omega) : Du \in L^{p(x)}(\Omega, \mathrm{C}\ell_n, \omega) \Big\},$$

with the norm

$$\|u\|_{W^{D,p(x)}(\Omega, \mathrm{C}\ell_n, \omega)} = \|u\|_{L^{p(x)}(\Omega, \mathrm{C}\ell_n, \omega)} + \|Du\|_{L^{p(x)}(\Omega, \mathrm{C}\ell_n, \omega)}.$$

and the Sobolev space $W^{1,p(x)}(\Omega, \mathrm{C}\ell_n, \omega)$ is defined by

$$W^{1,p(x)}(\Omega, \mathrm{C}\ell_n, \omega) = \Big\{ u \in L^{p(x)}(\Omega, \mathrm{C}\ell_n, \omega) : |\nabla u| \in L^{p(x)}(\Omega, \mathrm{C}\ell_n, \omega) \Big\},$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega, \mathrm{C}\ell_n, \omega)} = \|\nabla u\|_{L^{p(x)}(\Omega, \mathrm{C}\ell_n, \omega)} + \|u\|_{L^{p(x)}(\Omega, \mathrm{C}\ell_n, \omega)}. \tag{3.1}$$

Let $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n, \omega)$ be the completion of $C_0^\infty(\Omega, \mathrm{C}\ell_n)$ in $W^{1,p(x)}(\Omega, \mathrm{C}\ell_n, \omega)$ with respect to the norm (3.1). The space $W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n, \omega)$ is defined as the dual of the space $W_0^{1,p'(x)}(\Omega, \mathrm{C}\ell_n, \omega)$.

Theorem 3.1. *If $p(x) \in \mathcal{P}(\Omega)$, then the space $L^{p(x)}(\Omega, \mathrm{C}\ell_n, \omega)$ and $W^{D,p(x)}(\Omega, \mathrm{C}\ell_n, \omega)$ are reflexive Banach spaces.*

As an application, the existence and uniqueness of weak solutions to the scalar parts of nonhomogeneous A-Dirac equations $DA(x, Du) + B(x, u) = 0$ in spaces $W^{D,p(x)}(\Omega, \mathrm{C}\ell_n, \omega)$ were obtained under some natural assumptions. That is to say, there exists a unique Clifford valued solution $u \in W^{D,p(x)}(\Omega, \mathrm{C}\ell_n, \omega)$ to scalar part of the above equations such that

$$\int_\Omega \Big[\overline{A(x, Du)} D\varphi + \overline{B(x, u)} \varphi \Big]_0 dx = 0,$$

for any $\varphi \in W_0^{D,p(x)}(\Omega, C\ell_n, \omega)$. See [20] for the details.

Notice that $W^{1,p(x)}(\Omega, C\ell_n, \omega) \subset W^{D,p(x)}(\Omega, C\ell_n, \omega)$. Therefore, in order to replace Dirac Sobolev spaces $W^{D,p(x)}(\Omega, C\ell_n, \omega)$ by $W^{1,p(x)}(\Omega, C\ell_n)$, by the theory of Calderón-Zygmund in variable exponent spaces, the operator theory in classic Sobolev spaces of Clifford-valued functions was generalized to variable exponent spaces of Clifford-valued functions as follows, see [32]. In what follows, we always assume (unless declared specially) that $p \in \mathcal{P}^{\log}(\Omega)$ and $1 < p_- =: \inf_{x\in\bar{\Omega}} p(x) \le p(x) \le \sup_{x\in\bar{\Omega}} p(x) < +\infty$.

Theorem 3.2. *Let $p(x) \in \mathcal{P}(\Omega)$. Then $W^{1,p(x)}(\Omega, C\ell_n)$ is a separable and reflexive Banach space.*

Theorem 3.3. *If $u \in W_0^{1,p(x)}(\Omega, C\ell_n)$, then $\|u\|_{W_0^{1,p(x)}(\Omega, C\ell_n)}$ and $\|Du\|_{L^{p(x)}(\Omega, C\ell_n)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega, C\ell_n)$.*

As an application of above-mentioned spaces theory, the existence of weak solutions to the scalar parts of inhomogeneous Dirac systems $\overline{DA(x,u,Du)} = B(x,u,Du)$ in spaces $W_0^{1,p(x)}(\Omega, C\ell_n)$ was obtained under certain conditions, see [21, Theorem 3.1].

It is natural to consider the homogeneous A-Dirac questions $DA(x,Du) = 0$ under more natural conditions:

(A1) $A(x,\xi)$ is measurable with respect x for $\xi \in C\ell_n$ and continuous with respect to ξ for a.e. $x \in \Omega$;

(A2) $|A(x,\xi)| \le C_1|\xi|^{p(x)-1}$ for a.e. $x \in \Omega$ and $\xi \in C\ell_n$;

(A3) $\left[\overline{A(x,\xi)}\xi\right]_0 \ge C_2|\xi|^{p(x)}$ for a.e. $x \in \Omega$ and $\xi \in C\ell_n$;

(A4) $\left[\overline{(A(x,\xi_1) - A(x,\xi_2))}(\xi_1 - \xi_2)\right]_0 \ge 0$ for a.e. $x \in \Omega$ and $\xi_1, \xi_2 \in C\ell_n$.

In [48], the existence of weak solutions to the scalar part of homogeneous A-Dirac equations $DA(x,Du) = 0$ in spaces $W_0^{1,p(x)}(\Omega, C\ell_n)$ was proved under the assumptions (A1)–(A4).

It is crucial to consider the existence of weak solutions to nonlinear A-Dirac equations. For this, Fu, Zhang and Rădulescu [22] established a Hodge-type decomposition of variable exponent Lebesgue spaces of Clifford-valued functions as follows.

Theorem 3.4. *The space $L^{p(x)}(\Omega, C\ell_n)$ allows the Hodge-type decomposition*

$$L^{p(x)}(\Omega, C\ell_n) = (\ker\widetilde{D} \cap L^{p(x)}(\Omega, C\ell_n)) \oplus DW_0^{1,p(x)}(\Omega, C\ell_n) \tag{3.2}$$

with respect to the Clifford-valued product (2.2), where the operator $\widetilde{D} : L^{p(x)}(\Omega, C\ell_n) \to W^{-1,p(x)}(\Omega, C\ell_n)$ can be considered as a unique continuous linear extension of the Dirac operator.

Beginning with this decomposition we can get the following projections

$$\begin{cases} P : L^{p(x)}(\Omega, C\ell_n) \to \ker\widetilde{D} \cap L^{p(x)}(\Omega, C\ell_n) \\ Q : L^{p(x)}(\Omega, C\ell_n) \to DW_0^{1,p(x)}(\Omega, C\ell_n). \end{cases}$$

For $p(x) \equiv 2$, these are orthoprojections. Notice that from the proof of Theorem 3.7 we know

$$Q = D\Delta_0^{-1}\widetilde{D}, P = I - Q. \tag{3.3}$$

It follows from (3.3) that the operator Q as well as P maps the space $L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ into itself.

With the aid of Theorem 3.4, we obtain the following useful corollary.

Corollary 3.1. *The space $L^{p(x)}(\Omega, \mathrm{C}\ell_n) \cap \mathrm{im}Q$ is a closed subspace of $L^{p(x)}(\Omega, \mathrm{C}\ell_n)$.*

Corollary 3.2. $\left(L^{p(x)}(\Omega, \mathrm{C}\ell_n) \cap \mathrm{im}Q\right)^* = L^{p'(x)}(\Omega, \mathrm{C}\ell_n) \cap \mathrm{im}Q$.

With the aid of this decomposition, together with the Minty-Browder theorem, existence and uniqueness of a weak solution to the A-Dirac equations $DA(Du) = 0$ were obtained under the following assumptions:

(H1) $|A(\xi) - A(\eta)| \leq C_1{}'(|\xi| + |\eta|)^{p(x)-2}|\xi - \eta|$;

(H2) $\left[\overline{\left(A(\xi) - A(\eta)\right)}(\xi - \eta)\right]_0 \geq C_2{}'(|\xi| + |\eta|)^{p(x)-2}|\xi - \eta|^2$;

(H3) $A(0) \in L^{p'(x)}(\Omega, \mathrm{C}\ell_n)$,

where ξ and η are arbitrary elements from $\mathrm{C}\ell_n$, both $C_1{}'$ and $C_2{}'$ are positive constants independent of ξ and η. Notice that when $A(\xi) = |\xi|^{p(x)-2}\xi$, then the equations $DA(Du) = 0$ generalizes the important case of the equation $D\left(|Du|^{p(x)-2}Du\right) = 0$.

Obviously, the conditions (A1)–(A4) are weaker than those of (H1)–(H3) in a sense. Therefore, one may ask: does there exists a unique weak solution to the homogeneous A-Dirac equations under the assumptions (A1)–(A4)? In [47], Zhang, Molica Bisci and Rădulescu gave the positive answer to the homogeneous A-Dirac equations $DA(x, Du) = 0$. Furthermore, they proceeded to consider the inhomogeneous A-Dirac equations $DA(x, Du) = Df$, where $f \in L^{p'(x)}(\Omega, \mathrm{C}\ell_n)$. Their result reads as follows:

Theorem 3.5. *Under the assumptions* (*A*1), (*A*2), (*A*3) *and* (*A*4), *for each $f \in L^{p'(x)}(\Omega, \mathrm{C}\ell_n)$, there exists a weak solution $u \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$ to the A-Dirac equations $DA(x, Du) = Df$, that is to say, there exists a Clifford-valued function $u \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$ such that*

$$\int_\Omega \overline{A(x, Du)}Dvdx = \int_\Omega \overline{f}Dvdx$$

for any $v \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$. Furthermore, the solution to the scalar part of the above equations is unique up to a monogenic function.

In [23], Fu and Guo obtained the existence of weak solutions to nonhomogeneous A–harmonic equations $d^*A(x, du(x)) + B(x, u(x)) = 0$ in space $W_0^{1,p(x)}(\Omega, \wedge^{l-1})$, where $\wedge^{l-1}$ denotes differential forms of degree $l - 1$. Inspired by [23, 47], one may ask the following question:

Open Question 3.1. Does there exists a unique weak solution to the nonhomogeneous A-Dirac equations $DA(x, Du) = B(x, Du)$? where $A : \Omega \times \mathrm{C}\ell_n \to \mathrm{C}\ell_n$ satisfies the conditions (A1)–(A4) and $B : \Omega \times \mathrm{C}\ell_n \to \mathrm{C}\ell_n$ satisfies the growth condition.

It is worth pointing out that an A-harmonic equation $\mathrm{div}A(x, \nabla u) = 0$ is the scalar part of the equations (1.1) under appropriate identifications, see [20, 29]. When u is a real function, Du can be identified with ∇u. Hence the equations $DA(x, Du) = B(x, Du)$ corresponds to the nonhomogeneous A-harmonic equation $\mathrm{div}A(x, \nabla u) = B(x, \nabla u)$. For detailed account we refer to [5, 8, 46] and references therein.

More generally, one may consider the following question:

Open Question 3.2. Does there exists a unique weak solution to the nonhomogeneous A-Dirac equations $DA(x, u, Du) = B(x, u, Du)$? where $A : \Omega \times \mathrm{C}\ell_n \to \mathrm{C}\ell_n$ and $B : \Omega \times \mathrm{C}\ell_n \to \mathrm{C}\ell_n$ satisfies the certain growth conditions.

Another important subject for nonlinear A-Dirac equations is regularity. Alkhutov has proved in [6] that bounded supersolutions satisfy Harnack's weak inequality and solutions satisfy Harnack's inequality. Harjulehto, Kinnunen and Lukkari [30] generalize his result to unbounded supersolutions. Both papers apply Moser's iteration which heavily depends on the Caccioppoli estimate. For the detailed account we refer to the excellent survey [28] and references therein.

It is natural to ask if there has the corresponding Caccioppoli-type estimate for A-Dirac equations. In [47], Zhang, Molica Bisci and Rădulescu obtain the following Caccioppoli-type estimate.

Theorem 3.6. *Let $p(x) \in \mathcal{P}(\Omega)$. If u be a weak solution to the scalar part of $DA(x, Du) = 0$ and $\eta \in C_0^\infty(\Omega)$ with $0 < \eta \leq 1$, then*

$$\int_\Omega |Du|^{p(x)} \eta^{p_+} dx \leq \Big(1 + \frac{2C_1 p_+}{C_2}\Big)^{p_+} \int_\Omega |u|^{p(x)} |\nabla \eta|^{p(x)} dx.$$

In [7], Chen and Wang considered the inhomogeneous A-Dirac equations $DA(x, Du) = f(x, Du)$ in space $W_0^{1,p}(\Omega, \mathrm{C}\ell_n)$. It is showed that under suitable conditions, the solutions to the inhomogeneous A-harmonic equations if f satisfies the controllable growth condition in fact is a scalar part of weak solutions to the corresponding inhomogeneous A-Dirac equations, see [7, Theorem 1.1]. The proof applied a suitable Caccioppoli-type estimate for the above equations. Motivated by their works, we may ask the following questions:

Open Question 3.3. Does there has a Caccioppoli-type estimate for the inhomogeneous A-Dirac equations $DA(x, Du) = f(x, Du)$ in space $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$ under certain growth conditions?

In [32], Y. Lu and G. Bao were concerned with the regularity properties of weak solutions in $W_0^{1,p}(\Omega, \mathrm{C}\ell_n^k)$ to the obstacle problem for homogeneous A-Dirac equations, such as a global reverse Hölder inequality and stability. In [47], Zhang, Molica Bisci and Rădulescu considered the following stability for the inhomogeneous A-Dirac equations $DA(x, Du) = Df$ if the condition (A4) is replaced by the following strong monotonicity:

$$(A4') \quad \big[\overline{(A(x, \xi_1) - A(x, \xi_2))}(\xi_1 - \xi_2)\big]_0 \geq C_3 |\xi_1 - \xi_2|^{p(x)},$$

where $C_3 > 0$ is a constant.

Theorem 3.7. *Under the assumptions (A1), (A2), (A3) and (A4′), for given $f, g \in L^{p'(x)}(\Omega, \mathrm{C}\ell_n)$, each of the two equations*

$$\begin{cases} DA(x, Du) = Df \\ u \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n), \end{cases} \qquad \begin{cases} DA(x, Dv) = Dg \\ v \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n), \end{cases}$$

has a unique weak solution and

$$\min\left\{ \|u - v\|^{p_+ - 1}_{W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)}, \|u - v\|^{p_- - 1}_{W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)} \right\} \leq C(n, p, \Omega) \|f - g\|_{L^{p'(x)}(\Omega, \mathrm{C}\ell_n)}.$$

The usual problem is that whether the result can be extended to the stability case for the inhomogeneous A-Dirac equations $DA(x, Du) = B(x, Du)$. This is an interesting topic to explore.

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