

New results on two dimensional \mathbb{R} -complex Finsler spaces

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Abstract. In this paper we study two - dimensional \mathbb{R} complex Hermitian Finsler space. Using the Berwald frame $\{l, m, \bar{l}, \bar{m}\}$, we find two vertical curvature invariants $\mathbf{I}_1, \mathbf{I}_2$ which characterize the holomorphic sectional curvatures in direction l and m respectively, of a such space. Finally, some classes of two - dimensional \mathbb{R} complex Hermitian Finsler spaces are studied. In this case, we obtain that the notions of Kähler and strongly Kähler coincide.

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Key words: \mathbb{R} - complex Hermitian Finsler space, \mathbb{R} - complex Berwald frame, vertical curvature invariant.

1 Introduction

The notion of \mathbb{R} complex Finsler space was introduced in [8], p. 176, in order to obtain a Legendre duality between complex Randers and Kropina metrics, but their study is quite new and it has been initiated in [9]. The concept of \mathbb{R} complex Finsler manifold is a direct extension of the complex Finsler one, reducing the scalars to $\lambda \in \mathbb{R}$, in the homogeneity condition.

In two-dimensional real and complex Finsler geometry are known important results ([7, 3, 2]) and a powerful instrument like Berwald frame, is well known.

The aim of this paper is to investigate two-dimensional \mathbb{R} complex Finsler spaces, by means of a Berwald frame in which the orthogonality is with respect to Hermitian structure, defined by the fundamental metric tensor of a two-dimensional \mathbb{R} complex Hermitian Finsler space.

In section 2, we recall the main properties of the n - dimensional \mathbb{R} complex Finsler spaces, which are completed with some others needed.

In section 3 we give the constuction of the Berwald frame of a two-dimensional \mathbb{R} complex Hermitian Finsler space (M, F) , which have important properties. Also, we find the expression of the Cartan with respect to this frame and obtain the explicit

form of the \mathbb{R} complex Finsler metric F of a two-dimensional \mathbb{R} complex Hermitian Finsler space (Theorem 3.1).

Considering $v\bar{v}$ - Riemann type tensor, we find two vertical curvature invariants \mathbf{I}_1 and \mathbf{I}_2 and by means of these we define and compute the vertical holomorphic sectional curvatures in direction l and m . (Theorem 4.1). There results are in Section 4.

In Section 5 we study some classes of \mathbb{R} complex Hermitian Finsler space. We prove that in a two-dimensional \mathbb{R} complex Hermitian Finsler space the notions of Kähler and strongly Kähler coincide (Theorem 5.1). Also, we give some necessary conditions under which this class of spaces is a Berwald one, (Theorem 5.2).

2 \mathbb{R} complex Hermitian Finsler space

Let M be a n - dimensional complex manifold, $z = (z^k)_{k=1, \dots, n}$ are the complex coordinates in a local chart. The holomorphic tangent bundle $T'M$ is itself a complex manifold, and the coordinates in a local chart will be denoted by $u = (z^k, \eta^k)_{k=1, \dots, n}$.

Definition 2.1. A \mathbb{R} complex Finsler space is a pair (M, F) , where F is a continuous function $F : T'M \longrightarrow \mathbb{R}_+$ satisfying:

- i) $F := L^2$ is smooth on $\widetilde{T'M} := T'M \setminus \{0\}$;
- ii) $F(z, \eta) \geq 0$ the equality holds if and only if $\eta = 0$;
- iii) $F(z, \lambda\eta, \bar{z}, \lambda\bar{\eta}) = |\lambda| F(z, \eta, \bar{z}, \bar{\eta}), \forall \lambda \in \mathbb{R}$.

We use the following tensors

$$(2.1) \quad g_{ij} = \frac{\partial^2 L}{\partial \eta^i \partial \eta^j} ; g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j} ; g_{\bar{i}j} = \frac{\partial^2 L}{\partial \bar{\eta}^i \partial \eta^j}$$

and some consequences of the homogeneity condition *iii*), [9],

$$(2.2) \quad \begin{aligned} \frac{\partial L}{\partial \eta^i} \eta^i + \frac{\partial L}{\partial \bar{\eta}^i} \bar{\eta}^i &= 2L ; g_{ij} \eta^i + g_{\bar{j}i} \bar{\eta}^i = \frac{\partial L}{\partial \eta^j} ; \\ 2L &= g_{ij} \eta^i \eta^j + 2g_{i\bar{j}} \eta^i \bar{\eta}^j + g_{\bar{i}j} \bar{\eta}^i \eta^j ; \\ \frac{\partial g_{ik}}{\partial \eta^j} \eta^j + \frac{\partial g_{ik}}{\partial \bar{\eta}^j} \bar{\eta}^j &= 0 ; \frac{\partial g_{i\bar{k}}}{\partial \eta^j} \eta^j + \frac{\partial g_{i\bar{k}}}{\partial \bar{\eta}^j} \bar{\eta}^j = 0. \end{aligned}$$

Also, we consider the Cartan tensors

$$(2.3) \quad C_{ijk} = \frac{\partial g_{ij}}{\partial \eta^k} ; C_{i\bar{j}\bar{k}} = \frac{\partial g_{ij}}{\partial \bar{\eta}^k} = \frac{\partial g_{i\bar{k}}}{\partial \eta^j},$$

and their conjugates, which have the properties

$$(2.4) \quad \begin{aligned} C_{ijk} &= C_{ikj} = C_{jik}, & C_{i\bar{j}\bar{k}} &= C_{\bar{j}i\bar{k}} ; \\ C_{ij0} + C_{i\bar{j}0} &= 0, & C_{i0\bar{j}} + C_{i\bar{j}0} &= 0, \end{aligned}$$

together their conjugates, where $C_{ij0} := C_{ijk} \eta^k$ and $C_{i\bar{j}0} := C_{i\bar{j}\bar{k}} \bar{\eta}^k$.

If in an \mathbb{R} complex Finsler space, we suppose that F satisfies the regularity conditions: $g_{i\bar{j}}$ is nondegenerated, (i.e., $\det(g_{i\bar{j}}) \neq 0$, in any $u \in \widetilde{T'M}$), and it defines

a positive definite Levi-form for all $z \in M$, then a such class of spaces is called \mathbb{R} complex Hermitian Finsler space, [9].

Consider the sections of the complexified tangent bundle of $T'M$. Let $VT'M \subset T'(T'M)$ be the vertical bundle, locally spanned by $\{\frac{\partial}{\partial \eta^k}\}$, and $VT''M$ its conjugate. The idea of complex nonlinear connection, briefly (*c.n.c.*), is an instrument in 'linearization' of the geometry of the manifold $T'M$. A (*c.n.c.*) is a supplementary complex subbundle to $VT'M$ in $T'(T'M)$, i.e. $T'(T'M) = HT'M \oplus VT'M$. The horizontal distribution $H_u T'M$ is locally spanned by $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\}$, where $N_k^j(z, \eta)$ are the coefficients of the (*c.n.c.*). The pair $\{\delta_k := \frac{\delta}{\delta z^k}, \dot{\delta}_k := \frac{\partial}{\partial \eta^k}\}$ will be called the adapted frame of the (*c.n.c.*), which obey the change rules $\delta_k = \frac{\partial z'^j}{\partial z^k} \delta'_j$ and $\dot{\delta}_k = \frac{\partial z'^j}{\partial z^k} \dot{\delta}'_j$. By conjugation everywhere we obtain an adapted frame $\{\delta_{\bar{k}}, \dot{\delta}_{\bar{k}}\}$ on $T''(T'M)$. The dual adapted bases are $\{dz^k, \delta \eta^k\}$ and $\{d\bar{z}^k, \delta \bar{\eta}^k\}$.

A complex nonlinear connection for \mathbb{R} complex Hermitian Finsler space (M, F) , called Chern-Finsler (*c.n.c.*), is given by

$$(2.5) \quad N_j^h := g^{\bar{i}h} \left(\frac{\partial g_{\bar{i}1}}{\partial z^j} \bar{\eta}^r + \frac{\partial g_{s1}}{\partial z^j} \eta^s \right).$$

In an \mathbb{R} complex Hermitian Finsler space, we have recovered the *Chern-Finsler complex linear connection*, ([9]), which is metrical of $(1, 0)$ - type, and it is given by

$$(2.6) \quad L_{jk}^i = g^{\bar{m}i} \delta_j(g_{k\bar{m}}); \quad C_{jk}^i = g^{\bar{m}i} \dot{\delta}_j(g_{k\bar{m}}); \quad L_{j\bar{k}}^i = C_{j\bar{k}}^i = 0,$$

where here and further on δ_k is the adapted frame of the Chern-Finsler (*c.n.c.*) (2.5). Moreover, we have

$$(2.7) \quad L_{jk}^i = \dot{\delta}_j N_k^i; \quad N_k^i = L_{jk}^i \eta^j + (\dot{\delta}_{\bar{r}} N_k^i) \bar{\eta}^r; \quad 2G^i = N_j^i \eta^j.$$

In [4] are introduced the following notions. A \mathbb{R} complex Hermitian Finsler space (M, F) is *strongly Kähler* iff $T_{jk}^i = 0$, *Kähler* iff $T_{jk}^i \eta^j = 0$ and *weakly Kähler* iff $g_{i\bar{l}} T_{jk}^i \eta^j \bar{\eta}^l = 0$, where $T_{jk}^i := L_{jk}^i - L_{kj}^i$. Also, (M, F) is called *Berwald* if is Kähler with the coefficients L_{jk}^i depend only on z .

Theorem 2.1. [4]. *Let (M, F) be an \mathbb{R} complex Hermitian Finsler spaces. If G^i are holomorphic with respect to η and (M, F) is Kähler, then it is Berwald.*

The Sasaki type lift of the metric tensor $g_{i\bar{j}}$, defined on $T_C(T'M)$, is

$$(2.8) \quad \mathcal{G} = g_{i\bar{j}} dz^i \otimes d\bar{z}^j + g_{i\bar{j}} \delta \eta^i \otimes \delta \bar{\eta}^j,$$

with respect to Chern-Finsler (*c.n.c.*).

3 \mathbb{R} complex Berwald frames

Further on, let M be a 2 - dimensional complex manifold, $(z^k, \eta^k)_{k=1,2}$ are the complex coordinates on tangent bundle $T'M$, the indices $i, j, k \dots$ run over $\{1, 2\}$.

We set $a^2 := g_{i\bar{j}}\eta^i\bar{\eta}^j$ and $l := l^i\dot{\partial}_i$ a unit vector (i.e., $\mathcal{G}(l, \bar{l}) = 1$), with its dual form $\omega = l_i\delta\eta^i$, where

$$(3.1) \quad l^i = \frac{1}{a}\eta^i \text{ and } l_i = \frac{1}{a}g_{i\bar{j}}\bar{\eta}^j = g_{i\bar{j}}\bar{l}^j.$$

Using the same technique as in [3], we construct an orthonormal frame in the vertical bundle $VT'M$ which is 2 - dimensional in any point. So, $VT'M = \{l\} \oplus \{l\}^\perp$, where $\{l\}^\perp$ is spanned by a complex unit vector m .

Requiring the orthogonality conditions $\mathcal{G}(l, \bar{m}) = 0$ and $\mathcal{G}(m, \bar{m}) = 1$, and using $m_i = g_{i\bar{j}}\bar{m}^j$, we get a linear system:

$$\begin{cases} l_1m^1 + l_2m^2 = 0 \\ m_1m^1 + m_2m^2 = 1. \end{cases}$$

This doesn't admit an unique solution. If we solve this system formally ([3], [6]) as one linear, we get "the solutions":

$$m^1 = -\frac{l_2}{\Delta}, \quad m^2 = \frac{l_1}{\Delta}, \quad m_1 = -\Delta l^2, \quad m_2 = \Delta l^1,$$

where $\Delta = l_1m_2 - l_2m_1$. They are not completely determined, because Δ depends on m_i . A straightforward computation give us $|\Delta| = \sqrt{g}$ and $\Delta' = \tau\Delta$, under a change of the local coordinates $(z^k, \eta^k)_{k=1,2}$ in $(z'^k, \eta'^k)_{k=1,2}$, where $g = \det(g_{i\bar{j}}) > 0$ and $\tau = \det\left(\frac{\partial z^i}{\partial z'^j}\right)$. Our statement will be made for a fixed choice of Δ and $\{l, m, \bar{l}, \bar{m}\}$, with

$$m = \frac{1}{\Delta}(-l_2\dot{\partial}_1 + l_1\dot{\partial}_2),$$

which will be called the \mathbb{R} - *complex Berwald frame*. The dependence for the chosen Δ will be analyzed everywhere.

When we work in a fixed local chart, we can take $\Delta = \sqrt{g}$, and so, in this case we have

$$(3.2) \quad m = \frac{1}{\sqrt{g}}(-l_2\dot{\partial}_1 + l_1\dot{\partial}_2)$$

for a fixed local chart.

Then $\{l, m, \bar{l}, \bar{m}\}$, with m from (3.2), will be called the *local \mathbb{R} - complex Berwald frame* of the \mathbb{R} - complex Hermitian Finsler space.

Although m from (3.2) depends of the change of the local coordinates, it is very important for our study in a fixed chart. An important remark is that the quantities $m_i m^j$, $m^i m_j$, $m_i \bar{m}^j$, $m_i \bar{m}$ are independent of the chosen chart.

With respect to the local \mathbb{R} - complex Berwald frame, we have

$$(3.3) \quad \dot{\partial}_i = l_i l + m_i m \quad \text{and} \quad g_{i\bar{j}} = l_i \bar{l}_j + m_i \bar{m}_j.$$

Taking into account (2.2) and (2.4), with respect to the local \mathbb{R} - complex Berwald frame, the Cartan tensors (2.3) are decomposed as it follows

$$(3.4) \quad \begin{aligned} C_{ijk} &= Al_i l_j l_k + B(m_i l_j l_k + l_i m_j l_k + l_i l_j m_k) \\ &\quad + C(l_i m_j m_k + m_i l_j m_k + m_i m_j l_k) + Dm_i m_j m_k; \\ C_{i\bar{j}\bar{k}} &= (-A\bar{l}_k + \bar{B}m_{\bar{k}})l_i l_j + (-B\bar{l}_k + Wm_{\bar{k}})(l_i m_j + m_i l_j) \\ &\quad + (-C\bar{l}_k + Vm_{\bar{k}})m_i m_j, \end{aligned}$$

with $A = -\bar{A}$ and $W = -\bar{W}$, and from here we can express the vertical coefficients of Chern-Finsler complex linear connection

$$C_{ij}^{s\bar{s}} = g^{\bar{k}s} C_{ij\bar{k}} = (-Al^s + \bar{B}m^s)l_i l_j + (-Bl^s + Wm^s)(l_i m_j + m_i l_j) + (-Cl^s + Vm^s)m_i m_j,$$

and $A = -l^j l^k l_h C_{jk}^h$, $B = -l^j m^k l_h C_{jk}^h$, $C = -m^j m^k l_h C_{jk}^h$, $\bar{B} = l^j l^k m_h C_{jk}^h$, $V = m^j m^k m_h C_{jk}^h$, $W = l^j m^k m_h C_{jk}^h$.

Denoting $b^2 := g_{ij}\eta^i \eta^j$ and $\bar{b}^2 := g_{\bar{i}\bar{j}}\bar{\eta}^i \bar{\eta}^j$, and using (2.2) we have following relation:

$$2L = b^2 + \bar{b}^2 + 2a^2.$$

After several computations, we deduce the following results

$$(3.5) \quad \dot{\partial}_k a^2 + \dot{\partial}_k \bar{b}^2 = al_k \quad \text{and} \quad \dot{\partial}_k a^2 + \dot{\partial}_k b^2 = al_k + 2g_{ik}\eta^i.$$

Contracting (3.5) by l^k and m^k , it follows

$$l(a^2) + l(\bar{b}^2) = a, \quad l(a^2) + l(b^2) = a + \frac{2b^2}{a} \\ m(a^2) + m(\bar{b}^2) = 0, \quad m(a^2) + m(b^2) = 2g_{ik}\eta^i m^k.$$

But,

$$l(a^2) = l(g_{i\bar{j}}\eta^i \bar{\eta}^j) = C_{i\bar{j}k} l^k \eta^i \bar{\eta}^j + \frac{1}{a} g_{i\bar{j}} \eta^i \bar{\eta}^j = -a^2 A + a \\ m(a^2) = m(g_{i\bar{j}}\eta^i \bar{\eta}^j) = m^k C_{i\bar{j}k} \eta^i \bar{\eta}^j + m^k g_{i\bar{j}} \eta^i \bar{\eta}^j = -a^2 B.$$

Summarizing, we get the following results

$$(3.6) \quad \begin{aligned} l(a^2) &= -a^2 A + a; & l(b^2) &= a^2 A + \frac{2b^2}{a}; & l(\bar{b}^2) &= a^2 A; \\ m(a^2) &= -a^2 B; & m(b^2) &= a^2 B + 2g_{rj} m^r \eta^j; & m(\bar{b}^2) &= a^2 B; \\ l(L) &= \frac{a^2 + \bar{b}^2}{a}; & m(L) &= g_{rj} m^r \eta^j, \end{aligned}$$

and their conjugates.

Proposition 3.1. *With respect to the local \mathbb{R} - complex Berwald frame, we have*

$$(3.7) \quad \begin{aligned} l(l^i) &= \left(\frac{1}{2a} + \frac{A}{2}\right) l^i; & \bar{l}(l^i) &= \left(-\frac{1}{2a} + \frac{\bar{A}}{2}\right) l^i; \\ l(l_i) &= \left(-\frac{A}{2} - \frac{1}{2a}\right) l_i - Bm_i; & \bar{l}(l_i) &= \left(\frac{1}{2a} - \frac{\bar{A}}{2}\right) l_i - Bm_i; \\ l(m_i) &= \left(\frac{W}{2} + \frac{1}{2a}\right) m_i; & \bar{l}(m_i) &= \left(\frac{\bar{W}}{2} - \frac{1}{2a}\right) m_i; \\ l(m^i) &= \left(-\frac{W}{2} - \frac{1}{2a}\right) m^i + Bl^i; & \bar{l}(m^i) &= \left(-\frac{\bar{W}}{2} + \frac{1}{2a}\right) m^i - Bl^i \\ m(l_i) &= -\frac{B}{2} l_i - Cm_i; & \bar{m}(l_i) &= -\frac{\bar{B}}{2} l_i + \left(\frac{1}{a} + \bar{W}\right) m_i; \\ m(l^i) &= \frac{B}{2} l^i + \frac{1}{a} m^i; & \bar{m}(l^i) &= \frac{\bar{B}}{2} l^i; \\ m(m_i) &= -\frac{1}{a} l_i + \frac{V}{2} m_i; & \bar{m}(m_i) &= \frac{\bar{V}}{2} m_i; \\ m(m^i) &= Cl^i - \frac{V}{2} m^i; & \bar{m}(m^i) &= -\left(\frac{1}{a} + \bar{W}\right) l^i - \frac{\bar{V}}{2} m^i, \end{aligned}$$

and their conjugates.

For the proof we use the same technique for the formulas (3.6) from [3], but with (3.6).

Lemma 3.1. *With respect to the local \mathbb{R} - complex Berwald frame are hold*

$$\bar{l}(A) + l(A) = -\frac{A}{a}; \quad \bar{l}(W) + l(W) = -\frac{W}{a}; \quad \bar{l}(C) + l(C) = -\frac{C}{a}.$$

Proof. $\bar{l}(A) + l(A) = \bar{l}(-l^j l^k l_h C_{jk}^h) + l(-l^j l^k l_h C_{jk}^h)$
 $= \frac{1}{a} C_{kj}^h l^j l^k l_h - C_{jk}^h (-2Bm_h) l^j l^k = \frac{1}{a} C_{kj}^h l^j l^k l_h = -\frac{A}{a}$ and, we can prove the others relations in the same way. \square

Further on, our goal is to give the representation of the metric tensor $g_{ij} = \frac{\partial^2 L}{\partial \eta^i \partial \eta^j}$ in terms of the local \mathbb{R} - complex Berwald frame. Since $b^2 := g_{ij} \eta^i \eta^j$, then

$$(3.8) \quad g_{ij} = \frac{b^2}{a^2} l_i l_j + Y(l_i m_j + m_i l_j) + X m_i m_j,$$

where Y and X are functions on $T'M$ which will be determinate.

Firstly, we obtain that $\det g_{ij} = (\frac{b^2}{a^2} X - Y^2)g$ and so, g_{ij} is invertible iff $\frac{b^2}{a^2} X \neq Y^2$.

Secondly, differentiating (3.8) with respect to η and $\bar{\eta}$, we obtain

$$\begin{aligned} C_{ijk} &= \dot{\partial}_k g_{ij} = A l_i l_j l_k + (m_i l_j l_k + m_j l_i l_k) \left(-\frac{Bb^2}{a^2} + l(Y)\right) + l_i m_j l_k \left(\frac{YW}{2} - \frac{AY}{2}\right) \\ &+ [l(X) + XW + \frac{X}{a} - BY] m_i m_j l_k + [m(Y) - \frac{Cb^2}{a^2} - \frac{X}{a}] (l_i m_j m_k + m_i l_j m_k) + \\ &+ \left(\frac{YV}{2} - \frac{BY}{2}\right) m_j l_i m_k + l_i l_j m_k \left(B - \frac{Y}{a}\right) + m(X) + XV - YC) m_i m_j m_k \\ C_{ij\bar{k}} &= \dot{\partial}_{\bar{k}} g_{ij} = A l_i l_j l_{\bar{k}} - \frac{b^2 B}{a^2} (m_i l_j l_{\bar{k}} + m_j l_i l_{\bar{k}}) + [\bar{l}(X) - XW - \frac{X}{a}] m_i m_j l_{\bar{k}} + \\ &+ B l_i l_j m_{\bar{k}} + \frac{b^2}{a^2} \left(\frac{1}{a} - W\right) (l_i m_j m_{\bar{k}} + m_i l_j m_{\bar{k}}) + [\bar{m}(X) + X\bar{V}] m_i m_j m_{\bar{k}} \end{aligned}$$

which together with (3.4) lead to

$$(3.9) \quad \begin{aligned} Y &= 0; \quad B = 0; \quad W = \frac{b^2}{a(a^2 + b^2)}; \\ l(X) &= 2C \frac{a^2 + b^2}{a^2}; \quad \bar{l}(X) = -2C \frac{a^2 + b^2}{a^2}; \\ X &= -\frac{a^2 + b^2}{a} C; \quad D = m(X) + XV; \quad V = \bar{m}(X) + X\bar{V}. \end{aligned}$$

Now, we distinguish two cases:

1. If $C = 0$ then $X = 0$ and so, $D = V = 0$. These give that g_{ij} is not invertible.

It mince $g_{ij} = \frac{b^2}{a^2} l_i l_j$ and

$$(3.10) \quad C_{ijk} = A l_i l_j l_k \quad \text{and} \quad C_{ij\bar{k}} = -A l_{\bar{k}} l_i l_j + W m_{\bar{k}} (l_i m_j + m_i l_j),$$

with $W = \frac{b^2}{a(a^2 + b^2)}$.

2. If $C \neq 0$, then $g_{ij} = \frac{b^2}{a^2} l_i l_j - \frac{a^2 + b^2}{a} C m_i m_j$ and $\det g_{ij} = -\frac{b^2(a^2 + b^2)}{a^3} Cg$, i.e., g_{ij} is invertible. But, that the tensor g_{ij} to define a positive definite quadratic form for all $z \in M$, we must to make a suitable choice for the function C . Due to Lemma 3.1, it must to satisfy the equation $\bar{l}(C) + l(C) = -\frac{C}{a}$. A solution of this is $C = -\frac{\bar{b}^2}{a(a^2 + b^2)}$,

which gives $\det g_{ij} = \frac{b^2 \bar{b}^2}{a^4} g > 0$ and $X = \frac{\bar{b}^2}{a^2}$. Using (3.6) with $B = 0$, we have $\bar{m}(X) = m(X) = 0$. Hence $V = X\bar{V}$, which gives $V = 0$ and so, $D = 0$. In this case,

$$(3.11) \quad \begin{aligned} C_{ijk} &= Al_i l_j l_k + C(l_i m_j m_k + m_i l_j m_k + m_i m_j l_k); \\ C_{i\bar{j}\bar{k}} &= -A\bar{l}_i \bar{l}_j \bar{l}_k + W m_{\bar{k}}(l_i m_j + m_i l_j) - C\bar{l}_i m_j m_k, \end{aligned}$$

with $W = \frac{b^2}{a(a^2+b^2)}$ and $C = -\frac{\bar{b}^2}{a(a^2+\bar{b}^2)}$.

Moreover, we obtained that $W = \frac{b^2}{a(a^2+b^2)}$, in both cases. But, $W = -\bar{W}$ which implies $a^2 = -\frac{2b^2 \bar{b}^2}{b^2 + \bar{b}^2}$, with $b^2 + \bar{b}^2 < 0$, because $a^2 > 0$.

Therefore, we have proved:

Theorem 3.2. *Let (M, F) be a 2 - dimensional \mathbb{R} complex Hermitian Finsler space. Then, $F^2 = \frac{(b^2 - \bar{b}^2)^2}{2(b^2 + \bar{b}^2)}$ and either $g_{ij} = \frac{b^2}{a^2} l_i l_j$ or $g_{ij} = \frac{b^2}{a^2} l_i l_j - \frac{a^2 + b^2}{a} C m_i m_j$, $C \neq 0$, with $b^2 + \bar{b}^2 < 0$.*

Using above results (with $C = 0$ or $C = -\frac{\bar{b}^2}{a(a^2+b^2)}$), by a direct computation we obtain formulas for vertical covariant derivatives of l, m, \bar{l} and \bar{m} with respect to the Chern-Finsler complex linear connection (2.6):

$$(3.12) \quad \begin{aligned} l_i |_{\bar{j}} &= \left(\frac{A}{2} - \frac{1}{2a} \right) l_i l_{\bar{j}}; \quad m_i |_{\bar{j}} = \left(\frac{\bar{W}}{2} - \frac{1}{2a} \right) m_i l_{\bar{j}}; \\ l_i |_{\bar{j}} &= \left(\frac{1}{2a} - \frac{\bar{A}}{2} \right) l_i l_{\bar{j}} + \left(\frac{1}{a} + \bar{W} \right) m_i m_{\bar{j}}; \\ m_i |_{\bar{j}} &= \left(\frac{1}{2a} - \frac{W}{2} \right) l_j m_i + \left(-\frac{1}{a} - W \right) l_i m_j; \\ l^i |_{\bar{j}} &= \left(\frac{1}{2a} - \frac{A}{2} \right) l^i l_{\bar{j}} + \left(\frac{1}{a} + W \right) m^i m_j; \\ l^i |_{\bar{j}} &= \left(-\frac{1}{2a} + \frac{\bar{A}}{2} \right) l^i l_{\bar{j}}; \quad m^i |_{\bar{j}} = \left(-\frac{1}{2a} + \frac{W}{2} \right) m^i l_{\bar{j}}; \\ m^i |_{\bar{j}} &= \left(\frac{1}{2a} - \frac{\bar{W}}{2} \right) m^i l_{\bar{j}} + \left(-\frac{1}{a} - \bar{W} \right) l^i m_{\bar{j}}, \end{aligned}$$

and their conjugates.

4 Vertical curvature invariants

Our next aim is to give the expression of the vertical Riemann curvature tensors corresponding to the Chern-Finsler connection, with respect to the local \mathbb{R} - complex Berwald frame. The nonzero curvatures coefficients of the Chern-Finsler connection are:

$$(4.1) \quad \begin{aligned} R_{j\bar{h}k}^i &= -L_{jk|\bar{h}}^i - N_{k|\bar{h}}^l C_{jl}^i; \quad \Xi_{j\bar{h}k}^i = -C_{jk|\bar{h}}^i = \Xi_{k\bar{h}j}^i; \\ P_{j\bar{h}k}^i &= -L_{jk|\bar{h}}^i - N_{k|\bar{h}}^l C_{jl}^i; \quad S_{j\bar{h}k}^i = -C_{jk|\bar{h}}^i = S_{k\bar{h}j}^i. \end{aligned}$$

Then the Riemann tensors have the following $h\bar{h}-$, $h\bar{v}-$, $v\bar{h}-$, $v\bar{v}-$ components: $R_{j\bar{i}\bar{h}k} := g_{l\bar{j}} R_{i\bar{h}k}^l$; $P_{j\bar{i}\bar{h}k} := g_{l\bar{j}} P_{i\bar{h}k}^l$; $\Xi_{j\bar{i}\bar{h}k} := g_{l\bar{j}} \Xi_{i\bar{h}k}^l$; $S_{j\bar{i}\bar{h}k} := g_{l\bar{j}} S_{i\bar{h}k}^l$.

We study the $v\bar{v}$ - Riemann type tensor $S_{\bar{r}j\bar{h}k}$.

Firstly, we suppose that $C = -\frac{\bar{b}^2}{a(a^2+b^2)}$. In this case, by (3.6), we have $C|_{\bar{s}}m^{\bar{s}} = 0$.

Now, taking into account (3.12), it results

$$\begin{aligned} S_{\bar{r}j\bar{h}k} &= -C_{j\bar{r}k}|_{\bar{h}} = (A|_{\bar{s}}l^{\bar{s}} + \frac{A}{2a} + \frac{A^2}{2})l_{\bar{h}}l_{\bar{r}}l_jl_k + (A|_{\bar{s}}m^{\bar{s}})m_{\bar{h}}l_{\bar{r}}l_jl_k \\ &- (W|_{\bar{s}}l^{\bar{s}} + \frac{W}{2a} + \frac{WA}{2})l_{\bar{h}}m_{\bar{r}}(m_jl_k + m_kl_j) \\ &+ m_{\bar{h}}m_{\bar{r}}m_jm_k \left(-\frac{2W}{a} + 2W^2\right) + \left(\frac{A}{a} - AW + \frac{W}{a} - W^2\right) m_{\bar{h}}l_{\bar{r}}(l_jm_k + m_jl_k) \\ &+ l_{\bar{h}}l_{\bar{r}}m_jm_k \left(C|_{\bar{s}}l^{\bar{s}} - \frac{CA}{2} - \frac{3C}{2a} - CW\right), \text{ because } W|_{\bar{s}}m^{\bar{s}} = 0. \end{aligned}$$

But $S_{\bar{r}j\bar{h}k}$ is symmetric in j, k and \bar{r}, \bar{h} . Thus, we have

$$(4.2) \quad A|_{\bar{s}}m^{\bar{s}} = 0; \quad W|_{\bar{s}}l^{\bar{s}} = -\frac{3W}{2a} + \frac{WA}{2} - \frac{A}{a} + W^2.$$

In the another case, ($C = 0$), a similar computation gives

$$\begin{aligned} S_{\bar{r}j\bar{h}k} &= -C_{j\bar{r}k}|_{\bar{h}} = (A|_{\bar{s}}l^{\bar{s}} + \frac{A}{2a} + \frac{A^2}{2})l_{\bar{h}}l_{\bar{r}}l_jl_k + (A|_{\bar{s}}m^{\bar{s}})m_{\bar{h}}l_{\bar{r}}l_jl_k \\ &- (W|_{\bar{s}}l^{\bar{s}} + \frac{W}{2a} + \frac{WA}{2})l_{\bar{h}}m_{\bar{r}}(m_jl_k + m_kl_j) + m_{\bar{h}}m_{\bar{r}}m_jm_k \left(-\frac{2W}{a} + 2W^2\right) \\ &+ \left(\frac{A}{a} - AW + \frac{W}{a} - W^2\right) m_{\bar{h}}l_{\bar{r}}(l_jm_k + m_jl_k), \end{aligned}$$

and requiring the symmetry properties for $S_{\bar{r}j\bar{h}k}$, we again obtain the conditions (4.2).

We make the following notations:

$$\mathbf{I}_1 = A|_{\bar{r}}l^{\bar{r}} + \frac{A}{2}\left(\frac{1}{a} + A\right); \quad \mathbf{I}_2 = -2W\left(\frac{1}{a} - W\right),$$

in which \mathbf{I}_1 and \mathbf{I}_2 are invariable to the changes of the local coordinates because $S_{\bar{r}j\bar{h}k}$, $l_{\bar{h}}l_{\bar{r}}l_jl_k$ and $m_jm_{\bar{h}}m_k m_{\bar{r}}$ are tensors. As in [3], we call the functions \mathbf{I}_1 and \mathbf{I}_2 *vertical curvature invariants*.

Also, by analogy with [3], we define *the vertical holomorphic sectional curvature* in direction l

$$(4.3) \quad K_{F,l}^v(z, \eta) := 2\mathbf{R}(l, \bar{l}, l, \bar{l})$$

and *the vertical holomorphic sectional curvature* in direction m

$$(4.4) \quad K_{F,m}^v(z, \eta) := 2\mathbf{R}(m, \bar{m}, m, \bar{m}).$$

Now, using Lemma 3.1 and the properties $A = -\bar{A}$, $W = -\bar{W}$, the second equation from (4.2), yields $A = -W$. So, we can prove the following results.

Theorem 4.1. *Let (M, F) be a 2 - dimensional \mathbb{R} complex Hermitian Finsler space. Then,*

- i) $K_{F,l}^v(z, \eta) = 2\mathbf{I}_1 = 0$;
- ii) $K_{F,m}^v(z, \eta) = 2\mathbf{I}_2 = -\frac{4b^2}{(a^2+b^2)^2}$;
- iii) If $g_{ij} = \frac{b^2}{a^2}l_il_j$ then,

$$(4.5) \quad \begin{aligned} C_{ijk} &= Al_il_jl_k; \quad C_{ij\bar{k}} = -A(l_{\bar{k}}l_il_j + m_{\bar{k}}l_im_j + m_{\bar{k}}m_il_j), \\ S_{\bar{r}j\bar{h}k} &= \mathbf{I}_2m_{\bar{h}}m_{\bar{r}}m_jm_k; \end{aligned}$$

iv) If $g_{ij} = \frac{b^2}{a^2}l_i l_j + \frac{\bar{b}^2}{a^2}m_i m_j$ then,

$$(4.6) \quad \begin{aligned} C_{ijk} &= Al_i l_j l_k + C(l_i m_j m_k + m_i l_j m_k + m_i m_j l_k); \\ C_{ij\bar{k}} &= -A(l_{\bar{k}} l_i l_j + m_{\bar{k}} l_i m_j + m_{\bar{k}} m_i l_j) - Cl_{\bar{k}} m_i m_j; \\ S_{\bar{r}j\bar{h}k} &= \mathbf{I}_2 m_{\bar{h}} m_{\bar{r}} m_j m_k + \frac{\bar{b}^2 - b^2}{(a^2 + b^2)^2} l_{\bar{h}} l_{\bar{r}} m_j m_k, \end{aligned}$$

with $A = -\frac{b^2}{a(a^2+b^2)}$ and $C = -\frac{\bar{b}^2}{a(a^2+b^2)}$.

Proof. i) Since $A = -W$, we have $\bar{l}(A) = \bar{l}(-\frac{b^2}{a(a^2+b^2)}) = -\frac{A}{2}(\frac{1}{a} + A)$. But, $\mathbf{I}_1 = \bar{l}(A) + \frac{A}{2}(\frac{1}{a} + A)$ and so, it results $\mathbf{I}_1 = 0$ and $K_{F,l}^v(z, \eta) = 2\mathbf{I}_1 = 0$.

ii) By direct computation, $\mathbf{I}_2 = -2W(\frac{1}{a} - W) = \frac{-2b^2}{(a^2+b^2)^2}$ and from here, $K_{F,m}^v(z, \eta) = 2\mathbf{I}_2 = -\frac{4b^2}{(a^2+b^2)^2}$

iii) If $g_{ij} = \frac{b^2}{a^2}l_i l_j$ then $C = 0$ and by (3.11), we have

$$C_{ijk} = Al_i l_j l_k \text{ and } C_{ij\bar{k}} = -A(l_{\bar{k}} l_i l_j + m_{\bar{k}} l_i m_j + m_{\bar{k}} m_i l_j).$$

Now, because $A = -W$ and $\mathbf{I}_1 = 0$ it follows, $S_{\bar{r}j\bar{h}k} = \mathbf{I}_2 m_{\bar{h}} m_{\bar{r}} m_j m_k$.

iv) Using (3.11), it results the first two relation from (4.6). Since $g_{ij} = \frac{b^2}{a^2}l_i l_j + \frac{\bar{b}^2}{a^2}m_i m_j$ and $C = -\frac{\bar{b}^2}{a(a^2+b^2)}$ we have $C|_s l^{\bar{s}} - \frac{CA}{2} - \frac{3C}{2a} - CW = \frac{\bar{b}^2 - b^2}{(a^2+b^2)^2}$ which leads to the third relation of (4.6). \square

5 Some classes of \mathbb{R} complex Finsler spaces

With respect to the local \mathbb{R} - complex Berwald frame, the local coefficients of the Chern-Finsler (*c.n.c*) can be written in the following form

$$N_j^i = \alpha l_j l^i + \beta l_j m^i + \gamma m_j l^i + \delta m_j m^i,$$

where $\alpha, \beta, \gamma, \delta$ are functions on $T'M$. Taking into account (2.7) and (3.7), after some computations, we obtain

$$(5.1) \quad \begin{aligned} G^i &= \frac{a}{2}(\alpha l^i + \beta m^i); \\ L_{jk}^i &= l(\alpha)l_j l_k l^i + (l(\gamma) + \frac{\gamma}{a})l_j m_k l^i + \left(l(\beta) - \frac{\beta}{a}\right)l_j l_k m^i + l(\delta)l_j m_k m^i \\ &\quad + (m(\alpha) + \beta C - \frac{\gamma}{a})m_j l_k l^i + (-\alpha C + m(\gamma) + \delta C)m_j m_k l^i \\ &\quad \left(\frac{\alpha}{a} + m(\beta) - \frac{\delta}{a}\right)m_j l_k m^i + (-\beta C + \frac{\gamma}{a} + m(\delta))m_j m_k m^i, \end{aligned}$$

with conditions

$$(5.2) \quad \begin{aligned} \bar{l}(\alpha) + l(\alpha) &= \frac{\alpha}{a}; \quad \bar{l}(\beta) + l(\beta) = \frac{\beta}{a}; \\ \bar{l}(\gamma) + l(\gamma) &= \frac{\gamma}{a}; \quad \bar{l}(\delta) + l(\delta) = \frac{\delta}{a}. \end{aligned}$$

Proposition 5.1. *Let (M, F) be a 2 - dimensional \mathbb{R} complex Hermitian Finsler space. Then,*

i) *It is weakly Kähler if and only if*

$$(5.3) \quad m(\alpha) - l(\gamma) = \frac{2\gamma}{a} - \beta C;$$

ii) *It is strongly Kähler if and only if it is weakly Kähler and*

$$(5.4) \quad l(\delta) - m(\beta) = \frac{\alpha - \delta}{a};$$

iii) *It is Kähler if and only if it is strongly Kähler.*

Proof. To prove i), we must compute the expression $g_{i\bar{l}}T_{jk}^i\eta^j\bar{\eta}^l$. Due to (5.1), we obtain

$$g_{i\bar{l}}T_{jk}^i\eta^j\bar{\eta}^l = a^2T_{jk}^i l_i l^j = a^2[m(\alpha) - l(\gamma) - \frac{2\gamma}{a} + \beta C]m_k.$$

So, $g_{i\bar{l}}T_{jk}^i\eta^j\bar{\eta}^l = 0$ if and only if $m(\alpha) - l(\gamma) = \frac{2\gamma}{a} - \beta C$, i.e. i).

$$\text{ii) } L_{jk}^i - L_{kj}^i = [m(\alpha) + \beta C - \frac{2\gamma}{a} - l(\gamma)]l^i l_k m_j + [l(\gamma) + \frac{2\gamma}{a} - m(\alpha) - \beta C]l^i l_j m_k \\ + \{l(\delta) - m(\beta) - \frac{\alpha}{a} + \frac{\delta}{a}\}m^i m_j l_k + \{m(\beta) + \frac{\alpha}{a} - \frac{\delta}{a} - l(\delta)\}m^i l_j m_k.$$

Hence, $L_{jk}^i - L_{kj}^i = 0$ if and only if $m(\alpha) - l(\gamma) = \frac{2\gamma}{a} - \beta C$ and $l(\delta) - m(\beta) = \frac{\alpha - \delta}{a}$.

For iii), using again (5.1), we have $(L_{jk}^i - L_{kj}^i)\eta^j = 0$ if and only if $m(\alpha) - l(\gamma) = \frac{2\gamma}{a} - \beta C$ and $l(\delta) - m(\beta) = \frac{\alpha - \delta}{a}$. \square

Theorem 5.2. *Let (M, F) be a 2 - dimensional \mathbb{R} -complex Hermitian Finsler space.*

If it satisfies (5.3), (5.4) and

$$(5.5) \quad \bar{l}(\alpha) = \bar{m}(\beta) = 0 ; \bar{l}(\beta) = -\frac{\beta}{a} ; \bar{m}(\alpha) = \frac{\beta}{a} + \beta A.$$

then (M, F) is a Berwald space. In this case, $l(\alpha) = \frac{\alpha}{a}$ and $l(\beta) = \frac{\beta}{a}$.

Proof. Using (3.7) and (5.1), we obtain

$$2\dot{\partial}_{\bar{r}}G^i = l_{\bar{r}}[a\bar{l}(\alpha)l^i + (a\bar{l}(\beta) + \beta)m^i] + m_{\bar{r}}[(a\bar{m}(\alpha) - \beta - a\beta A)l^i + a\bar{m}(\beta)m^i].$$

If $\dot{\partial}_{\bar{r}}G^i = 0$, it results (5.5). Now, by applying Theorem 2.1 and (5.2), the claim follows. \square

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