The existence and the number of critical points in an abstract perturbation problem

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Abstract. We use the relative homology groups of the sublevel sets associated to some functions which appear in an abstract perturbation problem in order to prove some criticality criteria. It is also proved the stability of critical groups. We give an estimation for the minimal number of critical points of a function using the category of the space with respect to the perturbed function.

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1 Introduction

This paper is devoted to the study of the existence of critical points in an abstract perturbation problem under compactness assumptions of Palais-Smale type. A motivation is that perturbations arrive in many problems in Nonlinear Analysis; often it is difficult to study the function directly and we need to perturb it. Examples can be found in [1]. For Banach-Finsler manifolds, this kind of problems, in the presence of Palais-Smale type condition of [19]-[20], is studied in [21]. On the other hand, the use of relative homology in the study of the stability of critical values for two functions close enough, satisfying the Palais-Smale condition, goes back to [17]. In the last years, the classical critical point theory was extended to continuous functions on metric spaces; see [7]-[13]. The origin and evolution of the Palais-Smale condition in critical point theory is emphasized in [18], while [11] contains a purely topological approach of Morse theory.

It is well-known the fact that the minimum cardinality of the set of critical points of a function is difficult to compute. Estimating methods for this number go from the classical Lusternik-Schnirelmann category of the space, see [16], to φ-category, see [2]. Under compactness conditions, the number of critical points of a function can be estimated from below by the Lusternik-Schnirelmann category of the space; in [3]

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the authors introduced the category of a space with respect to a given function and it was proved that, in suitable hypothesis, this number is a better lower bound for the number of critical points of the function.

In this paper we use the relative homology groups of the sublevel sets associated to some functions which appear in an abstract perturbation problem in order to prove some criticality criteria. In Section 2 we review some of the standard facts on critical point theory on metric spaces and we briefly introduce the (Lusternik-Schnirelmann) category depending on a functional. We also prove some technical lemmas. Section 3 is devoted to the study of the existence and the number of critical points in our abstract perturbation problem. More precisely, we are interested in finding conditions such that a given interval is critical/non-critical for a function but not for the associated perturbed function respectively conditions such that a given interval is critical for both the original function and its perturbation. We also prove the stability of critical groups under perturbations and we give an estimation for the minimal number of critical points of the function using the category of the space with respect to the perturbed function.

2 Preliminaries

2.1 Critical point theory on metric spaces

Let X be a metric space endowed with the metric d. If $x \in X$ and $r > 0$, then $B_r(x)$ denotes the open ball in X of center x and radius r. Let $f : X \to \mathbb{R}$ be a continuous function. Recall from [7]-[13] basic notions and properties concerning the critical point theory on metric spaces.

The *weak slope* of f at x, denoted by $|df|(x)$, is the supremum of all $\sigma \in [0, \infty)$ such that there exist $\delta > 0$ and a continuous map $\mathcal{H} : B_\delta(x) \times [0, \delta] \longrightarrow X$ which satisfies the conditions

$$d(\mathcal{H}(y,t), y) \leq t; \quad f(\mathcal{H}(y,t)) \leq f(y) - \sigma t,$$

for all $y \in B_\delta(x)$ and all $t \in [0, \delta]$.

We call a point $x \in X$ a *critical point* of f if $|df|(x) = 0$. A real number c is called a *critical value* of f if there exists $x \in X$ such that $|df|(x) = 0$ and $f(x) = c$.

If X is a C^1-Finsler manifold and $f \in C^1(X, \mathbb{R})$, then $|df|(x) = \|df_x\|$, for any $x \in X$; in this case the notion of critical point agrees with the classical one, i.e. $x \in X$ a critical point of f if $df_x = 0$.

We use the notation $K[f] = \{x \in X | \ |df|(x) = 0\}$ for the *critical set* of f. If c is a real number, then $K_c[f] = K[f] \cap f^{-1}(c)$ is the *critical set of level* c of f and $X_c(f) = \{x \in X | \ f(x) < c\}$ denotes the *set of sublevel* c of f.

We say that f satisfies the *Palais-Smale condition*, denoted by (PS), if any sequence (x_n) in M such that $(f(x_n))$ is bounded and $|df(x_n)| \to 0$ has a convergent subsequence. We say that f satisfies the *Palais-Smale condition at level* c, denoted by $(PS)_c$, if any sequence (x_n) in M such that $f(x_n) \to c$ and $|df(x_n)| \to 0$ has a convergent subsequence.

If (PS) is satisfied, then the local condition $(PS)_c$ holds for all $c \in \mathbb{R}$, while the converse is not true. On the other hand, the function $x \mapsto |df|(x)$ is lower semicontinuous and this implies that a limit point of a subsequence (x_{n_k}) as above is a critical point of f.

For $x \in X$ denote $c = f(x)$ and let $V \subset X$ be a neighborhood of x. Set

$$C_q(f;x) = H_q(X_c(f) \cap V, (X_c(f) \setminus \{x\}) \cap V), q \in \mathbb{N},$$

where $H_q(Y,X)$ denotes the n^{th} relative singular homology group of the pair (Y,X), $X \subset Y$, and let $C_*(f;x) = \{C_q(f;x)\}_{q\in\mathbb{N}}$. The vector space $C_q(f;x)$ is called the q th *critical group* of f at x; the excision property of singular homology proves that the definition of $C_*(f;x)$ does not depend of the choice of the neighborhood V. The terminology is justified by the fact that if $C_q(f;x)$ is nontrivial for some q, then x is a critical point of f, see [8].

2.2 The category depending on a functional

Let X be a topological space and let $f : X \to \mathbb{R}$ be a fixed functional. Recall from [3] the definition and some properties of f-category of a space.

We say that A is (f,ε)-*contractible* in X, where A is a subset of X and $\varepsilon > 0$, if there exist $\widetilde{x} \in X$ and a continuous deformation $\eta : A \times [0,1] \to X$ such that

$$\eta(x,0) = x; \quad \eta(x,1) = \widetilde{x}; \quad f(\eta(x,t)) \le f(x) + \varepsilon,$$

for all $x \in A$ and all $t \in [0,1]$.

We define the f-*category* of B in X, where B is a subset of X, by

$$fcat_X(B) = \sup_{\varepsilon>0} n_\varepsilon^f(B,X);$$

here $n_\varepsilon^f(B,X)$ denotes the smallest integer n such that there exist closed subsets $A_1, \ldots, A_n$ which are (f,ε)-contractible in X and satisfy $B \subset \bigcup_{i=1}^n A_i$. Otherwise we define $n_\varepsilon^f(B,X) = \infty$. If $B = \varnothing$, we take $fcat_X(B) = 0$. Define the f-category of X by taking $fcat(X) = fcat_X(X)$.

2.3 Tools from algebraic topology

Denote by $H_n(Y,X)$ the n^{th} relative singular homology group of the pair (Y,X), $X \subset Y$, with real coefficients. Recall that for a (weak) deformation retract X of Y we have, for any n, $H_n(Y,X) = 0$. See [14]. Let A, B, C and D be topological spaces such that $A \subset B \subset C \subset D$. We prove the following auxiliary result:

Lemma 2.1. *If $H_n(C,A) = 0$ for any n, then there exists, for any n, an injective homomorphism $H_n(D,A) \to H_n(D,B)$.*

Proof. We consider the exact sequence of the triple (D,C,A). Then we have:

$$\cdots \longrightarrow H_n(C,A) \longrightarrow H_n(D,A) \xrightarrow{i_n} H_n(D,C) \longrightarrow H_{n-1}(C,A) \longrightarrow \cdots$$

Because $H_n(C,A) = 0$ for any n, it follows that i_n is an isomorphism. Denote α_n and β_n the homomorphisms induced by the inclusions : $(D,A) \hookrightarrow (D,B)$ and $(D,B) \hookrightarrow (D,C)$ respectively. Then we have $\beta_n \circ \alpha_n = i_n$, which shows that α_n is injective. $\square$

Lemma 2.2. *If $H_n(D,B)=0$ for any n, then there exists, for any n, an injective homomorphism $H_n(B,A)\to H_n(C,A)$.*

Proof. The conclusion follows from the equality $\delta_n\circ\gamma_n=j_n$, where γ_n,δ_n are the homomorphisms induced by the inclusions $(B,A)\hookrightarrow(C,A)$ and $(C,A)\hookrightarrow(D,A)$ respectively and $j_n:H_n(B,A)\to H_n(D,A)$ is an isomorphism. $\square$

Lemma 2.3. *If $H_n(B,A)=0$ and $H_n(D,C)=0$ for any n, then we have*

$$H_n(C,A)\cong H_n(C,B)\cong H_n(D,B).$$

Proof. We consider the exact sequences of the triple (D,C,B) :

$$\ldots\longrightarrow H_{n+1}(D,C)\longrightarrow H_n(C,B)\xrightarrow{i_n}H_n(D,B)\longrightarrow H_n(D,C)\longrightarrow\ldots$$

We have $H_{n+1}(D,C)=H_n(D,C)=0$, which implies that i_n is an isomorphism.

In the same way, by using the exact sequences of the triple (C,B,A), we have:

$$\ldots\longrightarrow H_n(B,A)\longrightarrow H_n(C,A)\xrightarrow{j_n}H_n(C,B)\longrightarrow H_{n-1}(B,A)\longrightarrow\ldots$$

Because $H_n(B,A)=H_{n-1}(B,A)=0$, we obtain that j_n is an isomorphism. $\square$

3 The existence and the number of critical points

3.1 The existence of critical points on metric spaces

Let X be a complete metric space. Recall the second deformation theorem, which we need proving the main results of this paper:

Theorem 3.1. ([7]) *Let $f:X\to\mathbb{R}$ be a continuous function, $a\in\mathbb{R}$ and $b\in\mathbb{R}\cup\{+\infty\}$ with $a<b$. Assume that for any $u\in[a,b)$, f satisfies the (PS)-condition on $f^{-1}([a,u])$, f has no critical point x with $a<f(x)<b$ and either $K_a[f]=\varnothing$ or the connected components of $K_a[f]$ are single points. Then $X_a(f)\cup K_a[f]$ is a weak deformation retract of $X_b(f)$.*

We consider two continuous functions $f,g:X\to\mathbb{R}$ with $g>0$. For any $\varepsilon>0$ enough small, the perturbation of f with the function g is given by

$$f^\varepsilon=f+\varepsilon g.$$

In [5] we obtained, for Finsler manifolds, sufficient conditions for the existence of a critical value for the perturbed function f^ε in a given interval, with suitable assumptions, including (PS). In [6], a stability property of critical values for continuous functions on metric spaces was proved.

In this section we study the existence of critical points of f and f^ε, assuming that the following condition is fulfilled:

$$a+\varepsilon g(x)\le b,\ \forall\ x\in X. \tag{3.1}$$

We can now formulate our main results.

Theorem 3.2. *Let $[a,b]$ be a non-critical interval for f^ε, i.e. f^ε has no critical values in $[a,b]$. Suppose that both f and f^ε satisfy the $(PS)_c$ condition for any $c \in [a,b]$. If there exists m such that $H_m(X_b(f), X_a(f^\varepsilon)) \neq 0$, then $K[f] \cap f^{-1}([a,b]) \neq \varnothing$.*

Proof. From (3.1) we obtain the next inclusions:

$$X_a(f^\varepsilon) \subseteq X_a(f) \subseteq X_b(f^\varepsilon) \subseteq X_b(f). \tag{3.2}$$

Because $[a,b]$ is a non-critical interval for f^ε and f^ε satisfies the $(PS)_c$ condition for any $c \in [a,b]$, the Theorem 3.1 implies that $X_a(f^\varepsilon)$ is a weak deformation retract of $X_b(f^\varepsilon)$. This shows that for any n we have $H_n(X_b(f^\varepsilon), X_a(f^\varepsilon)) = 0$. We can apply the Lemma 2.1 and we obtain that, for any n, there exists an injective homomorphism

$$H_n(X_b(f), X_a(f^\varepsilon)) \to H_n(X_b(f), X_a(f)). \tag{3.3}$$

By contradiction, assume that $K[f] \cap f^{-1}([a,b]) = \varnothing$. From Theorem 3.1 it follows that $X_a(f)$ is a weak deformation retract of $X_b(f)$; consequently $H_n(X_b(f), X_a(f)) = 0$ for any n. On the other hand, the homomorphism from (3.3) being injective, we obtain that $H_n(X_b(f), X_a(f^\varepsilon)) = 0$ for any n. But this contradicts the fact that there exists m such that $H_m(X_b(f), X_a(f^\varepsilon)) \neq 0$. □

Remark 3.1. Let f be as above, satisfying the $(PS)_c$ condition for any $c \in [a,b]$ and let us consider the family of perturbations of f,

$$\left\{ f^\varepsilon = f + \varepsilon g |\ 0 < \varepsilon,\ 0 < g < \frac{b-a}{\varepsilon},\ f^\varepsilon \text{ satisfies } (PS)_c,\ \forall\ c \in [a,b] \right\}.$$

The Theorem 3.2 shows that if we can find a function f^ε in this family and an integer m such that f^ε has no critical values in $[a,b]$ and $H_m(X_b(f), X_a(f^\varepsilon)) \neq 0$, then f has at least a critical value between a and b.

In the above theorem, we obtained sufficient conditions such that the interval $[a,b]$ is non-critical for the perturbed function f^ε, but it is critical for f. The next theorem gives sufficient conditions such that the interval $[a,b]$ is non-critical for f, but it is critical for the perturbed function f^ε.

Theorem 3.3. *Let $[a,b]$ be a non-critical interval for f. Assume that both f and f^ε satisfy the $(PS)_c$ condition for any $c \in [a,b]$ and there exists m such that $H_m(X_a(f), X_a(f^\varepsilon)) \neq 0$. Then $K[f^\varepsilon] \cap (f^\varepsilon)^{-1}([a,b]) \neq \varnothing$.*

Proof. If $[a,b]$ is a non-critical interval for f and f satisfies the $(PS)_c$ condition for any $c \in [a,b]$, then the Theorem 3.1 implies that $H_n(X_b(f), X_a(f)) = 0$ for any n. From (3.2) and Lemma 2.2, there exists, for any n, an injective homomorphism

$$H_n(X_a(f), X_a(f^\varepsilon)) \to H_n(X_b(f^\varepsilon), X_a(f^\varepsilon)). \tag{3.4}$$

Suppose that $K[f^\varepsilon] \cap (f^\varepsilon)^{-1}([a,b]) = \varnothing$. Therefore by Theorem 3.1 we get that $X_a(f^\varepsilon)$ is a weak deformation retract of $X_b(f^\varepsilon)$. Then $H_n(X_b(f^\varepsilon), X_a(f^\varepsilon)) = 0$, for any n. But the homomorphism from (3.4) is injective; this contradicts the fact that there exists m such that $H_m(X_a(f), X_a(f^\varepsilon)) \neq 0$. □

Remark 3.2. The above theorem is a generalization of [5, Theorem 3.1.(i)].

Now we establish sufficient conditions such that the interval $[a, b]$ is non-critical for both f and f^ε.

Theorem 3.4. *Let f and f^ε be as above satisfying the $(PS)_c$ condition for any $c \in [a, b]$. Assume that $H_n(X_a(f), X_a(f^\varepsilon)) = 0$ and $H_n(X_b(f), X_b(f^\varepsilon)) = 0$ for any n. If there exists m such that $H_m(X_b(f^\varepsilon), X_a(f)) \neq 0$, then $K[f] \cap (f^\varepsilon)^{-1}([a, b]) \neq \varnothing$ and $K[f^\varepsilon] \cap (f^\varepsilon)^{-1}([a, b]) \neq \varnothing$.*

Proof. We have from Lemma 2.3 and (3.2) that

$$H_n(X_b(f^\varepsilon), X_a(f^\varepsilon)) \cong H_n(X_b(f^\varepsilon), X_a(f)) \cong H_n(X_b(f), X_a(f)).$$

The conclusion follows from Theorem 3.1, by contradiction. □

We obtain now sufficient conditions for the existence of critical values in the given interval $[a, b]$ for f and for the perturbed function f^ε respectively, by using the relative homology involving the level set $(f^\varepsilon)^{-1}(a)$.

Theorem 3.5. *Let f and f^ε be as above satisfying the $(PS)_c$ condition for any $c \in [a, b]$.*

(i) Suppose that $H_n(X_a(f), (f^\varepsilon)^{-1}(a)) = 0$ for any n and that there exists some m such that $H_m(X_b(f^\varepsilon), (f^\varepsilon)^{-1}(a)) \neq 0$. Then $K[f^\varepsilon] \cap (f^\varepsilon)^{-1}([a, b]) \neq \varnothing$.

(ii) Suppose that $H_n(X_b(f^\varepsilon), (f^\varepsilon)^{-1}(a)) = 0$ for any n and that there exists some m such that $H_m(X_b(f), (f^\varepsilon)^{-1}(a)) \neq 0$. Then $K[f] \cap f^{-1}([a, b]) \neq \varnothing$.

Proof. From (3.1) we obtain the inclusions

$$(f^\varepsilon)^{-1}(a) \subseteq X_a(f^\varepsilon) \subseteq X_a(f) \subseteq X_b(f^\varepsilon);$$

$$(f^\varepsilon)^{-1}(a) \subseteq X_a(f) \subseteq X_b(f^\varepsilon) \subseteq X_b(f).$$

Then we use Lemma 2.1. □

As a limit case, we can look at the whole space X as a sublevel set of a function; for example, we can write $X = X_\infty(f^\varepsilon)$. Then we get the following result:

Theorem 3.6. *Let f and f^ε be as above satisfying the $(PS)_c$ condition for any $c \in [a, b]$. Suppose that $H_n(X, X_b(f^\varepsilon)) = 0$, for any n and there exists some m such that $H_m(X_b(f), X_a(f^\varepsilon)) \neq 0$. Then $K[f^\varepsilon] \cap (f^\varepsilon)^{-1}([a, b]) \neq \varnothing$.*

Proof. The conclusion follows from Lemma 2.2 and $X_a(f^\varepsilon) \subseteq X_b(f^\varepsilon) \subseteq X_b(f) \subseteq X$. □

3.2 The case of Banach-Finsler manifolds

Let M be a C^1-Finsler manifold and let $f : M \to \mathbb{R}$ be of C^1 class. If $K[f]$ denotes the critical set of f and $c \in \mathbb{R}$ a critical value of f, the critical set of level c of f is $K_c[f] = K[f] \cap f^{-1}(c)$ and $M_c(f)$ is the set of sublevel c of f. The second deformation theorem has the following formulation:

Theorem 3.7. ([4]) *Let M be a C^2-Finsler manifold and let $f \in C^1(M)$ be such that f satisfies the $(PS)_c$ condition for any $c \in [a,b]$ and $K[f] \cap f^{-1}([a,b]) = \varnothing$. Then $M_a(f)$ is a strong deformation retract of $M_b(f)$.*

Let M be a C^2-Finsler manifold and let f and g be real functions defined on M, of C^1 class, such that $g > 0$. Denote $f^\varepsilon = f + \varepsilon g$, where $\varepsilon > 0$ is enough small and the condition (3.1) is fulfilled. It is obvious that Theorems 3.2-3.6 remains true in this context.

The problem of finding information about the critical set of the perturbed function f^ε, where f is bounded from below and dg is bounded on sets on which g is bounded, is studied in [21]. We need the following result:

Theorem 3.8. ([21]) *Let f, g and f^ε be as above. Assume that f is bounded from below, dg is bounded on sets on which g is bounded, f^ε satisfies the (PS) condition and $K_c[f^\varepsilon] = \varnothing$ for any $\varepsilon \in (0, \varepsilon_0]$, where ε_0 is enough small and c is a given real number. Then $M_c(f^{\varepsilon_0})$ is a strong deformation retract of $M_c(f)$.*

We obtain now sufficient condition such that the interval $[a,b]$ is critical for f and also for the perturbed function f^{ε_0}.

Theorem 3.9. *Let f, g and f^ε be as above. Asuume that f is bounded from below, dg is bounded on sets on which g is bounded, f and f^ε satisfy the (PS) condition and f^ε has no critical points at level a and b, for any $\varepsilon \in (0, \varepsilon_0]$. If there exists m such that $H_m(M_b(f^{\varepsilon_0}), M_a(f)) \neq 0$, then $K[f^{\varepsilon_0}] \cap (f^{\varepsilon_0})^{-1}([a,b]) \neq \varnothing$ and $K[f] \cap f^{-1}([a,b]) \neq \varnothing$.*

Proof. It is obvious that

$$M_a(f^{\varepsilon_0}) \subseteq M_a(f) \subseteq M_b(f^{\varepsilon_0}) \subseteq M_b(f).$$

We apply the Theorem 3.8 twice:
if $K_a[f^\varepsilon] = \varnothing$ for any $\varepsilon \in (0, \varepsilon_0]$, then $H_n(M_a(f), M_a(f^{\varepsilon_0})) = 0$;
if $K_b[f^\varepsilon] = \varnothing$ for any $\varepsilon \in (0, \varepsilon_0]$, then $H_n(M_b(f), M_b(f^{\varepsilon_0})) = 0$.

Now we use the Lemma 2.3 and, for any n, we have:

$$H_n(M_b(f^{\varepsilon_0}), M_a(f^{\varepsilon_0})) \cong H_n(M_b(f^{\varepsilon_0}), M_a(f)) \cong H_n(M_b(f), M_a(f)).$$

The conclusion follows from Theorem 3.7 by contradiction. □

Remark 3.3. The assumptions of Theorem 3.9 imply that

$$H_n(M_a(f), M_a(f^{\varepsilon_0})) = H_n(M_b(f), M_b(f^{\varepsilon_0})) = 0$$

for any n. We conclude that the Theorem 3.9 is a particular case of Theorem 3.4.

3.3 The stability of critical groups

Let X be a complete metric space. Given a function $f : X \to \mathbb{R}$, we set:

$$\|f\|_\infty = \sup_X |f|; \quad Lip\ (f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)};$$

$$\|f\|_{1,\infty} = \max\{\|f\|_\infty, Lip\ (f)\}.$$

It is obvious that $\|f\|_{1,\infty} < \infty$ only if f is bounded and Lipschitz.

The homotopical stability of a minimum is studied in [15]; on the other hand, in [10] the authors extend the study to isolated critical points of continuous functions. We need the following result:

Theorem 3.10. ([10]) *Let U be an open subset of X and $x \in U$ the only critical point of f in $\overline{U}$. Assume that f satisfy the (PS) condition in $\overline{U}$. Then there exists $\delta > 0$ such that for any $\widetilde{f} : X \to \mathbb{R}$ continuous, having an unique critical point $\widetilde{x}$ in $\overline{U}$, satisfying the (PS) condition in $\overline{U}$ and $\|\widetilde{f}|_U - f|_U\|_{1,\infty} \leq \delta$, we have $C_*(\widetilde{f}, \widetilde{x}) = C_*(f, x)$.*

Let $f, g : X \to \mathbb{R}$ be continuous, $g > 0$; define $f^\varepsilon = f + \varepsilon g$, $\varepsilon > 0$ small; we do not need the assumption (3.1). We have the following stability property of critical groups.

Theorem 3.11. *Let U be an open subset of X and $x \in U$ the only critical point of f in $\overline{U}$. Assume that f satisfy the (PS) condition in $\overline{U}$. Then there exists $\delta > 0$ with the following property: for any $\varepsilon > 0$ such that f^ε has an unique critical point x_ε in $\overline{U}$, satisfies the (PS) condition in $\overline{U}$ and $\varepsilon\|g|_U\|_{1,\infty} \leq \delta$, we have $C_*(f^\varepsilon, x_\varepsilon) = C_*(f, x)$, i.e. the critical group does not depend on ε.*

Proof. We can apply the Theorem 3.10 for f and f^ε because the equalities

$$\|f^\varepsilon|_U - f|_U\|_\infty = \|\varepsilon g|_U\|_\infty = \sup_X(\varepsilon g|_U) = \varepsilon \sup_X g|_U = \varepsilon\|g|U\|_\infty$$

and

$$Lip\ (f^\varepsilon|_U - f|_U) = Lip\ (\varepsilon g|_U) = \varepsilon Lip\ (g|_U)$$

proves the fact that

$$\|f^\varepsilon|_U - f|_U\|_{1,\infty} = \max\{\varepsilon\|g\|_\infty, \varepsilon Lip\ (g)\} = \varepsilon\|g|_U\|_{1,\infty}.$$ □

3.4 The number of critical points

Let X be a topological space and $B \subseteq X$. We have the following property:

Proposition 3.12. ([3]) *Let $f_1, f_2 : X \to \mathbb{R}$ be two continuous functions. For any $k \in \mathbb{N}$ such that $f_1 cat_X(B) \geq k$, there exists $\delta > 0$ such that if $\sup_{x\in B} |f_1(x) - f_2(x)| \leq \delta$, then $f_1 cat_X(B) \leq f_2 cat_X(B)$.*

From [3, Theorem 5.8] and comments from [3, Section 5.4], we have:

Theorem 3.13. ([3]) *Let X be a locally contractible metric space and let $f : X \to \mathbb{R}$ satisfying the (PS) condition. If f is bounded from below, then f has at least $f cat(X)$ critical points.*

Let $f, g : X \to \mathbb{R}$ be continuous functions with $g > 0$. For any $\varepsilon > 0$ enough small we consider the perturbation $f^\varepsilon = f + \varepsilon g$ such that the condition (3.1) of Section 3.1 is fulfilled.

We estimate the minimal number of critical points of f by using the general category of the space. We prove first the following property.

Proposition 3.14. *Let $k \in \mathbb{N}$ such that $fcat_X(B) \geq k$. Then there exists a perturbation f^ε of f such that $f^\varepsilon cat_X(B) \geq k$.*

Proof. Proposition 3.12 shows that there exists $\delta > 0$ such that if $\sup_{x \in B} |f^\varepsilon(x) - f(x)| \leq \delta$, then $fcat_X(B) \leq f^\varepsilon cat_X(B)$.

The hypothesys (3.1) implies, in fact, that εg is bounded on X; then we can choose, from the family of perturbations of f, a function f^ε such that $\sup_{x \in B} |f^\varepsilon(x) - f(x)| = \varepsilon \sup_{x \in B} |g(x)| \leq \delta$, i.e. we can choose $\varepsilon > 0$ enough small such that $\varepsilon \sup_{x \in B} |g(x)| \leq \delta$. □

Proposition 3.15. *Let $k \in \mathbb{N}$ be such that $f^\varepsilon cat_X(B) \geq k$, for any $\varepsilon > 0$. Then $fcat_X(B) \geq k$.*

Proof. We apply the Proposition 3.12. □

We can now formulate the main result of this section.

Theorem 3.16. *Let X be a locally contractible metric space and let $k \in \mathbb{N}$ be such that $f^\varepsilon cat_X(B) \geq k$, for any $\varepsilon > 0$. If f is bounded from below and satisfies the (PS) condition, thenf has at least k critical points.*

Proof. From Proposition 3.15 we obtain that $fcat_X(B) \geq k$. Then we apply the Theorem 3.13. □

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